A Workshop on

Spatial Vector Algebra

The Easy Way to do Rigid Body Dynamics

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Spatial vector algebra is a concise vector notation for describing rigid–body velocity, acceleration, inertia, etc., using 6D vectors and tensors.

- fewer quantities
- fewer equations
- less effort
- fewer mistakes

Mathematical Structure

spatial vectors inhabit two vector spaces:

- \( M^6 \) — motion vectors
- \( F^6 \) — force vectors

with a scalar product defined between them

\[ \mathbf{m} \cdot \mathbf{f} = \text{work} \]

\[ \left\langle \cdot , \cdot \right\rangle : M^6 \times F^6 \to \mathbb{R} \]

Bases

A coordinate vector \( \mathbf{m} = [m_1, ..., m_6]^T \) represents a motion vector \( \mathbf{m} \) in a basis \( \{d_1, ..., d_6\} \) on \( M^6 \) if

\[ \mathbf{m} = \sum_{i=1}^{6} m_i \mathbf{d}_i \]

Likewise, a coordinate vector \( \mathbf{f} = [f_1, ..., f_6]^T \) represents a force vector \( \mathbf{f} \) in a basis \( \{e_1, ..., e_6\} \) on \( F^6 \) if

\[ \mathbf{f} = \sum_{i=1}^{6} f_i \mathbf{e}_i \]
Bases

If \{\mathbf{d}_1, \ldots, \mathbf{d}_6\} is an arbitrary basis on \(M^6\) then there exists a unique reciprocal basis \{\mathbf{e}_1, \ldots, \mathbf{e}_6\} on \(F^6\) satisfying

\[
\mathbf{d}_i \cdot \mathbf{e}_j = \begin{cases} 
0 & : i \neq j \\
1 & : i = j
\end{cases}
\]

With these bases, the scalar product of two coordinate vectors is

\[
\mathbf{m} \cdot \mathbf{f} = \mathbf{m}^T \mathbf{f}
\]

Velocity

The velocity of a rigid body can be described by

- choosing a point, \(P\), in the body
- specifying the linear velocity, \(\mathbf{v}_P\), of that point
- specifying the angular velocity, \(\omega\), of the body as a whole

The body is then deemed to be translating with a linear velocity \(\mathbf{v}_P\) while simultaneously rotating with an angular velocity \(\omega\) about an axis passing through \(P\)

velocity \((\omega, \mathbf{v}_P)\) at \(P\)

where

\[
\mathbf{v}_O = \mathbf{v}_P + \overrightarrow{OP} \times \omega
\]
Spatial velocity: \( \hat{v}_O = \begin{bmatrix} \omega \\ v_O \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_{Ox} \\ v_{Oy} \\ v_{Oz} \end{bmatrix} \)

These are the Plücker coordinates of \( \hat{v} \) in the frame \( Oxyz \).

A general force acting on a rigid body is equivalent to the sum of

- a force \( \mathbf{f} \) acting along a line passing through a point \( P \), and
- a couple, \( \tau_P \)

General force \( (\mathbf{f}, \tau_P) \) at \( P \)

is equivalent to \( (\mathbf{f}, \tau_O) \) at \( O \)

where \( \tau_O = \tau_P + \overrightarrow{OP} \times \mathbf{f} \)

Spatial force: \( \hat{\mathbf{f}}_O = \begin{bmatrix} \tau_O \\ \mathbf{f} \end{bmatrix} = \begin{bmatrix} \tau_{Ox} \\ \tau_{Oy} \\ \tau_{Oz} \\ f_x \\ f_y \\ f_z \end{bmatrix} \)

These are the Plücker coordinates of \( \hat{\mathbf{f}} \) in the frame \( Oxyz \).
Plücker Coordinates

A Cartesian coordinate frame \( Oxyz \) defines \textit{twelve} basis vectors:

\[ \mathbf{d}_{Ox}, \mathbf{d}_{Oy}, \mathbf{d}_{Oz}, \mathbf{d}_x, \mathbf{d}_y, \mathbf{d}_z : \]
rotations about the \( Ox, Oy \) and \( Oz \) axes, translations in the \( x, y \) and \( z \) directions

\[ \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z, \mathbf{e}_{Ox}, \mathbf{e}_{Oy}, \mathbf{e}_{Oz} : \]
couples in the \( yz, zx \) and \( xy \) planes, and forces along the \( Ox, Oy \) and \( Oz \) axes

