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Cut-elimination for Weak Grzegorczyk Logic *Go*

Abstract. We present a syntactic proof of cut-elimination for weak Grzegorczyk logic *Go*. The logic has a syntactically similar axiomatisation to Gödel–Löb logic *GL* (provability logic) and Grzegorczyk's logic *Grz*. Semantically, *GL* can be viewed as the irreflexive counterpart of *Go*, and *Grz* can be viewed as the reflexive counterpart of *Go*. Although proofs of syntactic cut-elimination for *GL* and *Grz* have appeared in the literature, this is the first proof of syntactic cut-elimination for *Go*. The proof is technically interesting, requiring a deeper analysis of the derivation structures than the proofs for *GL* and *Grz*. New transformations generalising the transformations for *GL* and *Grz* are developed here.

Keywords: Proof theory, Syntactic cut-elimination, *Go*, Weak Grzegorczyk logic.

1. Introduction

The logic *Go* is the smallest normal modal logic containing *K* and the axioms $\Box p \supset \Box\Box p$ and $\Box(\Box(p \supset \Box p) \supset p) \supset \Box p$. The logic is sound and complete with respect to the class of transitive frames with no proper clusters and no proper ∞-$\mathcal{R}$-chains [8], and it is a proper sublogic of both Gödel–Löb logic *GL* (also known as provability logic) and Grzegorzyk's logic *Grz*. From a semantic viewpoint, *GL* can be viewed as the irreflexive counterpart of *Go*, and *Grz* can be viewed as the reflexive counterpart of *Go*.

The logic *Go* has received attention in the literature, sometimes under the name *weak Grzegorczyk logic wGrz*. Litak [10] gives a survey of results on *Go*. Litak observes the possibility of using this logic to study properties common to *GL* and *Grz* and also to investigate the role of reflexivity in the semantics of a given logic. The former observation seems to arise from the fact that the transformations presented here for syntactic cut-elimination for *Go* generalise the transformations for *GL* and *Grz*. Further semantic properties of *Go* have been established by Amebauer [1] and Gabelaia [6]. Esakia [5] presents results concerning the class of normal extensions of *Go*.

Presented by **Heinrich Wansing**; *Received* October 17, 2011

Studia Logica
DOI: 10.1007/s11225-012-9432-9

Published online: 06 December 2012

Table 1. The sequent calculus GoS. Note: $i \in \{1, 2\}$ in the rules $L\wedge$ and $R\vee$.

Initial sequents: $A \Rightarrow A$ for each formula A

Logical rules:

$$\frac{X \Rightarrow Y, A}{X, \neg A \Rightarrow Y}\ L\neg \qquad \frac{A, X \Rightarrow Y}{X \Rightarrow Y, \neg A}\ R\neg$$

$$\frac{A_i, X \Rightarrow Y}{A_1 \wedge A_2, X \Rightarrow Y}\ L\wedge \qquad \frac{X \Rightarrow Y, A_1 \qquad X \Rightarrow Y, A_2}{X \Rightarrow Y, A_1 \wedge A_2}\ R\wedge$$

$$\frac{A_1, X \Rightarrow Y \qquad A_2, X \Rightarrow Y}{A_1 \vee A_2, X \Rightarrow Y}\ L\vee \qquad \frac{X \Rightarrow Y, A_i}{X \Rightarrow Y, A_1 \vee A_2}\ R\vee$$

$$\frac{X \Rightarrow Y, A \qquad B, X \Rightarrow Y}{A \supset B, X \Rightarrow Y}\ L\supset \qquad \frac{A, X \Rightarrow Y, B}{X \Rightarrow Y, A \supset B}\ R\supset$$

Modal rule:

$$\frac{\Box X, X, \Box(A \supset \Box A) \Rightarrow A}{\Box X \Rightarrow \Box A}\ GoR$$

Structural rules:

$$\frac{X \Rightarrow Y}{A, X \Rightarrow Y}\ LW \qquad \frac{X \Rightarrow Y}{X \Rightarrow Y, A}\ RW$$

$$\frac{A, A, X \Rightarrow Y}{A, X \Rightarrow Y}\ LC \qquad \frac{X \Rightarrow Y, A, A}{X \Rightarrow Y, A}\ RC$$

Cut-rule:

$$\frac{X \Rightarrow Y, A \qquad A, U \Rightarrow W}{X, U \Rightarrow Y, W}\ cut$$

A sequent calculus GoS for Go (see Table 1) is obtained by extending a suitable calculus for classical propositional logic with the modal rule GoR [1]:

$$\frac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\ GoR$$

Observe that GoS contains the cut-rule. Showing that it is always possible via constructive transformations to eliminate the cuts in a given derivation to obtain a cutfree derivation of the same sequent is called *syntactic cut-elimination*. It is one of the most important results in the proof theory of a logic and the existence of such transformations is a highly desirable property for a sequent calculus. The first such proof was given by Gentzen [7]—for the classical and intuitionistic sequent calculi LK and LJ respectively—who recognised the importance of a constructive procedure in his celebrated *Hauptsatz* or 'main theorem'.

A proof of syntactic cut-elimination is known for Grz [3], and while there has been some controversy regarding Valentini's [15] proof for GL, the issues are now resolved [9]. Here we show syntactic cut-elimination for GoS. To our knowledge, this is the first proof of syntactic cut-elimination for Go. We observe that cut-elimination for Go is not just a simple variation of the proofs for GL and Grz. Indeed, although Valentini's [15] transformations for GL remain an inspiration for our transformations, the proof presented here generalises the methods used for GL and Grz and is technically interesting. The added complexity is reflected in the fact that our proof uses a quaternary induction measure (three induction variables suffice for GL and Grz).

In the *Hauptsatz*, Gentzen relied on a primary induction on the degree of the cut-formula and secondary induction on cut-height. Suppose that cut_1 and cut_2 denote two occurrences of the cut-rule in some derivation, with the degree of the cut-formula and cut-height given by the tuples (d_1, s_1) and (d_2, s_2) respectively. Write $cut_1 < cut_2$ to mean that either $d_1 < d_2$, or $d_1 = d_2$ and $s_1 < s_2$. If we attempt a proof for Go following the proof of the *Hauptsatz*, we find that the only case deserving special attention is the case when both the cut-premises are the conclusions of GoR rules. Consider the following derivation where we assume without loss of generality that the derivations of both premises of cut_0 are cutfree:

$$\dfrac{\dfrac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\,GoR \qquad \dfrac{\Box B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box B, \Box U \Rightarrow \Box C}\,GoR}{\Box X, \Box U \Rightarrow \Box C}\,cut_0$$

It is not obvious how to proceed from here. If we could obtain a *cutfree* derivation of $\Box X, X \Rightarrow B$ then we may proceed as shown below:

$$\dfrac{\dfrac{\dfrac{\Box X, X \Rightarrow B \qquad \dfrac{\dfrac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\,GoR \qquad \Box B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box X, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,cut_1}{\Box X, \Box X, X, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,cut_2}{\Box X, X, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,LC^*}{\Box X, \Box U \Rightarrow \Box C}\,GoR$$

Notice that $cut_1 < cut_0$ and $cut_2 < cut_0$ so, appealing to the induction hypothesis, we can obtain a cutfree derivation of $\Box X, \Box U \Rightarrow \Box C$ as required. Thus it suffices to obtain a cutfree derivation of $\Box X, X \Rightarrow B$. Certainly, it is easy to show that $\Box X, X \Rightarrow B$ is derivable whenever $\Box X, X, \Box(B \supset \Box B) \Rightarrow B$ is derivable. Consider the following derivation, where LW^* (resp. LC^*) denotes some number of applications of the left weakening (contraction) rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\,GoR}{\Box X, X, \Box((B \supset \Box B) \supset \Box(B \supset \Box B)), B \Rightarrow \Box B}\,LW^*}{\Box X, X, \Box((B \supset \Box B) \supset \Box(B \supset \Box B)) \Rightarrow B \supset \Box B}\,R\supset}{\Box X \Rightarrow \Box(B \supset \Box B)}\,GoR \qquad \Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\dfrac{\Box X, \Box X, X \Rightarrow B}{\Box X, X \Rightarrow B}\,LC^*}\,cut_3$$

However, since $cut_0 < cut_3$ we cannot appeal here to the induction hypothesis to eliminate cut_3.

In this paper we show how to resolve this difficulty by presenting a cutfree derivation of $\Box X, X \Rightarrow B$ from a cutfree derivation of $\Box X, X, \Box(B \supset \Box B) \Rightarrow B$. This situation parallels the approach to cut-elimination for the calculus GLS for GL [15,4,13,9] and $GrzS$ for Grz [3]. In GLS for example, the main task is to obtain a cutfree derivation of $\Box X, X \Rightarrow B$ from a cutfree derivation of $\Box X, X, \Box B \Rightarrow B$. In $GrzS$, the main task is to obtain a cutfree derivation of $\Box X \Rightarrow B$ from a cutfree derivation of $\Box X, \Box(B \supset \Box B) \Rightarrow B$. Thus the obvious approach for GoS would be to draw on the syntactic proofs of cut-elimination for GLS and $GrzS$. We discuss the difficulties in adapting those proofs to GoS in Sect. 4.

Finally, we remind the reader that it is straightforward to show that the cut-rule is redundant by proving that the calculus without the cut-rule is sound and complete for the frame semantics of Go (see [1]). However, a disadvantage with relying on a semantic proof is that the cutfree derivation obtained using this result (via a backward proof search, for example) bears little resemblance, structurally, to the original derivation. In contrast, by syntactic cut-elimination, the cutfree derivation is obtained from the original derivation by constructive transformations.

