Tableau metatheory for propositional and syllogistic logics

Part I: Basic notions: logic, arguments and schemas

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Introduction

- The aim of the lectures (slides: Part I, Part II, Part III, Part IV, Part V) is to present a tableau metatheory for logics that can be determined by generalized relational models.
- Tableau methods are usually presented in a rather intuitive, non–formal and didactic way.
- But we try to present these methods in a strictly formal way.
- By dint of the material we will cover it is possible to construct tableau systems for various logics and to prove almost automatically the adequacy to their semantics.
- However, we start with some basic notions and then we consider more and more complex problems.

Program of lecture

We are going to:

- introduce abbreviations, notations and notions we will use
- analyze a problem: what is logic as a science about?
- consider what arguments, their validity/invalidity and the relationship of both to criterions provided by logical systems are
- describe a problem of expressibility and resolving power of formal languages.

We will discuss all the known as well as unknown problems from a pretty philosophical, abstract perspective.

Abbreviations, notations and notions

In our lectures we will use some basic notions of set-theory:

(a) sets X, Y, Z, ... and relations: $\subset$, $\subseteq$, $\in$ etc

(b) operations on sets: $\cap$, $\cup$, $\setminus$, $\mathsf{P}(X)$

(c) Cartesian products: $X_1 \times \cdots \times X_n$, where $1 < n \in \mathbb{N}$

(d) n-ary relations as subsets of n-ary Cartesian products: $R \subseteq X_1 \times \cdots \times X_n$, where $n \in \mathbb{N}$

(e) functions.

Systems of logics we generally denote by roman font (textrm), for example: CPL, $\mathrm{S5}$, or K.

What is logic about?

Logic as a science examines (among others):

(1) logical structures:

   (a) logical systems (particular logics)
   (b) semantic structures for logical systems
   (c) methods of proving

(2) relationships between various logical structures.

Logical systems

- Logical systems (or particular logics) are always defined on some formal language L.
- Every logic can be identified with a relation $R_{\mathsf{L}} \subseteq \mathsf{P}(\mathsf{L}) \times \mathsf{L}$.
- Every logic combines sets of premises $X \subseteq \mathsf{L}$ with conclusions $A \in \mathsf{L}$.
- An ordered pair $\langle X, A \rangle \in \mathsf{P}(\mathsf{L}) \times \mathsf{L}$ is *an argument*.
- If $R_{\mathsf{L}}(X, A)$, then argument according to logic R_{L} is *valid*.
- If it is not that $R_{\mathsf{L}}(X, A)$, then argument according to logic R_{L} is *invalid*.

Reasoning

We will use terms: *argument*, *reasoning* and *inference* interchangeably.

In the case of finite numbers of premises we may represent arguments as follows:

$$\begin{array}{l} Premise_1 \\ . \\ . \\ . \\ Premise_n \\ \hline Conclusion \end{array}$$

for a natural number n.

How do we draw conclusions?

Every logic provides a criterion of being valid for an argument.

It means that an argument might be valid under some logic and at the same time invalid under other logic.

Hence, from some set of premises a conclusion may follow under one logic, but under other logic it may not follow.

Let us analyze few examples.

Logic as criterion: example of transitivity of conditional

(A0)

If Mark died, then Steve marries Mark's wife.
If Steve marries Mark's wife, then Mark kills Steve.

If Mark died, then he kills Steve.

Logic as criterion: example of transitivity of conditional

Let us look at the reasoning again:

$$\text{(A0)}\quad \frac{\begin{array}{l}\text{If Mark died, then Steve marries Mark's wife.}\\ \text{If Steve marries Mark's wife, then Mark kills Steve.}\end{array}}{\text{If Mark died, then he kills Steve.}}$$

Now we treat:

- If . . . , then . . . as an implication $\rightarrow$
- Mark died as atomic proposition p
- Steve marries Mark's wife as atomic proposition q
- Mark/he kills Steve as atomic proposition r.

Logic as criterion: example of transitivity of conditional

Now, let us look at the schema of the reasoning (A0):

$$(SA0) \qquad \frac{\begin{array}{c} p \to q \\ q \to r \end{array}}{p \to r}$$

The schema (SA0) is for example:

- valid by criterion provided by Classical Propositional Logic (CPL) and all of its extensions, especially modal extensions
- valid by criterion provided by three-valued Łukasiewicz logic ($\text{Ł}3$)
- but invalid by criterion provided by relating logic with non-transitive relation of relating.