If \( \mathbf{v}_O = \begin{bmatrix} \omega \\ \mathbf{v}_O \end{bmatrix} \) and \( \mathbf{f}_O = \begin{bmatrix} \tau_O \\ \mathbf{f} \end{bmatrix} \) are the Plücker coordinates of \( \mathbf{v} \) and \( \mathbf{f} \) in \( Oxyz \), then

\[ \mathbf{v} = \omega_x \mathbf{d}_{Ox} + \omega_y \mathbf{d}_{Oy} + \omega_z \mathbf{d}_{Oz} + \]
\[ + v_{Ox} \mathbf{d}_x + v_{Oy} \mathbf{d}_y + v_{Oz} \mathbf{d}_z \]

\[ \mathbf{f} = \tau_{Ox} \mathbf{e}_x + \tau_{Oy} \mathbf{e}_y + \tau_{Oz} \mathbf{e}_z + \]
\[ + f_x \mathbf{e}_{Ox} + f_y \mathbf{e}_{Oy} + f_z \mathbf{e}_{Oz} \]

and

\[ \mathbf{v} \cdot \mathbf{f} = \mathbf{v}_O^T \mathbf{f}_O \]

Coordinate Transforms

Transform from \( A \) to \( B \) for motion vectors:

\[ \mathbf{B} \mathbf{X}_A = \begin{bmatrix} \mathbf{E} & 0 \\ 0 & \mathbf{E} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{r}^T & 1 \end{bmatrix} \]
where \( \mathbf{r} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \)

Corresponding transform for force vectors:

\[ \mathbf{B} \mathbf{X}_A^F = (\mathbf{B} \mathbf{X}_A)^{-T} \]

Basic Operations with Spatial Vectors

- \textbf{Composition of velocities}

If \( \mathbf{v}_A = \) velocity of body \( A \)
\( \mathbf{v}_B = \) velocity of body \( B \)
\( \mathbf{v}_{BA} = \) relative velocity of \( B \) w.r.t. \( A \)

Then \( \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \)

- \textbf{Scalar multiples}

If \( \mathbf{s} \) and \( \dot{q} \) are a joint motion axis and velocity variable, then the joint velocity is \( \mathbf{v}_J = \mathbf{s} \dot{q} \)
- **Composition of forces**
  If forces $f_1$ and $f_2$ both act on the same body then their resultant is
  
  $$ f_{tot} = f_1 + f_2 $$

- **Action and reaction**
  If body $A$ exerts a force $f$ on body $B$, then body $B$ exerts a force $-f$ on body $A$ (Newton’s 3rd law)

**Spatial Cross Products**

There are two cross product operations: one for motion vectors and one for forces

$$ \hat{v}_o \times \hat{m}_o = \begin{bmatrix} \omega \\ v_o \end{bmatrix} \times \begin{bmatrix} m \\ m_o \end{bmatrix} = \begin{bmatrix} \omega \times m \\ \omega \times m_o + v_o \times m \end{bmatrix} $$

$$ \hat{v}_o \times \hat{f}_o = \begin{bmatrix} \omega \\ v_o \end{bmatrix} \times \begin{bmatrix} \tau_o \\ f \end{bmatrix} = \begin{bmatrix} \omega \times \tau_o + v_o \times f \\ \omega \times f \end{bmatrix} $$

where $\hat{v}_o$ and $\hat{m}_o$ are motion vectors, and $\hat{f}_o$ is a force.

**Differentiation**

- The derivative of a spatial vector is itself a spatial vector
- in general, $\frac{d}{dt} s = \lim_{\delta t \to 0} \frac{s(t+\delta t)-s(t)}{\delta t}$
- The derivative of a spatial vector that is fixed in a moving body with velocity $v$ is $\frac{d}{dt} s = v \times s$

**Differentiation in Moving Coordinates**

$$ \begin{bmatrix} \frac{d}{dt} s \end{bmatrix}_o = \frac{d}{dt} s_o + v_o \times s_o $$

velocity of coordinate frame

componentwise derivative of coordinate vector

coordinate vector representing $ds/dt$
Acceleration

\[ \mathbf{a} = \frac{d}{dt} \mathbf{v} = \begin{bmatrix} \dot{\mathbf{v}}_O \end{bmatrix} \]

is the rate of change of velocity:

but this is not the linear acceleration of any point in the body!