2. Basic Definitions and Notation

Formulae are constructed in the usual way from propositional variables using the logical connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and the modal operator $\Box$. Propositional variables are written using $p, q, \ldots$ and formulae are denoted by $A, B, C, \ldots$. Multisets of formulae are denoted by $X, Y, \ldots$ and also by listing the members of the multiset within $\langle\ \ \rangle$. We write $\boxtimes A$ to denote the multiset $\langle \Box A, A\rangle$. Let X be the multiset $\langle A_1, \ldots, A_n\rangle$. Then we write $\Box X$ and $\boxtimes X$ to mean the following multisets respectively:

$$\langle \Box A_1, \ldots, \Box A_n\rangle \qquad\qquad \langle \Box A_1, \ldots, \Box A_n, A_1, \ldots, A_n\rangle$$

The notation A^m denotes the multiset $\overbrace{\langle A, \ldots, A\rangle}^{\text{m occurrences}}$.

A *sequent* is a tuple (X, Y) of multisets X and Y and is written $X \Rightarrow Y$. The symbol $\cup$ is used to denote multiset union. The multiset X and Y are called respectively the *antecedent* and *succedent* of the sequent. The sequent calculus *GoS* is defined in Table 1. As usual, in the antecedent or succedent of a sequent, A, U denotes the multiset union of $\langle A\rangle$ and U, and more generally, U, V denotes the multiset union of U and V. An (inference) rule ρ is constructed from sequents $X_1 \Rightarrow Y_1 \ldots X_n \Rightarrow Y_n, X \Rightarrow Y$ for $n \geq 0$ and is depicted as $X_1 \Rightarrow Y_1 \ldots X_n \Rightarrow Y_n /^{\rho} X \Rightarrow Y$, and also as

$$\frac{X_1 \Rightarrow Y_1 \qquad \ldots \qquad X_n \Rightarrow Y_n}{X \Rightarrow Y}\,\rho$$

The sequents $X_1 \Rightarrow Y_1, \ldots, X_n \Rightarrow Y_n$ are called the *premise(s)* of the rule, and the sequent $X \Rightarrow Y$ is called the *conclusion* of the rule. When $n = 1$ (resp. $n = 2$) we say that ρ is a one-premise (two-premise) rule.

A *derivation* (in *GoS*) is defined with reference to Table 1 in the usual manner, as the smallest set satisfying the following: for any formula A, the initial sequent $A \Rightarrow A$ is a derivation; an application of any one-premise rule to the conclusion of a derivation is a derivation; an application of any two-premise rule to the conclusions of two derivations is a derivation.

For the logical and structural rules in *GoS* (Table 1), the multisets X and Y are called the *context*. In the modal rule *GoR*, the context is the multisets $\Box X, X$ in the premise, and $\Box X$ in the conclusion. Furthermore, in each premise of a rule, a formula not in the context will be called an *auxiliary formula*. Informally, the auxiliary formulae are those formulae in the premise that the rule acts upon. Finally, in the conclusion of a rule, the formula not in the context is called a *principal formula*. For example, in the *GoR* rule, the context is $\Box X, X$ in the premise and $\Box X$ in the the conclusion. The principal formula is $\Box A$ and the auxiliary formulae are $\Box(A \supset \Box A)$ (in the antecedent) and A (in the succedent). Using this terminology, given an instance ρ of a rule in a derivation, we can always identify the context, auxiliary and principal formulae of ρ. Following the terminology for *GLS* [12], the formula A in the *GoR* rule in Table 1 is called the *diagonal formula*. We write $\rho(A)$ to identify the principal formula (the diagonal formula in the case of *GoR*) in the rule instance ρ. Repeated consecutive applications of a rule ρ (not necessarily with the same principal formula) are denoted by ρ^*.

In the cut-rule in Table 1, the formula A is the *cut-formula*. A derivation is said to be *cutfree* if it contains no instances of the cut-rule. Viewing a derivation as a tree, we call the root of the tree the *endsequent* of the derivation. We use the phrase 'upwards' informally to mean the direction from the endsequent to the initial sequents. 'Downwards' is the direction towards the endsequent. The phrases 'above' and 'below' are used with respect to these directions. The rule having as conclusion the endsequent of a derivation is the *endrule* of the derivation. If there is a derivation with endsequent $X \Rightarrow Y$ we say that $X \Rightarrow Y$ is *derivable* in *GoS*.

Let $\bigwedge X$ ($\bigvee Y$) denote the conjunction (disjunction) of all formula occurrences in X (Y). It is straightforward to show that a sequent $X \Rightarrow Y$ is derivable in *GoS* iff the formula $\bigwedge X \supset \bigvee Y$ is a theorem of the logic *Go*. That is, *GoS* is sound and complete with respect to *Go* and thus *GoS* is a sequent calculus for *Go*. Our cut-elimination result for *GoS* shows that the calculus minus the cut-rule is (also) sound and complete for *Go*.

We define the height, cut-height, and degree of a formula as usual.

DEFINITION 2.1 (height, cut-height, degree). The height $h(\tau)$ of a derivation τ is the greatest number of successive applications of rules in it plus one. The *cut-height* s of an instance of the cut-rule with premise derivations τ_1 and τ_2 is $h(\tau_1) + h(\tau_2)$. The *degree* $|A|$ of a formula A is defined as the number of symbol occurrences in A from $\{\Box, \neg, \wedge, \vee, \supset\}$.

Following Belnap [2], for a formula occurrence in the conclusion of a rule instance, a *congruent formula occurrence* is defined in the obvious way as follows (see the definition of context and principal formula on page 5):

(i) a formula occurring in the context in the conclusion is congruent to the corresponding formula occurrence(s) in the context(s) in the premise(s)

(ii) the principal formula in the conclusion is congruent to the auxiliary formulae in the premise(s).

Suppose that an *occurrence* (denoted $\hat{A}$) of some formula A occurs in the antecedent of a sequent $\mathcal{S}$ of the form $A, X \Rightarrow Y$ in some derivation τ and simultaneously proceed analogously when the $\hat{A}$ appears in the succedent of a sequent. Motivated by Belnap's [2] definition of parametric ancestor for the display calculus, define the set $\mathcal{PA}$ of *parametric ancestors* of $\hat{A}$ recursively as the smallest set of formula occurrences satisfying the following:

base: the occurrence $\hat{A}$ is in $\mathcal{PA}$

inductive: if $\mathcal{S}$ is an initial sequent, there is nothing more to do. Otherwise, let δ (shown below) denote the subderivation of τ deriving $\mathcal{S}$ and let ρ denote the endrule in δ:

$$\dfrac{\dfrac{\delta_1}{X_1 \Rightarrow Y_1} \quad \dots \quad \dfrac{\delta_n}{X_n \Rightarrow Y_n}}{A, X \Rightarrow Y}\ \rho$$

Let $\hat{\hat{B}}_1, \dots, \hat{\hat{B}}_n$ denote the sets of occurrences of formulae congruent to $\hat{A}$, in the premises $X_1 \Rightarrow Y_1, \dots, X_n \Rightarrow Y_n$ of ρ respectively. Then the parametric ancestors of all the occurrences in $\hat{\hat{B}}_1, \dots, \hat{\hat{B}}_n$ are in $\mathcal{PA}$.

Observe that a formula occurrence $\hat{A}$ is congruent to two formulae occurrences in the premise(s) of rule ρ if (i) $\rho = GoR$, or (ii) ρ is a contraction rule or a two-premise rule and $\hat{A}$ is the principal formula.

For simplicity, from now on we will not explicitly distinguish between a formula A and an occurrence $\hat{A}$ of that formula. The surrounding text will indicate the case that is referred. Here is an example to illustrate the construction of this set. The parametric ancestors of the formula $\Box(p \wedge q)$ in the endsequent are highlighted in bold

$$\dfrac{\dfrac{\dfrac{p \Rightarrow \boldsymbol{p}}{\boxtimes p, \boxtimes q, \boldsymbol{\Box(p \wedge q \supset \Box(p \wedge q))} \Rightarrow \boldsymbol{p}}\ LW^* \quad \dfrac{q \Rightarrow \boldsymbol{q}}{\boxtimes p, \boxtimes q, \boldsymbol{\Box(p \wedge q \supset \Box(p \wedge q))} \Rightarrow \boldsymbol{q}}\ LW^*}{\boxtimes p, \boxtimes q, \boldsymbol{\Box(p \wedge q \supset \Box(p \wedge q))} \Rightarrow \boldsymbol{p \wedge q}}}{\Box p, \Box q \Rightarrow \boldsymbol{\Box(p \wedge q)}}\ GoR$$

Is the set $\mathcal{PA}$ uniquely defined? In general, no. Consider the following examples where we present two different parametric ancestor sets for the formula occurrence $A \wedge B$ in the endsequent:

$$\dfrac{\begin{array}{c}\vdots\\ \boldsymbol{A \wedge B}, A, B, X \Rightarrow Y\end{array}}{\boldsymbol{A \wedge B}, A \wedge B, X \Rightarrow Y}\ \wedge \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ A \wedge B, \boldsymbol{A}, \boldsymbol{B}, X \Rightarrow Y\end{array}}{\boldsymbol{A \wedge B}, A \wedge B, X \Rightarrow Y}\ \wedge$$

However, for our purposes, given a derivation and a formula occurrence A, it will be sufficient to select a set of parametric ancestors of A and stick with this set throughout the discussion. For this reason we refrain from formalising this notion any further.

