On the Spatial Localization of a Wireless Transmitter from a Multisensor Receiver

Glenn N. Dickins
Dolby Australia
16/233 Castlereagh Street
Sydney NSW 2000 Australia
Email: glenn.dickins@dolby.com

Rodney A. Kennedy
Department of Information Engineering, RSISE
The Australian National University
Canberra ACT 0200 Australia
Email: rodney.kennedy@anu.edu.au

Abstract—We consider the fundamental ability of multisensor receiver of finite spatial extent to localize a wireless transmitter in space. Spatial localization is identified with a multisensor receiver’s ability to distinguish source transmitters from different regions in space based on imperfect measurements within the finite spatial extent spanned by the receiver’s sensors. Two transmitters positioned within a determined spatial region cannot be distinguished or resolved according to a given measurement threshold. The space outside that occupied by the receiver is shown to be tiled by a finite number of such regions. Further, the regions depend on whether intensity information only is used or whether there is both magnitude and phase information available. Numerical examples are given based on a novel tiling algorithm.

Index Terms—wireless telecommunications, spatial localization, antennas and propagation

I. INTRODUCTION

The classic problem of determining direction of arrival (DOA) is well studied [1] with the statistical behaviour of algorithms given extensive treatment [2]–[4]. A recent publication dedicated to the topic demonstrates the continued and active interest in the area [5]. A result relating the uncertainty in direction of arrival to the dimensionality of the wave field in the observed region was previously presented by Williams et al. [6]. In this work we consider a novel approach to analysing the ability of a sensor array to resolve both the angle (in the spirit of DOA) and distance of a source transmitter. This is the spatial localization problem.

The formulation in this paper diverges from the more commonly studied spatial localization problems where location of a transmitter is estimated by a set of co-operating receivers widely distributed in space. The transmitter, whose location is to be estimated, is located within a region formed by the convex hull of the co-operating receivers using information regarding the time of arrival and/or power of the signals at the receivers. In this work, we consider a different problem—the ability of a single multisensor receiver to determine the location of transmitting objects outside of its limited physical extent. The receiver is assumed to have a number of sensors which are closely spaced within the limited physical spatial extent.

The main contributions of this work are:

1) to introduce a novel formulation and approach to the problem of source localization using a norm of the observed field over the bounded 2D measurement region corresponding to the physical extent of the multisensor receiver
2) to determine the finite number of possible distinct source transmitter regions as a measure of spatial resolution and complexity of the spatial localization task
3) to illustrate through examples the effect of receiver sensor arrangements and geometry on spatial localization

Section II contains our problem formulation. Some numerical analysis and investigations are detailed in Section III. Section IV develops some continuous sensor models to determine bounds for the number of distinct spatial localization regions. Section V considers the problem where the complete field information, intensity and phase, is used for localization. A discussion of the results and comparison to some other results in the literature is provided in Section VI.

II. PROBLEM FORMULATION

How well can a single device of limited physical extent and limited measurement resolution determine the location of a transmitter by using measurements over the physical extent of the receiver? Having limited physical extent and limited measurement resolution means that two sufficiently closely spaced transmitters cannot be distinguished. Therefore, starting with a fixed transmitter location, we can associate a region, encapsulating that transmitter, within which the receiver cannot resolve to a given accuracy/threshold. We shall see that such regions tile space and the fundamental questions relate to what the tiling looks like and how many tiles are there.

A set of \( Q \) sensors are located at positions \( r_q \in \mathbb{R}^2 \) for \( q = 1, \ldots , Q \) within a radius \( R \) such that \( \|x_q\| \leq R \). The sensors produce the measurement vector \( y = [y_1, \ldots , y_Q]^T \), where \( y_q \in \mathbb{R} \), \( y_q > 0 \) is a measure of the signal strength or intensity of the source field at the location \( x_q \).

For an arbitrary uncooperative source, there may be no information regarding the power level transmitted. Assume that the intensity of the signal received is normalised such that it is unity at the origin. In this sense the problem relates to the ability to detect the location of a source given that a
reasonable signal level is present at the receiver. In practice, the ability to detect source movement would decrease with the signal strength and consequently the source distance.

