The Condition Number of the Joint Space Inertia Matrix

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The condition number of a matrix measures its closeness to singularity:

\[ \kappa(A) \rightarrow \infty \text{ as } A \rightarrow \text{ singular} \]

If \( \kappa(A) \) is large then \( A \) is said to be ill–conditioned.

If a physical system is described by an equation like

\[ y = A \, x + b \]

and \( A \) is ill–conditioned, then it can be difficult to calculate \( y \) from \( x \), or \( x \) from \( y \), without loss of accuracy.
The joint–space inertia matrix (JSIM) of a kinematic tree is known to be a symmetric, positive–definite matrix.

- It is therefore nonsingular.

- But is it ill–conditioned? Yes!
Example:

Suppose we want to accelerate this planar 8R robot from rest with an acceleration of

\[ \ddot{q}_d = [1,1,1,1,1,1,1,1]^T \]

The equation of motion is

\[ \tau = H \ddot{q} + C \]

where

\( H \) is the JSIM, and \( C = 0 \) (gravity and velocity terms are zero).
The exact force required to produce an acceleration of $\ddot{q}_d$ is

$$\tau_d = H \ddot{q}_d = \begin{bmatrix}
302.0450 \\
250.2104 \\
200.5413 \\
151.0375 \\
105.6992 \\
64.5263 \\
31.5188 \\
8.6767
\end{bmatrix}$$

But what if the actual joint force differs very slightly from the theoretically exact force?
Let $\tau_a$ be the exact force rounded to three significant figures. The acceleration caused by an applied force of $\tau_a$ is

$$\ddot{q}_a = H^{-1} \tau_a = H^{-1} \begin{bmatrix} 302 \\ 250 \\ 201 \\ 151 \\ 106 \\ 64.5 \\ 31.5 \\ 8.68 \end{bmatrix} = \begin{bmatrix} 0.7917 \\ 1.0281 \\ 1.4904 \\ 0.6886 \\ 1.1026 \\ 1.0911 \\ 0.5626 \\ 1.2384 \end{bmatrix}$$

A force error of $< 0.5\%$ has caused an acceleration error of $50\%$. 
Measuring the Condition Number

The following graphs plot $\kappa(H)$ vs $N$ (number of bodies) for robots with

- revolute joints
- identical or tapering links
- in curled and zigzag configurations
- unbranched or branched connectivity
- fixed or floating bases
identical links, unbranched, zigzag
identical links, unbranched, curled

\[ \theta = 0, 0.012, 0.025, 0.05, 0.1, 0.2, 0.4, 0.8, 1.4, 2.4 \]
tapered links, unbranched, curled
inertia–weighted metric

\[ \rho = 0.9 \]

- scaled
- unscaled
branches (spherically symmetric robots)
floating base, unbranched, identical links
Summary

- in general, the JSIM is very ill-conditioned, and it gets worse as the number of bodies increases

- worst case: $\kappa(H) = 4N^4$

- tapering can increase or decrease ill-conditioning, depending on how you measure it

- branches reduce ill-conditioning

- a floating base can reduce ill-conditioning