2.1. Preliminary Results

LEMMA 2.2 (invertibility of $L\supset$). *Suppose that τ is a cutfree derivation of $A \supset B, X \Rightarrow Y$. Then there are transformations to obtain cutfree derivations of $X \Rightarrow Y, A$ and $B, X \Rightarrow Y$.*

PROOF. Because *GoS* contains contraction rules, it is helpful to prove the stronger statement: if τ is a cutfree derivation of $(A \supset B)^{m+1}, X \Rightarrow Y$

with $m \geq 0$ then there are cutfree derivations of $A^{m+1}, X \Rightarrow Y$ and $X \Rightarrow B^{m+1}, Y$. The argument is a standard induction on the height of τ. ∎

Note that the transformations do not necessarily preserve the height of the original derivation.

LEMMA 2.3. *Let τ be a cutfree derivation of $X, \Box(B \supset \Box B)^{m+1} \Rightarrow Y$ with $m \geq 0$. Then there is a transformation to a cutfree derivation of $X, (\Box B)^{m+1} \Rightarrow Y$.*

PROOF. Induction on the height of τ. If τ is an initial sequent of the form $\Box(B \supset \Box B) \Rightarrow \Box(B \supset \Box B)$, hence X is empty, replace with

$$\cfrac{\cfrac{\cfrac{\Box B \Rightarrow \Box B}{\Box B, B, B, \Box((B \supset \Box B) \supset \Box(B \supset \Box B)) \Rightarrow \Box B}\ LW^*}{\Box B, B, \Box((B \supset \Box B) \supset \Box(B \supset \Box B)) \Rightarrow B \supset \Box B}\ R{\supset}}{\Box B \Rightarrow \Box(B \supset \Box B)}\ GoR$$

Otherwise, consider the last rule in τ. We only present the cases where the last rule is a GoR rule or a LC rule.

When the last rule in τ is the GoR rule, τ has the following form:

$$\cfrac{\boxtimes X, \Box(B \supset \Box B)^{m+1}, (B \supset \Box B)^{m+1}, \Box(C \supset \Box C) \Rightarrow C}{\Box X, \Box(B \supset \Box B)^{m+1} \Rightarrow \Box C}\ GoR$$

Applying the induction hypothesis to the premise we get

$$\boxtimes X, (\Box B)^{m+1}, (B \supset \Box B)^{m+1}, \Box(C \supset \Box C) \Rightarrow C$$

By $m + 1$ applications of Lemma 2.2 we obtain a derivation of

$$\boxtimes X, (\Box B)^{m+1}, (\Box B)^{m+1}, \Box(C \supset \Box C) \Rightarrow C$$

By repeated application of $LC(\Box B)$, and then $LW(B)$, we obtain

$$\boxtimes X, \Box B, B, \Box(C \supset \Box C) \Rightarrow C$$

Applying the GoR rule gives a derivation of $\Box X, \Box B \Rightarrow \Box C$ as required.

When the last rule in τ is LC, τ has the following form:

$$\cfrac{X, \Box(B \supset \Box B)^{m+2} \Rightarrow Y}{X, \Box(B \supset \Box B)^{m+1} \Rightarrow Y}\ LC$$

The induction hypothesis gives a derivation of $X, (\Box B)^{m+2} \Rightarrow Y$. The result follows from an application of the LC rule.

The other cases are similar. ∎

From now on, we will use the following convention: we use the diagonal formula of a *GoR* rule as the label for that rule, writing "C is a *GoR* rule in τ" to refer to an occurrence of a *GoR* rule in τ with diagonal formula C.

Let C_1 and C_2 be two different occurrences of the *GoR* rule in τ. Then C_1 *is above* C_2 if, tracing upwards along the inference rules from conclusion to the premise (a premise), there is a path in τ from C_2 up to C_1. Suppose that C is an occurrence of a *GoR* rule in a derivation τ. Let $C{\downarrow}$ denote the set $\{B_i\}_{i \in I}$ of all *GoR* rules such that C is above B_i for each $i \in I$, and let $\Box Y_i \Rightarrow \Box B_i$ denote the conclusion of B_i. Then we define $\Pi_{C\downarrow}$ as the multiset union of $\{Y_i \cup \langle B_i \supset \Box B_i \rangle\}_{i \in I}$. So $\Box\Pi_{C\downarrow}$ is the multiset union of $\{\Box Y_i \cup \langle \Box(B_i \supset \Box B_i)\rangle\}_{i \in I}$.

Definition 2.4 (boxes persist upwards). Suppose that C is an occurrence of the *GoR* rule with conclusion $\Box Y \Rightarrow \Box C$ in a derivation τ. We say that *boxes persist upwards for* C if each formula occurrence in $\Box\Pi_{C\downarrow}$ occurs either (i) in $\Box Y$ or (ii) is introduced by left weakening immediately below C.

If boxes persist upwards for every *GoR* rule in τ, then we say that *boxes persist upwards in* τ.

Put in another way, if boxes persist upwards in τ and C is a *GoR* rule in τ with conclusion $\Box Y \Rightarrow \Box C$, then for every *GoR* rule D above C, each formula occurrence in the multiset $\Box Y \cup \langle \Box(C \supset \Box C)\rangle$ occurs either

(i) in the antecedent of the conclusion of D, or

(ii) via left weakening immediately below the conclusion of D.

Since $\Pi_{B\downarrow} = \emptyset$ for a bottom-most *GoR* rule B in a derivation, it is always the case that boxes persist upwards for B.

Definition 2.5 (primary parametric ancestors). Let τ be a derivation ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$. Then the *primary parametric ancestors* of τ are all the formula occurrences of the form $\Box B$ or $\Box(B \supset \Box B)$ that are parametric ancestors of the formula occurrence $\Box(B \supset \Box B)$ in the premise of the endrule.

For short, we write "ppa" for the term "primary parametric ancestor(s)". Observe that a ppa can only occur in the antecedent of a sequent.

Definition 2.6 (implication forced). Let τ be a derivation ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$. We say that a *GoR* rule C is *implication forced* if: whenever the conclusion of C contains a ppa $\Box(B \supset \Box B)$, the rule C appears in τ in the following form, where $\Box Y$ contains no ppa and $\bar{k} \in \{0, 1\}$:

$$\dfrac{\dfrac{(\boxtimes \mathbf{B})^{\bar{k}}, \Box(\mathbf{B} \supset \Box \mathbf{B}), \Box Y, Y, \Box(C \supset C) \Rightarrow C, B \qquad (\boxtimes \mathbf{B})^{\bar{k}}, \Box \mathbf{B}, \Box(\mathbf{B} \supset \Box \mathbf{B}), \Box Y, Y, \Box(C \supset C) \Rightarrow C}{(\boxtimes \mathbf{B})^{\bar{k}}, \Box(\mathbf{B} \supset \Box \mathbf{B}), \mathbf{B} \supset \Box \mathbf{B}, \Box Y, Y, \Box(C \supset C) \Rightarrow C} L\supset}{(\Box \mathbf{B})^{\bar{k}}, \Box(\mathbf{B} \supset \Box \mathbf{B}), \Box Y \Rightarrow \Box C} GoR$$

If every GoR rule whose conclusion contains a ppa $\Box(B \supset \Box B)$ is implication forced, then we say that τ is implication forced*.*

Notice that the term 'implication forced' is defined for a derivation τ ending with a *GoR* rule B. From the definition, it follows that a *GoR* rule whose conclusion does not contain a ppa $\Box(B \supset \Box B)$ (such as the endrule B, for example) is trivially implication forced. Furthermore, if the derivation τ is implication forced, then (i) the conclusion of each *GoR* rule C contains at most one ppa of the form $\Box(B \supset \Box B)$, and (ii) when the conclusion of C contains a ppa $\Box(B \supset \Box B)$, then the rule preceding (above) C is $L\supset(B \supset \Box B)$.

Note that in the above proof diagram, only ppa are highlighted in bold. Throughout this paper, **we continue to highlight only ppa (as opposed to all parametric ancestors) in bold.**

Definition 2.7 (normal). Let τ be a cutfree derivation ending with $\boxtimes X$, $\Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$. A *GoR* rule C in τ is said to be *normal* if boxes persist upwards for C and C is implication forced.

A cutfree derivation τ is a *normal* derivation if every *GoR* rule in τ is normal. An *abnormal* derivation is a derivation that is not normal.

Let τ be a cutfree abnormal derivation ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$. Since boxes persist upwards for the endrule B, and since B is implication forced, it follows trivially that B is normal. Now suppose that δ_C is a subderivation in τ whose endsequent is the premise of a *GoR* rule C in τ. We say that *an abnormal derivation τ is abnormal due to δ_C* if every abnormal *GoR* rule in τ occurs in δ_C. Informally, the subderivation δ_C is the (largest) portion of the derivation τ preventing it from being a normal derivation.

In the following we will use the notation $\tau\{\delta_C\}$ to mean the derivation obtained by 'filling the sequent hole' $\{\ \}$ in $\tau\{\ \}$ with δ_C. Then $\tau\{\delta'_C\}$ is obtained from $\tau\{\delta_C\}$ by replacing the subderivation δ_C with δ'_C. Of course, in order for $\tau\{\delta'_C\}$ to be well-defined, δ_C and δ'_C must have the same endsequent.