Given a source at position \( x \), the normalised intensity received by each sensor will be

\[
y_q = \frac{\| x \|}{\| x - x_q \|} \quad (1)
\]

where \( \| \cdot \| \) represents the euclidean distance. This matches our normalisation and encapsulates the radial decay of intensity that would be expected in space. Designate this multidimensional function as a vector

\[
y = f(x) = [y_1 \ldots y_Q]^T. \quad (2)
\]

Due to noise, or some other resolution limitation, the receiver is only able to distinguish sets of signals that differ by a certain threshold. That is the measurements \( y \) and \( y' \) are considered indistinguishable if

\[
\| y - y' \|^2_R = \frac{1}{Q} \sum_{q=1}^{Q} |y_q - y'_q|^2 < \varepsilon^2. \quad (3)
\]

The scaling by \( 1/Q \) is incorporated into this norm to account for the number of sensors present. The norm \( \| \cdot \|^2_R \) represents the root mean squared difference for the sensor array.

Given this arrangement, we are interested in studying the ability to determine the location of the source from such measurements. Specifically the questions to be addressed are:

- Is there some limit to the number of distinct locations that can be resolved or identified outside the observation region?
- What can we say about the shape of the source regions that can be discerned?
- How does this depend on the number and arrangement of the sensors?

III. NUMERICAL INVESTIGATION OF DISTINCT LOCALITIES

A. Proposed Tiling Algorithm

A first observation is that the number of distinct distinguishable transmitter regions will be unbounded if the source is allowed arbitrarily close to the sensor array. This is noted from (1) that \( \| f(x) \|_R \to \infty \) as \( x \to x_q \).

Consider the problem of the source and sensors lying in the same 2D plane. The sensors are located within a disc of radius \( R \). Define \( S \) as the infinite region excluding the sensor array being points of norm \( S \) or greater,

\[
S = \{ x \in \mathbb{R}^2 : \| x \| \geq S > R \}. \quad (4)
\]

We perform a tiling of the space \( S \) by constructing a (finite) set of discrete points \( P \) such that any point in \( S \) is not more than a certain measurement threshold, \( \varepsilon \), from a member of \( P \). That is, the set \( P \) is constructed such that there exists at least one \( p \in P \) for which

\[
\| f(x) - f(p) \|_R \leq \varepsilon, \quad \forall x \in S \quad (5)
\]

Each of the discrete points in \( P \) defines a "center" of a tile. Further, there is no requirement that there is only one such \( p \in P \) such that (5) holds. Indeed, for points in space near to the edge of a tile there will be more than one \( p \in P \).

Since we are interested in determining the number of distinct regions, we are looking for the finite set \( P \) with the least cardinality (size) that satisfies this property. This optimal set is not so easy to determine. We can, however, determine a near optimal set \( P \) by a constructive procedure which begins with the empty set, \( P = \{ \emptyset \} \) and progressively add points from \( S \). As points (representing the center of the tiles) are added to the tiling, we keep track of the accumulated tile region which is within \( \varepsilon \) of any point in \( P \),

\[
P = \bigcup_{p \in P} \{ x : \| f(x) - f(p) \|_R < \varepsilon \}. \quad (6)
\]

This is shown for a single point in Fig. 1. We can then add another point from the set obtained when set \( P \) is subtracted from set \( S \) which is written as set \( S \setminus P \). This process can be continued until \( S \setminus P = \{ \emptyset \} \). Implicitly, because the process is claimed to terminate, there should only be a finite number of points in \( P \). This is far from obvious since the \( S \) region is infinite (the whole 2D plane less finite disk \( S \)). This conjecture is proven in Section IV-A.

Whilst the process for selecting the next point in \( S \setminus P \) to add to \( P \) can be arbitrary, a systematic approach is to select a point with minimum radius,

\[
P = P \bigcup \arg \min_{x \in S \setminus P} \| x \|. \quad (7)
\]

In this way, the algorithm starts by selecting points on the inner radius \( S \), and proceeding outwards. This procedure creates a set of points that is a suboptimal \( \varepsilon \) covering of the set \( S \). However, we can be sure that (5) will be satisfied.

This process is shown over a small region of \( S \) in Fig. 2(a) with the addition to the set \( P \) shown for each of the four points in \( P \) for this tiling. The second part, Fig. 2(b), shows the boundaries of the regions for a sensor measurement of half
that used in the tiling algorithm. These regions do not overlap and in some cases just touch. This is a consequence of the fact that the norm used for determining a unique location, (3), is a valid norm and satisfies the triangle inequality.

Thus it can be seen that while the set $\Omega$ is a suboptimal covering of $S$ at level $\epsilon$, it is an insufficient set of points to cover $S$ at level $\epsilon/2$. If we calculate the number of points required for a tiling at level $\epsilon$, this will represent the number of regions for some optimal tiling at a smaller level between $\epsilon/2$ and $\epsilon$.