Logic as criterion: example of Law of Excluded Middle application

$$
\text{(A1)} \quad \frac{\begin{array}{l}\text{If it is raining today, then Mark is in Canberra.}\\ \text{If it is not raining today, then Mark is in Canberra.}\end{array}}{\text{Mark is in Canberra.}}
$$

Logic as criterion: example of Law of Excluded Middle application

Let us look at the reasoning again:

(A1)

$$\frac{\begin{array}{l}\text{If it is raining today, then Mark is in Canberra.}\\ \text{If it is not raining today, then Mark is in Canberra.}\end{array}}{\text{Mark is in Canberra.}}$$

Now, we can treat:

- If . . . , then . . . as an implication $\rightarrow$
- not as a negation $\neg$
- It is raining today as atomic proposition p
- Mark is in Canberra as atomic proposition q.

Logic as criterion: example of Law of Excluded Middle application

Let us look at the schema of the reasoning:

$$(\text{SA1}) \qquad \begin{array}{l} p \rightarrow q \\ \neg p \rightarrow q \\ \hline q \end{array}$$

The schema is for example:

- valid by criterion provided by CPL and all of its extensions, especially modal extensions
- invalid by criterion provided by three-valued $\mathrm{Ł3}$
- invalid by intuitionistic logic IL
- invalid by criterion provided by many relating logics.

Logic as criterion: example of modalities

$$\text{(A2)} \qquad \frac{\text{Mark is in Canberra.}}{\text{It is possible that Mark is in Canberra.}}$$

$$\text{(A3)} \qquad \frac{\text{Mark is in Canberra.}}{\text{It is possible that it is necessary that Mark is in Canberra.}}$$

Logic as criterion: example of modalities

Let us look at the reading again:

$$\text{(A2)}\qquad \frac{\text{Mark is in Canberra.}}{\text{It is possible that Mark is in Canberra.}}$$

$$\text{(A3)}\qquad \frac{\text{Mark is in Canberra.}}{\text{It is possible that it is necessary that Mark is in Canberra.}}$$

Now we can treat:

- it is possible that ... as modality $\Diamond$
- it is necessary that ... as modality $\Box$
- Mark is in Canberra as atomic proposition p.

Logic as criterion: example of modalities

Now we can look at the schemas of the reasoning:

$$(\mathrm{SA2}) \quad \frac{p}{\Diamond p} \qquad\qquad (\mathrm{SA3}) \quad \frac{p}{\Box\Diamond p}$$

The schemas are for example:

- valid by criterion provided by modal logic $\mathrm{S5}$
- (SA2) is valid in modal logic T, but (SA3) is invalid in T
- both schemas are invalid in modal logic K.

Logic as criterion and issue of expressibility

- A necessary condition for an argument to be valid from the point of view of a given logic is to be expressible in the language of that logic.
- Surely, it is not a sufficient condition: an argument can be expressible in a language of a given logic, but still invalid.

Logic as criterion and issue of expressibility

Let us analyze (A2) and (A3) once again:

(A2) $\dfrac{\text{Mark is in Canberra.}}{\text{It is possible that Mark is in Canberra.}}$

(A3) $\dfrac{\text{Mark is in Canberra.}}{\text{It is possible that it is necessary that Mark is in Canberra.}}$

Now we can treat:

- Mark is in Canberra as atomic proposition p
- It is possible that Mark is in Canberra as atomic proposition q
- It is possible that it is necessary that Mark is in Canberra as atomic proposition r.

Logic as criterion and issue of expressibility

$$\text{(SA2)'} \qquad \frac{p}{q} \qquad\qquad\qquad \text{(SA3)'} \qquad \frac{p}{r}$$

The schemas are:

- invalid almost in all propositional logics that are closed under substitution
- because propositions p, q, and r might have nothing in common; are logically independent
- there is at least one exception: trivial logic $R_{TR} = \mathrm{P(L)} \times \mathrm{L}$
- to express arguments (A2) and (A3) exactly a language must be equipped with two unary operators: $\Box$-like and $\Diamond$-like, as it is in modal logic
- in the enriched propositional language they can be expressed as schemas (SA2) and (SA3).

Expressibility and resolving power of language: examples

To represent properly arguments in a given logic its language should dispose of a suitably rich resolving power.

Let us look at some reasoning:

(A4)

All crocodiles are reptiles.
All reptiles are animals.

All crocodiles are animals.

(A5)

Some animal must be a crocodile.
Every crocodile is necessarily a reptile.

Some animal may be a reptile.

Expressibility and resolving power of language: examples

Let us look at the reasoning again:

(A4)

All crocodiles are reptiles.
All reptiles are animals.

All crocodiles are animals.

(A5)

Some animal must be a crocodile.
Every crocodile is necessarily a reptile.

Some animal may be a reptile.