Lemma 2.8. *Let $\tau\{\delta_{A_0}\}$ be a cutfree abnormal derivation ending as $\boxtimes X$, $\Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$, where δ_{A_0} is a subderivation in τ whose endsequent is the premise of a GoR rule A_0. Suppose that $\tau\{\delta_{A_0}\}$ is abnormal due to δ_{A_0}. Then there is a transformation from δ_{A_0} to δ'_{A_0} so that the derivation $\tau\{\delta'_{A_0}\}$ is a normal derivation.*

PROOF. Since the GoR rule A_0 is in $\tau\{\delta_{A_0}\}$ but not in δ_{A_0}, by the definition of 'abnormal due to', it follows that A_0 is normal in $\tau\{\delta_{A_0}\}$. Consider the following schematic representation of the situation, where $\{A_1, \ldots, A_r\}$ denote the GoR rules in an arbitrary branch of δ_{A_0} (we depict the case when $A_0 \not\equiv B$; when $A_0 \equiv B$, proceed in a similar fashion):

$$
\begin{array}{c}
p_i \Rightarrow p_i \\
\vdots \\
\text{no } GoR \text{ rules} \\
\vdots \\
\dfrac{\boxtimes Y_r, \Box(A_r \supset \Box A_r) \Rightarrow A_r}{\Box Y_r \Rightarrow \Box A_r}\ GoR \\
\vdots \\
\dfrac{\boxtimes Y_1, \Box(A_1 \supset \Box A_1) \Rightarrow A_1}{\Box Y_1 \Rightarrow \Box A_1}\ GoR \\
\vdots \\
\text{no } GoR \text{ rules} \\
\vdots \\
\dfrac{\boxtimes \Pi_{A_0\downarrow}, \boxtimes Y_0, \Box(A_0 \supset \Box A_0) \Rightarrow A_0}{\Box \Pi_{A_0\downarrow}, \Box Y_0 \Rightarrow \Box A_0}\ GoR \\
\vdots \\
\dfrac{\boxtimes X, \boldsymbol{\Box(B \supset \Box B)} \Rightarrow B}{\Box X \Rightarrow \Box B}\ GoR
\end{array}
$$

The idea is to simultaneously transform each such branch in the derivation to the following using appropriate weakening and contraction rules (see below). Notice that all transformations occur *inside* the subderivation δ_{A_0}. Denote the new subderivation we obtain by $\delta^1_{A_0}$ (whose endsequent is once again the premise of A_0).

$$
\begin{array}{c}
\dfrac{p_i \Rightarrow p_i}{\Pi_{A_0\downarrow}, \boxtimes Y_0, \boxtimes(A_0 \supset \Box A_0), \ldots, \boxtimes Y_{r-1}, \boxtimes(A_{r-1} \supset \Box A_{r-1}), p_i \Rightarrow p_i}\ LW^* \\
\vdots \\
\text{no } GoR \text{ rules} \\
\vdots \\
\dfrac{\dfrac{\boxtimes \Pi_{A_0\downarrow}, \boxtimes Y_0, \boxtimes(A_0 \supset \Box A_0), \ldots, \boxtimes Y_{r-1}, \boxtimes(A_{r-1} \supset \Box A_{r-1}), \boxtimes Y_r, \Box(A_r \supset \Box A_r) \Rightarrow A_r}{\Box \Pi_{A_0\downarrow}, \Box Y_0, \Box(A_0 \supset \Box A_0), \ldots, \Box Y_{r-1}, \Box(A_{r-1} \supset \Box A_{r-1}), \Box Y_r \Rightarrow \Box A_r}\ GoR}{\boxtimes \Pi_{A_0\downarrow}, \boxtimes Y_0, \boxtimes(A_0 \supset \Box A_0), \ldots, \boxtimes Y_{r-1}, \Box(A_{r-1} \supset \Box A_{r-1}), \Box Y_r \Rightarrow \Box A_r}\ LW^* \\
\vdots \\
\dfrac{\dfrac{\boxtimes \Pi_{A_0\downarrow}, \boxtimes Y_0, \boxtimes(A_0 \supset \Box A_0), \boxtimes Y_1, \Box(A_1 \supset \Box A_1) \Rightarrow A_1}{\Box \Pi_{A_0\downarrow}, \Box Y_0, \Box(A_0 \supset \Box A_0), \Box Y_1 \Rightarrow \Box A_1}\ GoR}{\boxtimes \Pi_{A_0\downarrow}, \boxtimes Y_0, \Box(A_0 \supset \Box A_0), \Box Y_1 \Rightarrow \Box A_1}\ LW^* \\
\vdots \\
\text{no } GoR \text{ rules} \\
\dfrac{\dfrac{\vdots}{\boxtimes \Pi_{A_0\downarrow}, \boxtimes Y_0, \Box(A_0 \supset \Box A_0) \Rightarrow A_0}\ LC^*(\boxtimes \Pi_{A_0\downarrow}, \boxtimes Y_0, \Box(A_0 \supset \Box A_0))}{\Box \Pi_{A_0\downarrow}, \Box Y_0 \Rightarrow \Box A_0}\ GoR \\
\vdots \\
\dfrac{\boxtimes X, \boldsymbol{\Box(B \supset \Box B)} \Rightarrow B}{\Box X \Rightarrow \Box B}\ GoR
\end{array}
$$

It is straightforward to verify that boxes persist upwards for each *GoR* rule in $\{A_1, \ldots, A_r\}$. We omit the details as the proof is straightforward, if tedious. In this manner, from $\tau\{\delta_{A_0}\}$ we can obtain a derivation $\tau\{\delta^1_{A_0}\}$ ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \ /^{GoR}\ \Box X \Rightarrow \Box B$ such that boxes persist upwards in $\tau\{\delta^1_{A_0}\}$.

Now, the conclusion of each *GoR* rule in $\tau\{\delta^1_{A_0}\}$ either contains a ppa $\Box(B \supset \Box B)$ or does not. To transform $\tau\{\delta^1_{A_0}\}$ to be implication forced, it suffices to consider the former case. Each such *GoR* rule must then have the following form, where $\Box Y$ does not contain any ppa and $k \geq 0, l \geq 1$:

$$\frac{\boldsymbol{(\Box B)^k}, B^k, \boldsymbol{\Box(B \supset \Box B)^l}, (B \supset \Box B)^l, \Box Y, Y, \Box(A \supset \Box A) \Rightarrow A}{\boldsymbol{(\Box B)^k}, \boldsymbol{\Box(B \supset \Box B)^l}, \Box Y \Rightarrow \Box A}\ GoR$$

By repeated application of the contraction rules we obtain a derivation of the following sequent, where $\bar{k} = 1$ if $k > 0$ otherwise $\bar{k} = 0$:

$$(\Box B)^{\bar{k}}, B^{\bar{k}}, \Box(B \supset \Box B), B \supset \Box B, \Box Y, Y, \Box(A \supset \Box A) \Rightarrow A$$

By invertibility of $L\supset$ (Lemma 2.2) we then obtain cutfree derivations of the following two sequents:

$$(\Box B)^{\bar{k}}, B^{\bar{k}}, \Box(B \supset \Box B), \Box Y, Y, \Box(A \supset \Box A) \Rightarrow A, B$$
$$(\Box B)^{\bar{k}}, B^{\bar{k}}, \Box(B \supset \Box B), \Box B, \Box Y, Y, \Box(A \supset \Box A) \Rightarrow A$$

Finally, replace the subderivation ending $(\Box B)^k, \Box(B \supset \Box B)^l, \Box Y \Rightarrow \Box A$ with the derivation:

$$\dfrac{\dfrac{\dfrac{(\boxtimes B)^{\bar{k}}, \Box(B \supset \Box B), \Box Y, Y, \Box(A \supset \Box A) \Rightarrow A, B \qquad (\boxtimes B)^{\bar{k}}, \Box(B \supset \Box B), \Box B, \Box Y, Y, \Box(A \supset \Box A) \Rightarrow A}{(\boxtimes B)^{\bar{k}}, \Box(B \supset \Box B), B \supset \Box B, \Box Y, Y, \Box(A \supset \Box A) \Rightarrow A}\ L\supset}{(\Box B)^{\bar{k}}, \Box(B \supset \Box B), \Box Y \Rightarrow \Box A}\ GoR}{(\Box B)^k, \Box(B \supset \Box B)^l, \Box Y \Rightarrow \Box A}\ LW^*$$

Observe that this transformation does not destroy the property of boxes persisting upwards. Notice that all transformations occur *inside* the subderivation $\delta^1_{A_0}$. Denote the corresponding new subderivation we obtain by $\delta^2_{A_0}$. In this manner, from $\tau\{\delta^1_{A_0}\}$ we can obtain a derivation $\tau\{\delta^2_{A_0}\}$ ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \ /^{GoR}\ \Box X \Rightarrow \Box B$ such that boxes persist upwards in $\tau\{\delta^2_{A_0}\}$ and $\tau\{\delta^2_{A_0}\}$ is implication forced. ∎

The following is a straightforward corollary of the above result.

Corollary 2.9 (normal derivation lemma). *Let τ be a cutfree derivation ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \ /^{GoR}\ \Box X \Rightarrow \Box B$. Then there is a* *transformation to a normal derivation* τ' *ending with* $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$.