Formally, define $N_\epsilon$ as the minimum integer for which there exists a set with $N_\epsilon$ elements that is a covering of $S$ at level $\epsilon$. The number of elements in the tiling $\Omega$ will be an upper bound for $N_\epsilon$ and a lower bound for $N_\epsilon/2$. The tiling algorithm is not likely to be the algorithm used to partition the space for a practical application, however it serves to provide an upper bound.

B. Numerical Examples of Location Tiling

The numerical analysis is carried out using a fine grid of points to represent the set membership of $\Omega$. Whilst this is not an accurate numerical method, it is suitable for investigating the flavour of the problem. Tracking the exact boundary of $\Omega$ would be an arduous task. A suitable level of detail is obtained by making the grid size small enough to reveal the smallest regions near the region boundary with radius $S$.

Fig. 3 shows a plot of such a point set obtained for an 8 element uniform circular array with radius $R = 1$ with minimum radius $S = 1.5$ for a value of $\epsilon = 0.1$. The boundaries shown on the plot represent the region around each point for which the level of distinguishable $\|y - y'\|_R = \epsilon/2$. These regions do not overlap since the distance between any two points in the tiling is at least $\epsilon$, $\|y - y'\|_R \geq \epsilon$, and the norm as defined in (3) is sub-additive.

The regions become densely packed near the sensor array and grow in size further away from the array. Beyond the limits shown in the figure, all points become indistinguishable with one region covering the entire range of $S$ beyond the regions shown. Thus the regions shown represent a complete tiling of the space $S$.

With the same configuration as the previous example, the sensor geometry is changed to a 16 element uniform circular array. The shape of the localization regions for the two different array geometries are compared in Fig. 4. The characteristics of the regions are not overly sensitive to the number of sensors. While there is some variation in the region size and shape closer to the sensor array, any difference

Fig. 3. An example of the distinguishable location regions for an 8 element uniform circular array or radius $R = 1$. Points are plotted for radii greater than $S = 1.5$. The signal is normalised to be unity at the origin. The points correspond to a minimum spacing of $\epsilon = 0.1$ with the contour shown representing $\epsilon/2$. There are 216 distinct localization regions.

Fig. 2. Demonstration of the tiling algorithm used to partition the space into resolvable locations. The simulation uses 8 sensors with unity radius and a minimum radius for $S$ of 1.5. The threshold for the distinguishable locations was $\epsilon = 0.2$. The first figure shows the tiling regions with $\|y - y'\|_R = |f(x) - f(x')|_R \leq \epsilon$ for each of the four points added to the tiling. The second region shows the boundary of the region for $\|y - y'\|_R \leq \epsilon/2$. Points are plotted for radii greater than $S = 1.5$.
Fig. 4. Comparison of the discernable region shapes for an 8 and 16 element UCA. The regions are fairly insensitive to the number of sensors, becoming almost identical for $R > 2$. The regions close to the array are slightly smaller. A complete tiling for the 16 element configuration would have 228 distinct regions.

Fig. 5. An example of the discernable location regions for an 8 element array with random sensor location on the circle with radius $R = 1$. The tiling covers points with radius greater than $S = 1.5$. The density of the indistinguishable regions varies with the sensor arrangement, and the total number is reduced to 161.

becomes negligible once the radius exceeds twice that of the sensor array.

Since the larger number of sensors offers an improved resolution close to the sensor array, the total number of distinguishable regions is increased. For the 16 element UCA the tiling has 228 points compared with 216 points for the 8 element UCA.

If a more random distribution of sensor locations is considered, the distortion in the region shapes becomes more apparent as shown in Fig. 5. Region sizes are smaller closer to the clustered sensors and become larger for the orientations where the sensors are further apart. While the region shapes have changed, the total number of distinguishable regions has not changed significantly. For the example presented, the tiling consists of 161 points compared with 216 for the 8 element uniform circular array.

The uniform circular array has desirable properties of symmetry and maximal minimum inter-element spacings. It is reasonable to expect that the tiling for the uniform circular array would provide an upper bound for the number of points in a tiling of an arbitrary array geometry confined to the same radius.

The previous examples used a measure of unique location detection, (3), being the root mean squared (RMS) of the difference in the intensity at the sensors. If the sensor measurements were in some way quantised, the indistinguishable region would be that for which the largest change in any sensor value was less than some threshold. This gives the norm and location resolution criterion

$$
\|\mathbf{y} - \mathbf{y}'\|_{R'} = \max_q |y_q - y_q'| < \epsilon'.
\tag{8}
$$

Fig. 6 compares the region shapes of this norm to the previous norm (3). The general characteristics of the regions are similar, after appropriate scaling. For the example presented, a value of $\epsilon' = 2\epsilon$ create regions of a similar size. The new norm creates regions smaller than the RMS norm close to the sensor array where the proximity to one sensor will dominate. Further away, the new regions are larger since the contribution from multiple sensors is not considered in the norm. The region shapes for the single sensor show abrupt corners where there is a change in the sensor dominating the norm.

