Solving a Two-Body Dynamics Problem using 6-D Vectors

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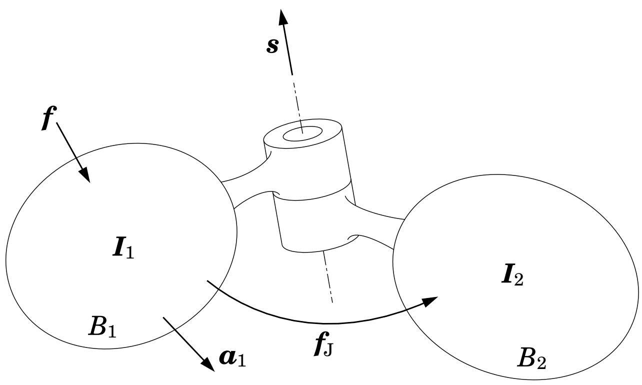
Problem Statement

We are given a rigid-body system consisting of two bodies, B_1 and B_2, connected by a revolute joint. The bodies have inertias of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$, respectively, and they are initially at rest. The joint's rotation axis is $\boldsymbol{s}$. A force $\boldsymbol{f}$ is applied to B_1, causing both bodies to accelerate. The problem is to calculate the acceleration of B_1 as a function of $\boldsymbol{f}$.

Diagram



Solution

Let $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ be the accelerations of the two bodies, and let $\boldsymbol{f}_\mathrm{J}$ be the force transmitted from B_1 to B_2 through the joint. The net forces acting on the two bodies are therefore $\boldsymbol{f} - \boldsymbol{f}_\mathrm{J}$ and $\boldsymbol{f}_\mathrm{J}$, respectively, and their equations of motion are

$$\boldsymbol{f} - \boldsymbol{f}_\mathrm{J} = \boldsymbol{I}_1\,\boldsymbol{a}_1 \tag{1}$$

and

$$\boldsymbol{f}_\mathrm{J} = \boldsymbol{I}_2\,\boldsymbol{a}_2\,. \tag{2}$$

(There are no velocity terms because the bodies are at rest.) The joint permits B_2 to accelerate relative to B_1 about the axis specified by $\boldsymbol{s}$; so $\boldsymbol{a}_2$ can be expressed in the form

$$\boldsymbol{a}_2 = \boldsymbol{a}_1 + \boldsymbol{s}\,\alpha \tag{3}$$

where α is the joint acceleration variable. (Again, there are no velocity terms because the bodies are at rest.) This motion constraint is implemented by $\boldsymbol{f}_\mathrm{J}$, which is the joint constraint force, so $\boldsymbol{f}_\mathrm{J}$ must satisfy

$$\boldsymbol{s}^\mathrm{T}\boldsymbol{f}_\mathrm{J} = 0 \tag{4}$$

(i.e., the constraint force does no work in the direction of motion allowed by the joint).

Given Eqs. 1 to 4, the problem is solved as follows. First, substitute Eq. 3 into Eq. 2, giving

$$\boldsymbol{f}_\mathrm{J} = \boldsymbol{I}_2\,(\boldsymbol{a}_1 + \boldsymbol{s}\,\alpha). \tag{5}$$

Substituting this equation into Eq. 4 gives

$$\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2\,(\boldsymbol{a}_1 + \boldsymbol{s}\,\alpha) = 0,$$

from which we get the following expression for α:

$$\alpha = -\frac{\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2\,\boldsymbol{a}_1}{\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2\,\boldsymbol{s}}\,. \tag{6}$$

Substituting Eq. 6 back into Eq. 5 gives

$$\boldsymbol{f}_\mathrm{J} = \boldsymbol{I}_2\left(\boldsymbol{a}_1 - \frac{\boldsymbol{s}\,\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2}{\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2\,\boldsymbol{s}}\,\boldsymbol{a}_1\right),$$

and substituting this equation back into Eq. 1 gives

$$\begin{aligned}\boldsymbol{f} &= \boldsymbol{I}_1\,\boldsymbol{a}_1 + \boldsymbol{I}_2\,\boldsymbol{a}_1 - \frac{\boldsymbol{I}_2\,\boldsymbol{s}\,\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2}{\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2\,\boldsymbol{s}}\,\boldsymbol{a}_1 \\ &= \left(\boldsymbol{I}_1 + \boldsymbol{I}_2 - \frac{\boldsymbol{I}_2\,\boldsymbol{s}\,\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2}{\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2\,\boldsymbol{s}}\right)\boldsymbol{a}_1\,.\end{aligned}$$

The expression in brackets is non-singular, and may therefore be inverted to express $\boldsymbol{a}_1$ in terms of $\boldsymbol{f}$:

$$\boldsymbol{a}_1 = \left(\boldsymbol{I}_1 + \boldsymbol{I}_2 - \frac{\boldsymbol{I}_2\,\boldsymbol{s}\,\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2}{\boldsymbol{s}^\mathrm{T}\boldsymbol{I}_2\,\boldsymbol{s}}\right)^{-1}\boldsymbol{f}\,. \tag{7}$$

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