Definition 2.10 ($\mathcal{S}_L(C), \mathcal{S}_R(C)$). Let C be an arbitrary occurrence of a *GoR* rule that is not an endrule in a normal derivation τ ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$. Also suppose that the antecedent of the conclusion of C contains a ppa $\Box(B \supset \Box B)$. Since τ is implication-forced, C must then appear in the following form, where $\Box Y$ does not contain any ppa (it is easy to see that $\Box C$ cannot be a ppa) and $\bar{k} \in \{0, 1\}$:

$$\dfrac{\dfrac{(\boldsymbol{\boxtimes B})^{\bar{k}}, \boldsymbol{\Box(B \supset \Box B)}, \boxtimes Y, \Box(C \supset \Box C) \Rightarrow C, B \qquad (\boldsymbol{\boxtimes B})^{\bar{k}}, \boldsymbol{\Box(B \supset \Box B)}, \Box B, \boxtimes Y, \Box(C \supset \Box C) \Rightarrow C}{(\boldsymbol{\boxtimes B})^{\bar{k}}, \boldsymbol{\Box(B \supset \Box B)}, B \supset \Box B, \boxtimes Y, \Box(C \supset \Box C) \Rightarrow C}\, L\supset}{(\boldsymbol{\Box B})^{\bar{k}}, \boldsymbol{\Box(B \supset \Box B)}, \Box Y \Rightarrow \Box C}\, GoR$$

We write $\mathcal{S}_L(C)$ *for 'left sequent of* C*' to denote the left premise of* $L\supset$*:*

$$(\Box B)^{\bar{k}}, B^{\bar{k}}, \Box(B \supset \Box B), \Box Y, Y, \Box(C \supset \Box C) \Rightarrow C, B$$

and $\mathcal{S}_R(C)$ *for 'right sequent of* C*' to denote the right premise of* $L\supset$*:*

$$(\Box B)^{\bar{k}}, B^{\bar{k}}, \Box(B \supset \Box B), \Box B, \Box Y, Y, \Box(C \supset \Box C) \Rightarrow C$$

Suppose that C_1 is an occurrence of the *GoR* rule above C. If C_1 occurs above the sequent $\mathcal{S}_L(C)$ then we say that C_1 is *left-above* C. Similarly, if C_1 occurs above the sequent $\mathcal{S}_R(C)$ then we say that C_1 is *right-above* C. If there is no *GoR* rule on the path between C_1 and C then we say that C_1 is *immediately* left-above (resp. right-above) C.

Definition 2.11 (topmost sequent). Let τ be a cutfree derivation ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$. Suppose that $\mathcal{S}$ is a sequent in τ and let *ppa*($\mathcal{S}$) denote the set of ppa of τ in $\mathcal{S}$. Then $\mathcal{S}$ is called *topmost* if no parametric ancestor of an occurrence in *ppa*($\mathcal{S}$) appears in the conclusion of a *GoR* rule.

Recall that a ppa must be a formula of the form $\Box B$ or $\Box(B \supset \Box B)$. However, a parametric ancestor of a ppa may be a proper subformula of the ppa if a *GoR* rule is encountered while tracing upwards. This happens if and only if there is some parametric ancestor that appears in the conclusion of a *GoR* rule. However, when $\mathcal{S}$ is a topmost sequent, the possibility that some parametric ancestor of a ppa in $\mathcal{S}$ appears in the conclusion of a *GoR* rule is excluded. This means that when tracing each ppa upwards along each branch of the derivation from the topmost sequent, we will encounter the initial sequents $\Box B \Rightarrow \Box B$ or $\Box(B \supset \Box B) \Rightarrow \Box(B \supset \Box B)$ or weakening rules $LW(\Box B)$ or $LW(\Box(B \supset \Box B))$ that 'introduce' the ppa *before* encountering a *GoR* rule.

EXAMPLE 2.12. Consider the following proof diagram of the derivation τ, where in the antecedent of the sequent $\Box B, \Box(B \supset \Box B) \Rightarrow \Box C$, it is assumed that the formula occurrence $\Box(B \supset \Box B)$, but not the formula occurrence $\Box B$, is a ppa (remember that only ppa are highlighted in bold):

$$\dfrac{\dfrac{\dfrac{\vdots}{\mathcal{S}_1 = \Box B, \boldsymbol{\Box(B \supset \Box B)}, \Box(C \supset \Box C) \Rightarrow C}\,GoR}{\boxtimes B, \boldsymbol{\Box(B \supset \Box B)}, \Box(C \supset \Box C) \Rightarrow B, C}\,LW^* \qquad \dfrac{\dfrac{\dfrac{\vdots}{\Box(C \supset \Box C) \Rightarrow C}\,GoR}{\boldsymbol{\Box(B \supset \Box B)}, \boldsymbol{\Box B}, \Box(C \supset \Box C) \Rightarrow C}\,LW^*}{\mathcal{S}_2 = \boxtimes B, \boldsymbol{\Box(B \supset \Box B)}, \boldsymbol{\Box B}, \Box(C \supset \Box C) \Rightarrow C}\,LW^*}{\dfrac{\mathcal{S}_3 = \boxtimes B, \boldsymbol{\Box(B \supset \Box B)}, B \supset \Box B, \Box(C \supset \Box C) \Rightarrow C}{\Box B, \boldsymbol{\Box(B \supset \Box B)} \Rightarrow \Box C}\,GoR}\,L\supset$$

$$\vdots$$

$$\dfrac{\boxtimes X, \boldsymbol{\Box(B \supset \Box B)} \Rightarrow B}{\Box X \Rightarrow \Box B.}\,GoR$$

First observe that the ppa occurrences in $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ are respectively the sets $\{\Box(B \supset \Box B)\}, \{\Box(B \supset \Box B), \Box B\}$ and $\{\Box(B \supset \Box B)\}$. Observe that $\mathcal{S}_1$ is *not* a topmost sequent because the ppa occurrence $\Box(B \supset \Box B)$ is itself a parametric ancestor appearing in the conclusion of a *GoR* rule. However $\mathcal{S}_2$ *is* a topmost sequent because both the ppa occurrences $\Box(B \supset \Box B)$ and $\Box B$ in the antecedent are introduced via weakening and there is no *GoR* rule in-between $\mathcal{S}_2$ and this introduction. Finally, $\mathcal{S}_3$ is *not* a topmost sequent because, tracing upwards, there is a branch above it (the left premise derivation of $L\supset$) where a parametric ancestor of the ppa $\Box(B \supset \Box B)$ appears in the conclusion of a *GoR* rule.

LEMMA 2.13 (topmost sequent lemma). *Let τ be a cutfree derivation ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \ /^{GoR}\ \Box X \Rightarrow \Box B$ and suppose that $\Gamma \Rightarrow \Delta$ is a topmost sequent in τ. Then there is a cutfree derivation of $\Gamma^- \Rightarrow \Delta$ where Γ^- is the multiset obtained from Γ by replacing each ppa with $\Box X$.*

PROOF. Since $\Gamma \Rightarrow \Delta$ is a topmost sequent, each $\Box B$ formula in Γ that is a ppa must have been introduced by a $LW(\Box B)$ weakening rule above $\Gamma \Rightarrow \Delta$, or can be traced to an initial sequent $\Box B \Rightarrow \Box B$. Similarly, each $\Box(B \supset \Box B)$ formula in Γ that is a ppa must have been introduced by a $LW(\Box(B \supset \Box B))$ weakening rule above $\Gamma \Rightarrow \Delta$, or can be traced to an initial sequent $\Box(B \supset \Box B) \Rightarrow \Box(B \supset \Box B)$.

A cutfree derivation of $\Gamma^- \Rightarrow \Delta$ can be obtained as follows.

Replace every weakening rule (resp. contraction rule) on ppa $\Box B$ or $\Box(B \supset \Box B)$ above the sequent $\Gamma \Rightarrow \Delta$ with $LW^*(\Box X)$ $(LC^*(\Box X))$. Replace occurrences of the initial sequent $\Box B \Rightarrow \Box B$ that introduce a ppa, with the given derivation τ of $\Box X \Rightarrow \Box B$. Finally, replace an initial sequent $\Box(B \supset \Box B) \Rightarrow \Box(B \supset \Box B)$ introducing a ppa, with the following derivation where we utilize the given derivation τ:

$$\dfrac{\dfrac{\dfrac{\overset{\tau}{\Box X \Rightarrow \Box B}}{B, \boxtimes X, \Box((B \supset \Box B) \supset \Box(B \supset \Box B)) \Rightarrow \Box B} LW^*}{\boxtimes X, \Box((B \supset \Box B) \supset \Box(B \supset \Box B)) \Rightarrow B \supset \Box B} R\supset}{\Box X \Rightarrow \Box(B \supset \Box B)} GoR$$

By inspection, the obvious derivation that can be obtained from these transformations is a cutfree derivation of $\Gamma^- \Rightarrow \Delta$. ■

Notice that the above lemma is not height-preserving since it utilizes the given derivation to obtain the new derivation. Thus, the use of this lemma in the proof of cut-elimination necessitates the use of an induction measure that does not rely solely on the degree of the cut-formula and the cut-height.

Example 2.14. In Example 2.12 we saw that the sequent $\mathcal{S}_2 = \boxtimes B, \boldsymbol{\Box(B \supset \Box B)}, \boldsymbol{\Box B}, \Box(C \supset \Box C) \Rightarrow C$ is a topmost sequent in τ containing the ppa $\{\Box(B \supset \Box B), \Box B\}$ (highlighted in bold). Then from Lemma 2.13 we can obtain a derivation of $\boxtimes B, \Box X, \Box X, \Box(C \supset \Box C) \Rightarrow C$.

3. Cut-elimination for *Go*

In this section, we consider exclusively a normal derivation τ ending with:

$$\dfrac{\boxtimes X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B} GoR$$

Recall that the (GoR) rule at the end of τ is called the *endrule*.

Definition 3.1 (leftflush, rightflush rules). Let C be some GoR rule in a normal derivation τ ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$. The set of GoR rules that are *leftflush* (resp. *rightflush*) rules wrt C is precisely the set defined by the following recursive definition:

- base(1): a GoR rule immediately above the endrule B is leftflush and rightflush wrt B
- base(2): a GoR rule that is left-above (right-above) a GoR rule C is leftflush (rightflush) wrt C

inductive: a *GoR* rule that is left-above (right-above) a rule that is itself leftflush (rightflush) wrt C is said to to be leftflush (rightflush) wrt C.

Note that **base(1)** is relevant only if we seek a leftflush or rightflush rule wrt the endrule B. Otherwise it is **base(2)** that is used. Intuitively, D is leftflush wrt C if D is encountered by repeatedly tracing through *GoR* rules left-above C. The intuition for rightflush is analogous.