Although the problem has been formulated with discrete sensors, the examples show that beyond some limit the number to the clustered sensors and become larger for the orientations where the sensors are further apart. While the region shapes have changed, the total number of distinguishable regions has not changed significantly. For the example presented, the tiling consists of 161 points compared with 216 for the 8 element uniform circular array.

The uniform circular array has desirable properties of symmetry and maximal minimum inter-element spacings. It is reasonable to expect that the tiling for the uniform circular array would provide an upper bound for the number of points in a tiling of an arbitrary array geometry confined to the same radius.

The previous examples used a measure of unique location detection, (3), being the root mean squared (RMS) of the difference in the intensity at the sensors. If the sensor measurements were in some way quantised, the indistinguishable region would be that for which the largest change in any sensor value was less than some threshold. This gives the norm and location resolution criterion

$$
\|\mathbf{y} - \mathbf{y}'\|_{R'} = \max_q |y_q - y_q'| < \epsilon'.
\tag{8}
$$

Fig. 6 compares the region shapes of this norm to the previous norm (3). The general characteristics of the regions are similar, after appropriate scaling. For the example presented, a value of $\epsilon' = 2\epsilon$ create regions of a similar size. The new norm creates regions smaller than the RMS norm close to the sensor array where the proximity to one sensor will dominate. Further away, the new regions are larger since the contribution from multiple sensors is not considered in the norm. The region shapes for the single sensor show abrupt corners where there is a change in the sensor dominating the norm.

Although the problem has been formulated with discrete sensors, the examples show that beyond some limit the number
of sensors is not significant to the ability to resolve the source location. A field across the measurement region is described by (1). This constraint is a consequence of the wave equation which describes the field characteristics across a region of space. The following sections will investigate this further by adopting a continuous spatial model of the signal space to address the questions posed in Section II.

IV. INTRINSIC LIMITS OF RESOLVING SPATIAL LOCATION

The previous numerical examples demonstrated that distinguishable region size increased with the source distance. As the source is moved away, the intensity measured by each sensor, (1), will approach unity. This suggests a “horizon” beyond which it is not possible to resolve the location of a source with any certainty under (3).

A. Localization Horizon

Consider the general case of \( Q \) sensors within a region of radius \( R \), and the measurement condition (3). A sufficient condition for all sources located at a distance \( H \) or greater to be indistinguishable at level \( \varepsilon \) will be

\[
\|1-y\|_R^2 \leq \frac{1}{Q} \sum_{q=1}^{Q} (1-y_q)^2 \leq \left(1 - \frac{H}{H-R}\right)^2 = \left(\frac{R}{H-R}\right)^2 \leq \varepsilon^2 \quad (9)
\]
equivalently,

\[H \geq R \left(1 + \frac{1}{\varepsilon}\right) \quad (10)\]

where the measurement vector \( y \) has elements \( y = f(x) = [y_1 \ldots y_Q]^T \).

The previous examples where \( R = 1 \) and \( \varepsilon = 0.1 \) will have a horizon with radius less than 11. This is a strict upper bound for the horizon based on a worst case geometry. If the sensors are spread evenly with radius \( R \) then

\[
\|1-y\|_R^2 \approx \frac{1}{2\pi} \int_0^{2\pi} \left(1 - \frac{H}{H-R(\cos \theta)}\right)^2 \, d\theta \quad (11)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{R(\cos \theta)}{H-R(\cos \theta)}\right)^2 \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2}{H-R(\cos \theta)} \, d\theta \leq \frac{1}{2} \left(\frac{R}{H-R}\right)^2 \leq \varepsilon^2 \quad (12)
\]

which can be simplified to yield the result

\[H \geq R \left(1 + \frac{1}{\sqrt{\varepsilon}}\right). \quad (14)\]

This provides a superior approximation to (10) when the sensors are evenly spaced on the edge of the region. Fig. 7 demonstrates the bound and approximation for a uniform and skewed distribution of sensors. For the uniform array, the actual horizon is approximately 7 units whilst the approximation is 8 and the bound is 11 units. For the skewed distribution the actual horizon is seen to approach the bound in some directions.

