Kaiser window based kernel for time-frequency distributions

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A novel sinc kernel, which generates a Kaiser window based time-frequency distribution for non-stationary signal analysis is introduced. The Kaiser distribution belongs to the Cohen class of time-frequency distributions, and satisfies desirable distribution properties. By controlling the shape of the Kaiser window, auto-term resolutions and cross-term magnitudes can be successfully traded.

Introduction: Time-frequency distributions (TFDs) have been used as a tool to locate the time dependency of the spectrum in analysing non-stationary signals. The Cohen class [1] generalises most existing TFDs, and the distinction between different distributions is made by a kernel function. Finding an optimum kernel function with maximum energy concentration and minimum cross-terms remains an open question.

In this Letter, we introduce a novel sinc kernel which generates a Kaiser window based TFD. The Kaiser window has been used extensively in designing finite impulse response digital filters, and in other digital signal processing applications [2, 3]. The Kaiser window is an asymptotic approximation to the prolate spheroidal wave functions, which have been shown to provide concentration of energy simultaneously in time and frequency [3]. We show that Kaiser distribution is a member of the Cohen class, and the corresponding kernel satisfies all the desirable kernel properties [4]. We also show that the shape of the Kaiser window can be used to control the trade-off between auto-term and cross-term energy.

Kaiser distribution: For kernel function $\phi(\cdot, \cdot)$, the Cohen class of TFDs [4] is given by

$$C_s(t, \omega, \phi) = \frac{1}{2\pi} \iint \exp(j(\omega t - \phi(\xi, \tau))) s(\mu + \frac{\xi}{2}) s^*(\mu - \frac{\xi}{2}) d\xi d\mu$$

where $s(\cdot)$ is the time signal to be analysed, $s^*(\cdot)$ is its complex conjugate, $t$ and $\omega$ are time and frequency, respectively, and $\tau$ and $\xi$ are time lag and frequency lag, respectively. The limit of each integral is from $-\infty$ to $\infty$.

The kernel function $\phi(\cdot, \cdot)$ defines a particular distribution of the class. Some choices for the kernel function are $\phi(\xi, \tau) = 1$ (Wigner-Ville distribution [1]), $\phi(\xi, \tau) = \exp(-\xi^2/\sigma)$ (Choi-Williams distribution [5]) and $\phi(\xi, \tau) = J_0(2\pi\sigma\xi)/\pi\sigma\xi$ (Bessel distribution [6]).

We introduce the sinc kernel

$$\phi(\xi, \tau) = \frac{\sin(\beta \sqrt{1 - \alpha^2 \xi^2 \tau^2})}{\sinh(\beta \sqrt{1 - \alpha^2 \xi^2 \tau^2})}$$

where $\alpha$ and $\beta$ are real parameters. When $\beta = 0$, the sinc kernel will be reduced to the sinc kernel [1], given by $\phi(\xi, \tau) = \sin(\alpha \xi \tau)/(\alpha \xi \tau)$. By substituting (2) into (1) and integrating with respect to $\xi$ we get

$$K_s(t, \omega) = \frac{1}{4\pi \alpha \tau \sinh(\beta)} \frac{1}{\tau} e^{-\alpha \tau} \int_0^\infty \left[ \beta \left( 1 - \frac{(\mu - \frac{\xi}{2})^2}{\alpha^2 \tau^2} \right) \right] d\xi$$

where

$$\Pi(\frac{\mu - \frac{\xi}{2}}{2\alpha \tau}) = 1 \quad |\mu - \xi| \leq \alpha |\tau|$$

$$0 \quad \text{elsewhere}$$

and $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind.

Fig. 1 Effect of changing $\beta$ on time correlated window function

Let us consider the following two component signal, which is a sum of a frequency-modulated (FM) signal and a chirp signal:

$$s(t) = \cos \left( 2\pi f_1 t + \frac{\Delta f}{f_0} \sin(2\pi f_0 t) \right) + \cos \left( 2\pi \left( f_2 + \frac{k}{2} t \right) \right)$$

where the FM signal has a maximum frequency deviation of $\Delta f$ and a modulation frequency of $f_0$, and the chirp signal has a starting frequency of $f_2$ and a chirp rate of $k$. We use Kaiser distribution to perform time-frequency analysis of signal (5) with two values of $\beta$ as shown in Figs. 1, 2 and 3, with $f_1 = 12$Hz, $\Delta f = 4$Hz, $f_2 = 0.04$Hz, $f_0 = 2$Hz, and $k = 0.21$Hz/s. It is clear that the low value of $\beta$ can smooth out the interference terms in the time-frequency plane, but at the expense of slightly reduced auto-term resolution.

Cross-term suppression: Owing to the quadratic nature of the Cohen class, cross-terms appear in the distribution for a multi-component signal. The parameter $\beta$ controls the shape of the time correlated window function as shown in Fig. 1, and it could be used to achieve a good trade-off between auto-term and cross-term energy concentrations.

Fig. 2 Kaiser distribution with $\beta = 12$, for sum of FM and chirp signals

Fig. 3 Kaiser distribution with $\beta = 4$, for sum of FM and chirp signals
Conclusion: A novel Kaiser window based TFD has been introduced for non-stationary signal analysis. Auto-term resolutions and cross-term magnitudes were successfully traded by controlling the shape of the Kaiser window.

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