Notice that it is never the case that C is leftflush (rightflush) wrt itself. Also the endrule B is not leftflush (rightflush) wrt to any rule, although from **base(1)**, every *GoR* rule immediately above the endrule is both leftflush and rightflush wrt B.

Definition 3.2 (left-, right-topmost). Suppose that τ is a normal derivation. A *GoR* rule C in τ is called *left-topmost* (resp. *right-topmost*) if the sequent $\mathcal{S}_L(C)$ ($\mathcal{S}_R(C)$) is a topmost sequent.

Definition 3.3 (MLL rule wrt C). Let C be some occurrence of the *GoR* rule in a normal derivation τ and suppose that D is simultaneously a leftflush rule (wrt C) and a left-topmost rule. Then D is called an *MLL rule wrt C* if there is no rule below D that is simultaneously a leftflush rule (wrt C) and a left-topmost rule.

The term MLL stands for 'minimal leftflush left-topmost'. Although a normal derivation may contain distinct rules D and E which are both MLL rules wrt C, it must be the case that D and E lie on different branches above C. Intuitively, an MLL rule is the leftflush left-topmost rule that is a minimal distance from C. Similarly we define:

Definition 3.4 (MRR rule wrt C). Let C be some occurrence of the *GoR* rule in a normal derivation τ and suppose that D is simultaneously a rightflush rule (wrt C) and a right-topmost rule. Then D is said to be an *MRR rule wrt C* if there is no rule below D that is simultaneously a rightflush rule (wrt C) and a right-topmost rule.

The term MRR stands for 'minimal rightflush right-topmost'. See Fig. 1 for an illustration of these terms.

Definition 3.5 (depth). The depth of a *GoR* rule ρ in a derivation τ is the number of *GoR* rules between the premise of ρ and the endsequent of τ.

Equivalently, this is the number of *GoR* rules below ρ plus 1. For example, in a derivation ending with a *GoR* rule ρ, the depth of ρ is 1.

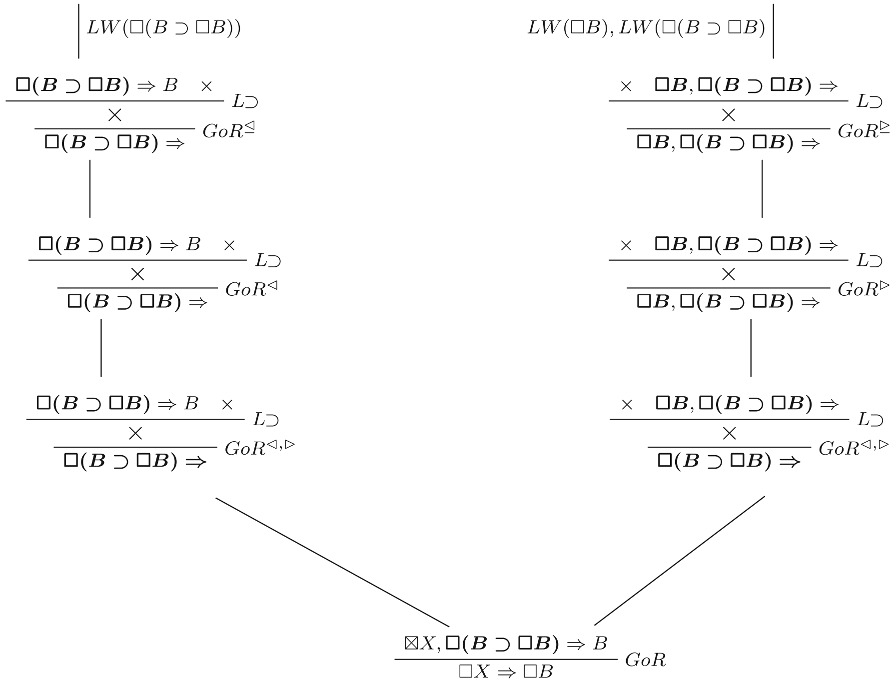

Figure 1. A schematic representation of a fragment of a normal derivation ending with $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$. For clarity, not all formulae and sequents have been included in the diagram. Formula multiplicities are also ignored. We write '$\times$' to indicate the presence of a sequent. The solid vertical lines represent portions of the derivation that are GoR-free and ppa are highlighted in bold. The symbol $\lhd$ denotes a leftflush rule and $\rhd$ denotes a rightflush rule (each wrt B). The symbols $\unlhd$ and $\unrhd$ respectively denote a MLL rule and a MRR rule (each wrt to B).

Definition 3.6 (leftwidth, rightwidth). Let C be a GoR rule in the normal derivation τ. The *leftwidth* $lw(C, \tau)$ is defined as the sum of the depths of each MLL rule (wrt C) in τ. Similarly, define the *rightwidth* $rw(C, \tau)$ as the sum of the depths of each MRR rule (wrt C) in τ.

Definition 3.7 (width of cut). The width of an instance of cut is defined when the left premise derivation δ of cut is a normal derivation ending in a GoR rule B, as the leftwidth $lw(B, \delta)$.

We are ready to prove the main lemma in the paper. Theorem 3.9 follows directly from this result. We first give an outline of the proof. Given a normal derivation τ of $\Box X \Rightarrow \Box B$, we want to obtain a cutfree derivation of $\Box X, X \Rightarrow B$. This result is immediate if $lw(B,\tau)=0$. If $lw(B,\tau)>0$, we show how to transform τ in order to reduce the leftwidth (induction on the leftwidth of B). We achieve this by selecting an arbitrary MLL rule C wrt to the endrule B and then transforming τ so that C is no longer an MLL rule wrt the endrule B. To effect this transformation, it suffices to replace the subderivation of

$$\frac{\boxtimes Y, \boldsymbol{\Box(B \supset \Box B)}, B \supset \Box B, \Box(C \supset \Box C) \Rightarrow C}{\Box Y, \boldsymbol{\Box(B \supset \Box B)} \Rightarrow \Box C}\ GoR$$

in τ with

$$\frac{\dfrac{\boxtimes Y, \Box(C \supset \Box C) \Rightarrow C}{\Box Y \Rightarrow \Box C}\ GoR}{\Box Y, \Box(B \supset \Box B) \Rightarrow \Box C}\ LW$$

Following this replacement, it is clear that C is no longer an MLL rule wrt the endrule B. So it remains to obtain the latter derivation. This derivation can be obtained directly from the sequent $\mathcal{S}_R(C)$ (below) in τ,

$$\boxtimes Y, \boldsymbol{\Box B}, \boldsymbol{\Box(B \supset \Box B)}, \Box(C \supset \Box C) \Rightarrow C$$

if $\mathcal{S}_R(C)$ is a topmost sequent. This is indeed the case if $rw(C,\tau)=0$. If $rw(C,\tau)>0$ we show how to transform τ in order to reduce the rightwidth (induction on the rightwidth of C). The proof is divided into three steps. Step I produces a derivation τ_I that is used in Step II to obtain a derivation where $\mathcal{S}_R(C)$ is a topmost sequent. In Step III, a cutfree derivation of $\Box X, X \Rightarrow B$ is obtained.

We now fill in all the details.

Lemma 3.8. *Let τ be a normal derivation with endsequent $\Box X \Rightarrow \Box B$. Assume (∗): an instance of cut in a derivation can be eliminated if the cut has degree $< |\Box B|$, or degree $|\Box B|$ and width $< lw(B,\tau)$. Then there is a transformation of τ to a cutfree derivation of $\Box X, X \Rightarrow B$.*

Proof. The derivation τ is a normal derivation so the endrule is GoR:

$$\frac{\boxtimes X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\ GoR$$

If $lw(B,\tau)=0$ this implies that every $\Box B$ and $\Box(B \supset \Box B)$ formula in the antecedent of $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B$ is introduced via weakening or initial sequents with no intervening GoR rule. Thus this sequent is a topmost sequent. The required derivation of $\boxtimes X \Rightarrow B$ follows from Lemma 2.13.

If $lw(B,\tau)>0$, let C be an arbitrary MLL rule wrt the endrule B.

To simplify the notation, in the following we omit writing the full antecedent of each sequent, dropping context non-ppa terms such as $\Box X$, and ignore formula multiplicities. For example, for the premise sequent $\Box X, X, \Box(B \supset \Box B) \Rightarrow B$ of a normal derivation we write $\Box(B \supset \Box B) \Rightarrow B$. Similarly, the rule

$$\frac{\boxtimes X, \boxtimes Y, \boxtimes(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow C}{\Box X, \Box Y, \Box(B \supset \Box B) \Rightarrow \Box C}\, GoR$$

will be written

$$\frac{\boxtimes(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow C}{\Box(B \supset \Box B) \Rightarrow \Box C}\, GoR$$

It is straightforward to extend the proof to the general case. In particular, bringing back the context non-ppa terms does not introduce new difficulties.
Step I. obtain a cut-free derivation of $\boxtimes(C \supset \Box C) \Rightarrow B$

By assumption, $lw(B, \tau) > 0$. We can schematically represent τ as:

$$\cfrac{\overset{\text{(topmost sequent)}}{\mathcal{S}_2 = \boldsymbol{\Box(B \supset \Box B)}, \Box(C \supset \Box C) \Rightarrow B, C} \qquad \overset{\vdots}{\mathcal{S}_3 = \boldsymbol{\Box B}, \boldsymbol{\Box(B \supset \Box B)}, \Box(C \supset \Box C) \Rightarrow C}}{\cfrac{\boldsymbol{\Box(B \supset \Box B)}, B \supset \Box B, \Box(C \supset \Box C) \Rightarrow C}{\mathcal{S}_1 = \boldsymbol{\Box(B \supset \Box B)} \Rightarrow \Box C}\, GoR}$$
$$\vdots$$
$$\frac{\boldsymbol{\Box(B \supset \Box B)} \Rightarrow B}{\Rightarrow \Box B}$$