B. Number of Distinct Localities

The numerical examples from Section III demonstrated that a finite number of points tiled the space \( S \) external to the sensor array. Since the regions are of finite size and need only fill the space from radius, \( S \), to the horizon, \( H < \infty \), it should be possible to bound the number of distinct localities. This provides useful information, for example the amount of storage or bits required to specify the source location as determined by the receiver.

A first approximation for this bound can be obtained from the space of measured signals. From (1) the sensor values are bounded, with the extremum occurring for a source with radius \( S \),

\[
\frac{S}{S+R} \leq y_q \leq \frac{S}{S-R} \quad \forall y_q, \ q = 1, \ldots, Q. \quad (15)
\]

Thus we can consider the \( Q \)-dimensional vector \( y = f(x) =...
\[ y_1 \ldots y_Q \] as lying in the \( Q \)-dimensional hypercube,
\[
y \in \left[ \frac{S}{S + R}, \frac{S}{S - R} \right]^Q.
\] (16)

A grid of hypercubes covering this space with stride \( a = 2\varepsilon/\sqrt{Q} \) will ensure every measurement lies within \( \varepsilon \) of a cube centre. The number of regions, \( N \), is bounded by the number of \( Q \)-dimensional cubes to tile the space,
\[
N \leq \left( \frac{S}{S - R} \right)^{Q} \leq \left( \frac{SR}{S^2 - R^2} \right)^{Q}. 
\] (17)

 Whilst this shows a finite bound, it is extremely conservative. For the example with \( R = 1 \), \( S = 1.5 \), \( Q = 8 \) and \( \varepsilon = 0.1 \), the bound is \( N < 7 \times 10^9 \). From the numerical investigation, Fig. 4, we know that \( N < 228 \). The bound (17) grows with the number of sensors, however Fig. 4 showed the regions are fairly independent of the number of sensors beyond some point. The bound is not particularly useful.

Since the bound is based on the sensor values being independent, it does not take into account the constraint of the continuous field across the region. The field cannot vary arbitrarily and must satisfy the sensor constraint given in (1). Only a small subset of the space in (16) can represent valid points. The problem is to find the number of points for an \( \varepsilon \) covering of this subset.

As an alternate approach, noting that the region sizes grow with increasing radius, at \( S \) the smallest region can be found from assuming a worst case of all sensors closest to the source,\[
\|y - y'\|^2 = \frac{1}{Q} \sum_{q=1}^{Q} (y_q - y'_q)^2 \leq \frac{\varepsilon (S - R)^2}{R - \varepsilon (S - R)} 
\] (18)

for two locations with radius \( S \) and \( S + \Delta S \). This leads to the bound \( \|x - x'\| = \Delta S \leq \Delta S \leq \frac{\varepsilon (S - R)^2}{R - \varepsilon (S - R)} \). (19)

For the example with \( R = 1 \), \( S = 1.5 \), and \( \varepsilon = 0.1 \) this corresponds to a radius of approximately 0.03 consistent with the plots in Fig. 4. The number of regions of this size covering the region from \( S \) to \( H \) will be
\[
N = \frac{\pi (R^2 - S^2)}{\pi d S^2} = \frac{R^2 (1 + \frac{1}{3})^2 - S^2}{(R^2 - S^2) d S^2} \leq \frac{R^2 (R + S) - 2}{2\varepsilon^2 (S - R)^4} 
\] (20)

This provides a bound on the number of distinct regions that is independent of the number or orientation of the sensors. For the previous example, the bound is \( N < 16 \times 10^3 \). Whilst this is a lower bound than (17) it is still very conservative since the growth in the region size with radius is not taken into consideration.

**C. Application of Continuous Spatial Model**

Sensor measurements reflect the variation of the wave field over a region of space. This continuous spatial field must satisfy the wave equation. This allows us to employ the approach of continuous spatial models where we can derive a natural set of basis functions to represent the field in the measurement region. Whilst the examples presented have considered two-dimensional space, we develop the continuous framework for the three-dimensional localization problem. For a source at position \( x \) and the sensor located at \( x_q \), the fundamental solution of the Helmholtz equation can be expanded [7, Theorem 2.10]
\[
e^{-ik|x - x_q|} = ik4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(k \|x\|) Y_n^m(\hat{x}) j_n(k \|x_q\|) Y_n^m(\hat{x}_q) 
\] (22)

where \( Y_n^m(\cdot) \) are the spherical harmonics defined on a unit vector argument, \( j_n(\cdot) \) is the \( n^{th} \) order spherical Bessel function of the first kind, and \( h_n(\cdot) \) is the \( n^{th} \) order spherical Hankel function of the first kind. The wave number \( k = 2\pi/\lambda \) is related to the rate of change of the wave phase across space.