Classical acceleration is

\[ \begin{bmatrix} \dot{\mathbf{v}}_O \\ \mathbf{v}_O \end{bmatrix} \]

where \( \mathbf{v}_O \) is the derivative of \( \mathbf{v}_O \)

taking \( O \) to be fixed in the body

Spatial acceleration is

\[ \begin{bmatrix} \dot{\mathbf{v}}_O \\ \mathbf{v}_O \end{bmatrix} \]

where \( \dot{\mathbf{v}}_O \) is the derivative of \( \mathbf{v}_O \)

taking \( O \) to be fixed in space

What’s the difference?

Spatial acceleration is a vector.

Acceleration Example

A body rotates with constant angular velocity \( \mathbf{\omega} \), so its spatial velocity is constant, so its spatial acceleration is zero; but almost every body-fixed point is accelerating.

Basic Operations with Accelerations

- **Composition**

  If \( \mathbf{a}_A = \) acceleration of body A
  \( \mathbf{a}_B = \) acceleration of body B
  \( \mathbf{a}_{BA} = \) acceleration of B w.r.t. A

  Then \( \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} \)

  Look, no Coriolis term!
Rigid Body Inertia

- mass: $m$
- CoM: $C$
- inertia at CoM: $I_C$

Spatial inertia tensor: \( \hat{I}_O = \begin{bmatrix} I_O & m \bar{c} \\ m \bar{c}^T & m \mathbf{1} \end{bmatrix} \)

where $I_O = I_C - m \bar{c} \bar{c}$

Basic Operations with Inertias

- Composition
  
  If two bodies with inertias $I_A$ and $I_B$ are joined together then the inertia of the composite body is
  
  $I_{tot} = I_A + I_B$

- Coordinate transformation formula
  
  $I_B = B X_A^F I_A A X_B = (A X_B)^T I_A A X_B$

Equation of Motion

\[ f = \frac{d}{dt}(I v) = I a + v \times I v \]

- $f$ = net force acting on a rigid body
- $I$ = inertia of rigid body
- $v$ = velocity of rigid body
- $I v$ = momentum of rigid body
- $a$ = acceleration of rigid body

Constrained Motion

A force, $f$, is applied to a body with inertia $I$ that is constrained to a subspace $S = \text{Range}(S)$ of $M^6$. Assuming the body is initially at rest, what is its acceleration?

velocity $v = S \alpha = 0$

acceleration $a = S \dot{\alpha} + \dot{S} \alpha = S \dot{\alpha}$

constraint force $S^T f_c = 0$
In the document, the following points are highlighted:

- **Inverse Dynamics**
  - joint velocity, acceleration & axis
  - link velocity and acceleration
  - force transmitted from link $i-1$ to $i$
  - joint force variable
  - link inertia

- **The Recursive Newton–Euler Algorithm**
  - Calculate the joint torques $\tau_i$ that will produce the desired joint accelerations $\ddot{q}_i$.

Mathematical equations and formulas are provided to support these points, such as:

- Equation of motion:
  \[ f + f_c = Ia + v \times Iv \]
  \[ = Ia \]

- Solution:
  \[ IS \dot{\alpha} = f + f_c \]
  \[ S^T IS \dot{\alpha} = S^T f \]
  \[ \dot{\alpha} = (S^T IS)^{-1} S^T f \]
  \[ a = S(S^T IS)^{-1} S^T f \]

- Velocity of link $i$ is the velocity of link $i-1$ plus the velocity across joint $i$:
  \[ v_i = v_{i-1} + s_i \dot{q}_i \]

- Acceleration is the derivative of velocity:
  \[ a_i = a_{i-1} + \dot{s}_i \dot{q}_i + s_i \ddot{q}_i \]

- Equation of motion:
  \[ f_i - f_{i+1} = I_i a_i + v_i \times I_i v_i \]

- Active joint force:
  \[ \tau_i = s_i^T f_i \]