Notice that in the above proof diagram, the sequents $\mathcal{S}_L(C)$ and $\mathcal{S}_R(C)$ have been denoted by $\mathcal{S}_2$ and $\mathcal{S}_3$ respectively. First, replace the subderivation of sequent $\mathcal{S}_1$ in τ with the derivation

$$\frac{\Box C \Rightarrow \Box C}{\Box(B \supset \Box B), \Box C \Rightarrow \Box C}\, LW$$

thus deleting an MLL rule from τ. We can then obtain a derivation γ of $\Box C \Rightarrow \Box B$ by mimicking the existing derivation below $\mathcal{S}_1$. Let E be the MLL rule below $\Box(B \supset \Box B), \Box C \Rightarrow \Box C$ in γ. Observe that E could not have been an MLL rule in τ because this would contradict the fact that C is an MLL in τ. Although E has now become an MLL rule, by inspection it is clear that the depth of E in γ is strictly less than the depth of the MLL rule C in τ. It follows that $lw(B, \gamma) < lw(B, \tau)$. From $\Box(B \supset \Box B) \Rightarrow B$ we can obtain a derivation of $\Box B \Rightarrow B$ by Lemma 2.3, and so we have

$$\frac{\overset{\gamma}{\Box C \Rightarrow \Box B} \qquad \Box B \Rightarrow B}{\Box C \Rightarrow B}\, cut$$

where the cut has width $< lw(B, \tau)$. Using assumption $(*)$ we obtain a cut-free derivation of this sequent.

Since C is an MLL, we know that $\mathcal{S}_L(C)(= \mathcal{S}_2)$ is a topmost sequent. Then by Lemma 2.13 we can obtain directly a derivation of $\Box(C \supset \Box C) \Rightarrow B, C$. We can now obtain the required derivation τ_I:

$$\dfrac{\Box(C \supset \Box C) \Rightarrow B, C \qquad \dfrac{\Box C \Rightarrow B}{\Box(C \supset \Box C), \Box C \Rightarrow B}\,LW}{\Box(C \supset \Box C), C \supset \Box C \Rightarrow B}\,L{\supset}$$

Step II. transform derivation above $\mathcal{S}_R(C)$ in τ so that $\mathcal{S}_R(C)$ becomes a topmost sequent

To show this we will prove a stronger statement:

– Whenever D is either the MLL rule C or a rightflush rule wrt to the MLL rule C in τ, then we can transform the subderivation above D in τ to obtain a derivation where $\mathcal{S}_R(D)$ is now a topmost sequent.

The proof is by induction on the rightwidth $rw(D, \tau)$. We treat separately the case when (1) D is the MLL rule C, and (2) D is a rightflush rule wrt C.
Case II.1 Suppose that D is the MLL rule C.

If $rw(C, \tau) = 0$, then it must be the case that there are no MRR rules wrt C. This implies that every $\Box B$ and $\Box(B \supset \Box B)$ formula in the antecedent of $\mathcal{S}_R(C)(= \mathcal{S}_3$; refer proof diagram on page 19) is introduced via weakening or initial sequents above $\mathcal{S}_3$ in every branch. Thus, $\mathcal{S}_R(C)$ is already a topmost sequent.

If $rw(C, \tau) > 0$, there must be some *GoR* rule F immediately right-above the MLL rule C—observe that F is a rightflush rule wrt C—and thus τ has the following form where the labelling of $\mathcal{S}_i$ $(i \in \{1, 2, 3\})$ is consistent with the proof diagram on page 19:

$$\dfrac{\dfrac{\times \qquad \mathcal{S}_4 = \boldsymbol{\Box B}, B, \boldsymbol{\Box(B \supset \Box B)}, \boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F}{\boldsymbol{\Box B}, B, \boldsymbol{\Box(B \supset \Box B)}, B \supset \Box B, \boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F}\,L{\supset}}{\mathcal{S}_5 = \boldsymbol{\Box B}, \boldsymbol{\Box(B \supset \Box B)}, \Box(C \supset \Box C) \Rightarrow \Box F}\,GoR$$

(no *GoR* rules)

$$\dfrac{\dfrac{\mathcal{S}_2 \quad \mathcal{S}_3 = \boldsymbol{\Box B}, \boldsymbol{\Box(B \supset \Box B)}, \Box(C \supset \Box C) \Rightarrow C}{\boldsymbol{\Box(B \supset \Box B)}, B \supset \Box B, \Box(C \supset \Box C) \Rightarrow C}\,L{\supset}}{\mathcal{S}_1 = \boldsymbol{\Box(B \supset \Box B)} \Rightarrow \Box C}\,GoR$$

$$\vdots$$

$$\Rightarrow \Box B$$

We will first show how to obtain a derivation of the sequent seqII.1:

$$B, \boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F$$

If $\mathcal{S}_R(F)(= \mathcal{S}_4)$ is a topmost sequent, then a derivation of seqII.1 can be obtained immediately from Lemma 2.13. Otherwise, observe that every MRR rule wrt F is also a MRR rule wrt C. Thus $rw(F,\tau) < rw(C,\tau)$. Now, by the induction hypothesis, we can transform the subderivation in τ deriving $\mathcal{S}_4$ to obtain a derivation where $\mathcal{S}_4$ is a topmost sequent. Then a derivation of seqII.1 follows from Lemma 2.13.

Now, together with the derivation τ_I we obtained on page 19:

$$\dfrac{\overset{\tau_I}{\boxtimes(C \supset \Box C) \Rightarrow B} \qquad B, \boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F}{\boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F}\ cut$$

Using assumption $(*)$ we can eliminate the cut-rule. Now proceed:

$$\dfrac{\dfrac{\boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F}{\Box(C \supset \Box C) \Rightarrow \Box F}\ GoR}{\mathcal{S}_5 = \boldsymbol{\Box B}, \boldsymbol{\Box(B \supset \Box B)}, \Box(C \supset \Box C) \Rightarrow \Box F}\ LW^*$$

$$\text{(mimic derivation below } \mathcal{S}_5 \text{ in } \tau)$$

$$\dfrac{\dfrac{\mathcal{S}_2 \quad \mathcal{S}_3 = \boldsymbol{\Box B}, \boldsymbol{\Box(B \supset \Box B)}, \Box(C \supset \Box C) \Rightarrow C}{\boldsymbol{\Box(B \supset \Box B)}, B \supset \Box B, \Box(C \supset \Box C) \Rightarrow C}\ L{\supset}}{\boldsymbol{\Box(B \supset \Box B)} \Rightarrow \Box C}\ GoR$$

$$\vdots$$

$$\Rightarrow \Box B$$

Although this new derivation of $\Rightarrow \Box B$ may be abnormal due to the subderivation deriving the premise of F (refer the terminology introduced above Lemma 2.8), using Lemma 2.8 we can transform *this subderivation* to obtain a normal derivation of $\Rightarrow \Box B$. This is a technical point that is required because the terms MLL and MRR are defined with respect to a normal derivation. Due to the left weakening rules deriving $\mathcal{S}_5$ in the above proof diagram, the rightwidth of C in the above derivation is $< rw(C,\tau)$, and thus by the induction hypothesis it follows that we can obtain a cut-free derivation τ_{II} such that $\mathcal{S}_R(C)$ is a topmost sequent.

Case II.2 Suppose that D is a rightflush rule wrt C. Let $\mathcal{S}_6$ denote $\mathcal{S}_R(D)$. If $rw(D,\tau) = 0$ then there are no MRR rules wrt D. Thus $\mathcal{S}_6$ is a topmost sequent and there is nothing more to do. Else, if $rw(D,\tau) > 0$, then there must be some GoR rule G immediately right-above D, so τ has the form:

$$\cfrac{\times \qquad \mathcal{S}_7 = \Box B, B, \Box(B \supset \Box B), \boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G}{\cfrac{\Box B, B, \boxtimes(B \supset \Box B), \boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G}{\mathcal{S}_8 = \Box B, \Box(B \supset \Box B), \Box(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow \Box G} GoR} L\supset$$

$$\vdots$$

$$\cfrac{\times \quad \mathcal{S}_6 = \Box(B \supset \Box B), \Box B, \boxtimes(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow D}{\cfrac{\boxtimes(B \supset \Box B), \boxtimes(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow D}{\Box(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow \Box D} GoR} L\supset$$

$$\vdots$$

$$\Rightarrow \Box B$$

We will first show how to obtain a derivation of the sequent seqII.2:

$$B, \boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G$$

If $\mathcal{S}_R(G)(= \mathcal{S}_7)$ is a topmost sequent, then a derivation of seqII.2 follows immediately from Lemma 2.13. Otherwise observe that every MRR rule wrt G is also a MRR rule wrt D. Thus $rw(G, \tau) < rw(D, \tau)$. By the induction hypothesis, we can transform the subderivation in τ deriving $\mathcal{S}_7$ to obtain a derivation where $\mathcal{S}_7$ is a topmost sequent. Then a derivation of seqII.2 follows from Lemma 2.13.