For the problem being considered, the sensors are only sensitive to the intensity of the field. This can be achieved by considering the limit of the fundamental solution as \( k \to 0 \). We can then consider small argument approximations for the spherical Bessel and Hankel functions,
\[
j_n(z) = \frac{z^n}{1 \cdot 3 \cdot \ldots (2n + 1)} (1 + O(z^2)) \quad z \to 0 
\] (23)
\[
h_n^{(1)}(z) = \frac{1 \cdot 3 \cdot \ldots (2n - 1)}{iz^{n+1}} (1 + O(z^2)) \quad z \to 0. 
\] (24)

Substituting these into (22) and adding the normalisation (1) we obtain
\[
y_q = \frac{\|x\|}{\|x - x_q\|} 
\] (25)
\[
= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{4\pi}{(2n + 1) \|x\|} Y_n^m(\hat{x}) \|x_q\| Y_n^m(\hat{x}_q). 
\] (26)

We are interested in the case where the source is some minimum distance from the receiver, \( \|x\| > S > R \). The signal observed by the receiver is constrained to \( \|x_q\| \leq R \). Using this we can write
\[
y_q = \sum_{n=0}^{\infty} \frac{4\pi R^n}{(2n + 1) \|x\|^n} \|x_q\|^n Y_n^m(\hat{x}) \beta_n^m(\hat{x}_q) 
\] (27)

where
\[
\beta_n^m(\hat{x}_q) = \frac{\|x_q\|^n}{R^n} Y_n^m(\hat{x}_q). 
\] (28)

Since the sensor region is constrained, \( \|x_q\| \leq R \), the basis functions \( \beta_n^m \) will be bounded. The coefficients decrease.
will decrease exponentially at a rate related to the ratio of the receiver and source radius \((R/||x||)^n\) with \(||x|| > S > R\). An expansion of the form (28) will be essentially finite dimensional.

From the problem definition in Section II, we know the problem has only two degrees of freedom. The field generated by a source is uniquely specified by the source position, which for the two-dimensional problem studied has two degrees of freedom. The problem lies in finding a representation of the field which reflects this dimensionality and also allows us to easily determine the number of distinguishable fields.

Consider the summation identity for the spherical harmonics, [7, Theorem 2.8]

\[
\sum_{m=-n}^{n} Y^m_n(\hat{x}) Y^m_n(\hat{x}_q) = \frac{2n+1}{4\pi} P_n(\cos \theta) \quad (29)
\]

where \(P_n\) is the Legendre function and \(\theta\) is the angle between the directions of \(\hat{x}\) and \(\hat{x}_q\). Using this in equation (26) we obtain,

\[
y_q = \frac{||x||}{||x - x_q||} = \sum_{n=0}^{\infty} P_n(\cos \theta) \frac{||x_q||}{||x||} \quad (30)
\]

Since \(|P_n(\cos \theta)| \leq 1\) [8], the terms contributing to \(y_q\) will decrease exponentially at least as fast as \((R/S)^n\). This expansion is not a basis function expansion since the argument of \(P_n(\cos \theta)\) is dependent on both the source and receiver position.

**D. Reflecting in the Circle**

Consider another approach to the problem where the sensor values are normalised

\[
||y||_R = \frac{1}{Q} \sum_{q=1}^{Q} y_q^2 = 1 \quad (31)
\]

and the distance between two measurements is calculated as the root mean squared sensor value, (3). This normalisation is equivalent to having a unit average signal intensity across the array independent of the source distance.

If the sensor is confined to the circle \(||x_q|| = R\), then for each position outside of the sensor array, there is a position within the sensor array with an equivalent normalized sensor output. This can be observed by considering the geometry of the problem as shown in Fig. 8. It is evident that these two points will lie on the same line extending from the origin of the circular array. From the radial source function (1) and normalisation (31) it can be seen that the measurements for points \(x\) and \(x'\) will be equivalent when

\[
\|y - y'\|_R = \frac{f(x)}{\|f(x)\|_R} - \frac{f(x')}{\|f(x')\|_R} = 0. \quad (32)
\]

Since each measurement \(y_q\) scales with the reciprocal of the distance between the source and sensor, this implies that the distances between the locations \(x\) and \(x'\) and any two points on the circle must be in the same ratio. We select two points, one being the intersection of the line extending from the origin through \(x\) and \(x'\), and the other at an arbitrary angle \(\theta\). Using the law of cosines for the associated triangles,

\[
\frac{R - x'}{\sqrt{R^2 + x'^2 - 2Rx' \cos \theta}} = \frac{x - R}{\sqrt{R^2 + x^2 - 2Rx \cos \theta}} \quad (33)
\]

This gives the quadratic equation to solve for the radius \(x'\) of the point \(x'\) as

\[
x'^2 - x' (R^2 + x^2) - R^2 x = 0, \quad \cos \theta \neq 1 \quad (34)
\]

for which it can be shown that \(x' = R^2 / x\).