Expressibility and resolving power of language: examples

Now we can treat:

- All crocodiles are reptiles as atomic proposition p
- All reptiles are animals as atomic proposition q
- All crocodiles are animals as atomic proposition r
- Some animal must be a crocodile as atomic proposition s
- Every crocodile is necessarily a reptile as atomic proposition t
- Some animal may be a reptile as atomic proposition u.

Expressibility and resolving power of language: examples

$$(SA4)\quad \frac{\begin{array}{c}p\\ q\end{array}}{r} \qquad\qquad (SA5)\quad \frac{\begin{array}{c}s\\ t\end{array}}{u}$$

The schemas are:

- invalid almost in all propositional logics that are closed under substitution
- because propositions p, q, and r; and propositions s, t, u have nothing in common; are logically independent
- again, there is at least one exception: trivial logic $R_{TR} = \mathrm{P}(\mathrm{L}) \times \mathrm{L}$
- we cannot express arguments (A4), (A5) in any propositional language, since no additional propositional operators/connectives are sufficient to describe the relations between these sentences
- we need a more subtle language: syllogistic or first order one.

Expressibility and resolving power of language: examples

Let us look at the arguments the last time:

(A4)

All crocodiles are reptiles.
All reptiles are animals.

All crocodiles are animals.

(A5)

Some animal must be a crocodile.
Every crocodile is necessarily a reptile.

Some animal may be a reptile.

Expressibility and resolving power of language: examples

Now we can treat:

- *crocodile*, *reptile*, *animal* as terms: P, Q, S, respectively
- All ... are ... as logical constant $\mathbf{a}$
- Some ... must be ... as logical constant $\mathbf{i}^{\Box}$
- Every ... is necessarily ... as logical constant $\mathbf{a}^{\Box}$
- Some ... may be ... as logical constant $\mathbf{i}^{\Diamond}$.

Expressibility and resolution power of language: examples

Let us look at the schemas:

$$\text{(SA4)'} \quad \frac{\begin{array}{c} P\mathbf{a}Q \\ Q\mathbf{a}S \end{array}}{P\mathbf{a}S} \qquad\qquad \text{(SA5)'} \quad \frac{\begin{array}{c} S\mathbf{i}^{\Box}P \\ P\mathbf{a}^{\Box}Q \end{array}}{S\mathbf{i}^{\Diamond}Q}$$

- schema (SA4)' is valid in Classical Syllogistic (CS)
- schema (SA5)' can not be expressed in Classical Syllogistic
- schema (SA5)' is valid in Classical Syllogistic with *de re* modalities.

Schema vs. argument: examples

Let us analyze two inferences:

(A6)

If John is a crocodile, then he is a reptile.
If John is a reptile, then he is an animal.
John is not an animal.

John is not a crocodile.

(A7)

If $1 > n$, then $2 > n$.
If $2 > n$, then $5 > n$.
$5 \not> n$

$1 \not> n$

Schema vs. argument: examples

- Inferences (A6), (A7) include propositions with a very different content.
- However, after some reduction to an artificial language they are instances of the same schema of this language.
- Logical systems are not about the content, but about schemas of reasoning that are expressible in the language of a given logic.

Schema vs. argument: examples

We assume that:

$$\ldots \not> \ldots$$

means:

$$\text{It is not that } \ldots > \ldots$$

Schema vs. argument

Let us have a look at (A6), (A7) again:

(A6)

If John is a crocodile, then he is a reptile.
If he is a reptile, then John is an animal.
John is not an animal.

John is not a crocodile.

(A7)

If $1 > n$, then $2 > n$.
If $2 > n$, then $5 > n$.
It is not that $5 > n$.

It is not that $1 > n$.

Schema vs. argument

We can treat:

- If ..., then ... as an implication $\rightarrow$
- not and It is not that as a negation $\neg$
- John is a crocodile and $1 > n$ as atomic proposition p
- He is a reptile and $2 > n$ as atomic proposition q
- John is an animal and $5 > n$ as atomic proposition r.

Schema vs. argument

Then, it turns out that both inferences have the same schema:

$$\text{(SA5\&6)} \qquad \begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \neg r \\ \hline \neg p \end{array}$$

- If (SA5&6) is valid according to some logic, then both arguments (A5) and (A6) are valid.
- Every argument that is an instance of (SA5&6) is valid exactly if the remaining instances of (SA5&6) are also valid.

Schema vs. argument: conlcusions

- In logic we examine schemas of reasoning, not particular instances.
- That is why logical systems are determined on artificial, formal languages.
- These languages are deprived of content as much as it is possible.
- Only logical constants have under a set of intended interpretations an invariable meaning.
- The meaning of the rest symbols (e.g. terms, propositions) of these languages may vary from one interpretation to another.

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