Now, together with the derivation τ_I we obtained on page 19:

$$\cfrac{\overset{\tau_I}{\boxtimes(C \supset \Box C) \Rightarrow B} \qquad B, \boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G}{\boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G} cut$$

Using assumption $(*)$ we can eliminate the cut-rule. Now proceed:

$$\cfrac{\cfrac{\boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G}{\Box(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow \Box G} GoR}{\mathcal{S}_8 = \mathbf{\Box B}, \mathbf{\Box(B \supset \Box B)}, \Box(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow \Box G} LW^*$$

$$\text{(mimic derivation below } \mathcal{S}_8 \text{ in } \tau)$$

$$\cfrac{\times \quad \mathcal{S}_6 = \mathbf{\Box B}, \mathbf{\Box(B \supset \Box B)}, \boxtimes(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow D}{\cfrac{\Box(B \supset \Box B), B \supset \Box B, \boxtimes(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow D}{\mathbf{\Box(B \supset \Box B)}, \Box(C \supset \Box C) \Rightarrow \Box D} GoR} L\supset$$

$$\vdots$$

$$\Rightarrow \Box B$$

Although this new derivation of $\Rightarrow \Box B$ may be abnormal due to the subderivation deriving the premise of G, using Lemma 2.8 we can transform this subderivation so that the derivation of $\Rightarrow \Box B$ becomes a normal derivation. Due to the left weakening rules deriving $\mathcal{S}_8$ in the above proof diagram, the rightwidth of D in the above derivation is $< rw(D, \tau)$, and thus by the induction hypothesis it follows that we can obtain a cut-free derivation τ_{II} such that $\mathcal{S}_R(D)(=\mathcal{S}_6)$ is a topmost sequent.

We have proved all the cases for the inductive step and so $(**)$ is proved.
Step III. obtain a cut-free derivation of $\Rightarrow \Box B$

In Step II we showed for an arbitrary MLL rule C wrt the endrule B in τ, how to transform the derivation so that $\mathcal{S}_R(C)$ is a topmost sequent. Using Lemma 2.13 we can delete the ppa in $\mathcal{S}_R(C)$ to obtain a derivation of $\Box(C \supset \Box C) \Rightarrow C$. Now consider the following derivation:

$$\dfrac{\dfrac{\Box(C \supset \Box C) \Rightarrow C}{\Rightarrow \Box C}\, GoR}{\Box(B \supset \Box B) \Rightarrow \Box C}\, LW$$

Replace the subderivation of $\mathcal{S}_1$ in τ (see proof diagram on page 19) with the above derivation to obtain ultimately a derivation τ' of $\Rightarrow \Box B$ where $lw(B,\tau') < lw(B,\tau)$. Then the following cut has width $< lw(B,\tau)$:

$$\dfrac{\overset{\tau'}{\Rightarrow \Box B} \qquad \Box B \Rightarrow B}{\Rightarrow B}\, cut$$

From $(*)$ we obtain a cut-free derivation of $\Rightarrow B$. Bringing back the context non-ppa terms and formula multiplicities that we omitted for simplicity of notation, and applying the weakening and contraction rules to these terms as required, we obtain the required cut-free derivation of $\Box X, X \Rightarrow B$. ■

Theorem 3.9. *Syntactic cut-elimination holds for GoS.*

Proof. Given a derivation τ, we obtain a cutfree derivation of the same sequent. Without loss of generality, we eliminate instances of the cut-rule whose premise derivations are cutfree ('topmost cuts'). Primary induction on the degree $|A|$ of the cut-formula A, secondary induction on the width w of the cut, and ternary induction on the cut-height s—ie. associate the ordered triple $(|A|, w, s)$ with the instance of cut. For instances cut_1 and cut_2 of the cut-rule, we write $cut_1 < cut_2$ to mean that cut_1 is less than cut_2 under this above measure. Since the proof of Lemma 3.8 uses an induction on rightwidth, this proof implicitly uses a quaternary induction measure.

When the cut-formula is not a boxed-formula (ie. not of the form $\Box B$) the standard transformations [7,14] suffice (we explain how to deal with the contraction rules below) — in particular, the presence of the width does not affect the correctness of these transformations. If the cut-formula is a boxed-formula, first transform the left premise derivation and then the right-premise derivation in the usual manner to obtain ultimately the following situation, where the cut-formula is principal by the GoR rule in both premises, and the premise derivations are cutfree (this is ensured by the fact that we repeatedly choose to transform topmost cuts):

$$\dfrac{\dfrac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\,GoR \qquad \dfrac{\Box B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box B, \Box U \Rightarrow \Box C}\,GoR}{\Box X, \Box U \Rightarrow \Box C}\,cut_0$$

By Corollary 2.9 we can write the left premise derivation as a normal derivation ending with $\Box X, X, \Box(B \supset \Box B) \Rightarrow B \;/^{GoR}\; \Box X \Rightarrow \Box B$. Noting that the induction hypothesis is precisely the condition $(*)$ in Lemma 3.8, use that result to obtain a cut-free derivation of $\Box X, X \Rightarrow B$. Then,

$$\dfrac{\dfrac{\dfrac{\Box X, X \Rightarrow B \qquad \dfrac{\Box X \Rightarrow \Box B \qquad \Box B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box X, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,cut_1}{\Box X, \Box X, X, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,cut_2}{\Box X, X, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,LC^*}{\Box X, \Box U \Rightarrow \Box C}\,GoR$$

Here, cut_1 has reduced cut-height (and identical degree and width compared to cut_0) and cut_2 has reduced degree of cut-formula compared to cut_0. Thus $cut_1 < cut_0$ and $cut_2 < cut_0$. Using the induction hypothesis, we can obtain a cutfree derivation of $\Box X, \Box U \Rightarrow \Box C$, thus eliminating the cut.

Since we use sequents built from multisets, we need to specify also how to deal with the case of contraction rules immediately above the instance of cut. In general, this case requires special attention [7,16]. If we are prepared to use the multicut rule below where $m, n > 0$

$$\dfrac{X \Rightarrow Y, A^m \qquad A^n, U \Rightarrow W}{X, U \Rightarrow Y, W}\,mcut$$

then we can obtain a cutfree derivation by taking a detour via the calculus $GoS + mcut$. This is the approach Gentzen [7] takes in his proof of the *Hauptsatz*. From a proof-theoretical viewpoint, a disadvantage with the $mcut$ rule is that it combines the structural rules of contraction and cut, hindering our ability to analyse the independent 'effect' of each rule. Instead of using the multicut rule, we can adapt the transformations described in [16] for classical logic to deal with contractions above cut. The only new case to deal with is a derivation of the following form:

$$\dfrac{\dfrac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\,GoR \qquad \dfrac{\dfrac{\Box B, \Box B, B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box B, \Box B, \Box U \Rightarrow \Box C}\,GoR}{\Box B, \Box U \Rightarrow \Box C}\,LC}{\Box X, \Box U \Rightarrow \Box C}\,cut_0$$

Then the following transformation suffices, where we use Lemma 3.8 to obtain a derivation of $\Box X, X \Rightarrow B$:

$$\dfrac{\Box X, X \Rightarrow B \qquad \dfrac{\dfrac{\Box X \Rightarrow \Box B \qquad \dfrac{\Box B, \Box B, B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box B, B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,LC}{B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,cut_1}{\Box X, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,LC}{\dfrac{\Box X, X, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box X, \Box U \Rightarrow \Box C}\,GoR}\,cut_2$$

Since $cut_1 < cut_0$ (reduced cut-height) and $cut_2 < cut_0$ (reduced degree of the cut-formula), we appeal to the induction hypothesis here. This is similar to the approach for avoiding multicut in cut-elimination for GLS [9]. ■

4. Conclusion

Although proofs of syntactic cut-elimination for the logics GL and Grz have appeared in the literature, this is the first proof of syntactic cut-elimination for Go. Each of the proofs in the traditional sequent calculus for GL [15,4,13,11,9] and Grz [3] make use respectively of the calculi GLS and $GrzS$, or minor variants of these systems. We conclude by presenting some observations on how these proofs relate to our proof for GoS.

The sequent calculus GLS for GL is obtained by substituting the GoR rule in Table 1 with the GLR rule:

$$\frac{\Box X, X, \Box B \Rightarrow B}{\Box X \Rightarrow \Box B}\,GLR$$

The sequent calculus $GrzS$ for Grz can be obtained by substituting the GoR rule in Table 1 with the following rules:

$$\frac{B, X \Rightarrow Y}{\Box B, X \Rightarrow Y}\,GRZa \qquad\qquad \frac{\Box X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\,GRZc$$

Aside from Mints [11] proof for GL, the main idea in each of the proofs for GLS and $GrzS$ is to transform a cutfree derivation of the premise ($\Box X, X, \Box B \Rightarrow B$ for GLS; $\Box X, \Box(B \supset \Box B) \Rightarrow B$ for $GrzS$) of the modal rule into a cutfree derivation of the sequent $\Box X, X \Rightarrow B$ (for GLS) and $\Box X \Rightarrow B$ (for $GrzS$).

Informally, the proof for GoS appears to be more intricate than the proofs for the calculus GLS because of the necessity for dealing with the formula $\Box(B \supset \Box B)$ as opposed to $\Box B$ in the premise antecedent of the respective modal rule. Although the $GRZc$ rule does contain the formula $\Box(B \supset \Box B)$

in the premise antecedent, the presence of the $GRZa$ rule in $GRzS$ enables us to directly transform any sequent of the form $C \supset \Box C, X \Rightarrow Y$ into $\Box(C \supset \Box C), X \Rightarrow Y$ and this greatly simplifies the proof. In GoS, we have only the GoR rule at our disposal to 'box' the $C \supset \Box C$ formula in a sequent of the form $C \supset \Box C, X \Rightarrow Y$, and must abide by the restrictions it places on the multisets X and Y. As a result, the proof for GoS requires a more detailed study of the structure of derivations in GoS, and a quaternary induction measure, whereas three induction variables suffice for GLS and $GrzS$. Moreover, the transformations for GoS witnessing syntactic cut-elimination generalise the transformations employed for those calculi.

Acknowledgements. The authors would like to thank the anonymous reviewers for their helpful comments.

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