Thus each point in the region \(S\) is mapped into the finite region bounded by a circle with radius \(R^2 / S\). This is convenient since the unbounded region \(S\) maps to a simple bounded region.

Fig.9 shows the regions of distinct localization for the case of a uniform circular array with \(R = 1\), \(Q = 8\), \(S = 1.5\) and \(\epsilon = 0.05\). For each distinct region in the space \(S\) there is a corresponding region within the array. Furthermore, the corresponding regions within the circular array are all approximately the same size. This provides a bounded region with two degrees of freedom across which the

---

**Fig. 8.** Geometry for the reflection of the location regions inside the uniform circular array.

**Fig. 9.** Reflection of the space of distinct localities for the uniform circular array. For each point outside the array, there is a corresponding point, resulting in the same measurement vector, located inside the array.
distance between regions of indistinguishability is relatively constant.

To determine the smallest region size, consider a region inside the circle at the reflection of radius $S$. For two points at radius $x$ and $x'$, the detected signal difference will be

$$
\| y - y' \|_R \approx \sum_{q=1}^{2} \left( \frac{1}{x^2 + R^2 - 2Rx \cos \theta_q} - \frac{1}{x'^2 + R^2 - 2Rx' \cos \theta_q} \right)^2
$$

(35)

where the approximation arises from the normalisation (31) being applied equally to both observations. This is valid for small perturbations $x \approx x'$. By numerical inspection, for the case of $R = 1$, $S = 1.5$ and $\varepsilon = 0.05$, the minimum region size is approximately 0.015. This is consistent with Fig. 9.

The reflected regions inside the circle will fill the region from the origin to a radius of $R^2/S < R$. This is a finite area for which we can place a bound on the number of reflected regions with the smallest region size. This is independent of the number of sensors. For the example given this bound is $N < 2000$.

The simulations for the uniform circular array with signal intensity normalisation in Fig. 9 consisted of 453 points in the tiling. This bound obtained by considering the reflected regions is within an order of magnitude of this result.

Whilst this approach leads to the best matching bound, it is specific to the case of a uniform circular array with the intensity normalisation. As can be seen from the figures, the size and number of distinct regions is comparable, thus this bound gives some indication of the number of localities for the original problem.

It can also be argued that this will be a firm upper bound since a wave field across a region of space can be determined from its value on the boundary [7].

V. LOCALIZATION WITH PHASE COHERENT RECEIVER

The problem considered initially was the ability to localise a source given a receiver was only able to detect the field amplitude or intensity. This corresponds to the practical situation of processing a set of received signals without coherent phase detection across the array region. It was anticipated that this would be a simpler problem than considering the complete field information.

Consider a configuration where the receiver has access to the field amplitude and phase across the sensor region. The phase information will improve the ability to resolve the direction of arrival and distance through the direction and curvature of the wavefront passing through the sensor region.

Assuming the amplitude and phase of the source is normalised at the origin, the signal model will be

$$
y_q = \frac{\| x \| e^{j2\pi \| x - x_q \| / \| x - x_q \| e^{j2\pi \| x \|} .}
$$

(36)

Fig. 10 shows the distinguishable region tiling for the case of a circular array with $R = 1$, $Q = 8$, $S = 1.5$ and $\varepsilon = 0.2$.

This can be compared to Fig. 3 which considered the same configuration without phase information. In Fig. 10 the space is more segmented in angle and the radial extent is comparable even though the detection threshold has been doubled.

For a distant source, the normalised field amplitude across the sensor region will be unity. For a continuous uniform circular array, the signal difference introduced by two distinct directions of arrival separated by an angle $\phi$ will be

$$
\frac{1}{2\pi} \int_0^{2\pi} |e^{jkR \cos \theta} - e^{jkR \cos (\theta - \phi)}|^2 d\theta < \varepsilon^2.
$$

(37)

For the value of $\varepsilon = 0.2$ in Fig. 10, the value of $\phi$ that achieves the bound (37) is approximately $2.5^\circ$. This corresponds to 144 distinct angular regions. This is consistent with the results presented from the numerical tiling in the figure.

The natural basis expansion for the three-dimensional narrow-band field was presented previously (22). With the sources at a distance $S > R$ this expansion can be truncated to a finite dimensional representation with $(N + 1)^2$ terms where $N \approx kR$. This result has been presented in other works [9].

The sensor signals can be written

$$
y_q = \frac{\| x \| e^{j2\pi \| x - x_q \| / \| x - x_q \| e^{j2\pi \| x \|.}
$$

(38)

where

$$
\beta_n^m = j_n^m (k \| x_q \|) \frac{\gamma_n^m (x_q)}{\| x \|.}
$$

(39)

1The problem formulation was for a two-dimensional observation region with the sources lying in the same plane. However, the fundamental solution for three dimensions was used, (1), with the field intensity varying with the reciprocal of the radius. The wave equation in two dimensions permits a fundamental solution where the field intensity varies with the square root of the source radius. Whilst this is not a problem when considering general multipath fields and far-field source distributions, it is significant in the determination of the distinct localization regions in the vicinity of the sensor array.
For this example there will be \((N + 1)^2 = 49\) degrees of freedom. However, the valid coefficients for a normalised point source will be constrained to

\[
\alpha_n^m = \frac{ik4\pi b_0^{(1)}(k \|x\|)Y_n^m(\hat{x})}{b_0^{(1)}(k \|x\|)}.
\]

(40)

From the problem geometry, these coefficients will lie in a two dimensional manifold which has an appropriate mapping from the two dimensional source region \(S\). The unique determination of locations will be related to a weighted distance between the vectors of \(\alpha_n^m\) coefficients. Thus the problem of determining the number of unique localization regions would be related to determining the area of this manifold in an appropriately scaled space.

In general, the ability to resolve the distance of a source given measurements over a finite region is rather limited. As could be seen in Fig. 10, the angular resolution provides a more numerous division of the space than the range resolution. This work is part of a larger body of work investigating the application of continuous spatial models to problems in multiple antenna signal processing.

VI. DISCUSSION AND FURTHER IDEAS

The distance from source to sensor can be approximated by

\[
\|x - x_q\| = \sqrt{\|x\|^2 + \|x_q\|^2 - 2\|x\|\|x_q\|\cos\theta}
\]

\[
\approx \|x\| - \|x_q\|\cos\theta + \frac{\|x_q\|^2}{2\|x\|}\sin^2\theta
\]

(41)

where \(\theta = \theta_{x} - \theta_{x_q}\) is the angle between the source and sensor directions. For a uniform linear array, this equation is quadratic in the sensor element number and is sometimes referred to as the Fresnel approximation. For sources in the Fresnel region where (41) is a reasonable assumption, this can be used to simplify the signal model. A further simplification can be made to neglect the signal intensity. If the received signal is normalised, the signal model becomes

\[
y_q = \exp(\|x_q\|\cos\theta + \|x_q\|^2\sin^2\theta/2\|x\|).
\]

(42)

This approach has been used to create an algorithm for passive localization of near field sources [10].

It is a common assumption that sources beyond a certain distance appear as far-field sources with a planar wave front across the sensor array [11], [12]. This is a similar concept to the localization horizon introduced in Section IV-A. For a uniform linear array of length \(2R\) and a maximum phase variance of \(\pi/8\) radians over the array, the far-field distance is \(8R^2/\lambda\). This distance will increase with increasing frequency of the narrow-band signal. This contrasts the intensity only horizon (10) which was frequency independent. This implies that as the wavelength decreases, the signal phase dominates the size and shape of the localization regions. This is consistent with the assumption of \(k \to 0\) for the field intensity expansion (26). For the example presented in Fig. 10 the effective far-field distance would be around 8 which is consistent with the numerical analysis.

The size of a sensor array for which the phase information will dominate localization can be determined by considering (10)

\[
\frac{8R^2}{\lambda} > R(1 + \frac{1}{\varepsilon}) \quad \Rightarrow \quad R > \frac{\lambda}{8}(1 + \frac{1}{\varepsilon}).
\]

(43)

For the case considered in the examples, this corresponds to a radius of around \(1.4\lambda\). Thus in the example there is still some contribution from the intensity information. Fig. 11 compares the distinguishable regions for the case of phase only and phase and intensity measurements. The regions with both phase and intensity are slightly smaller. It can also be seen that the regions at a radius beyond 8 are extended to cover all radii beyond this.

The Cramér-Rao bound for passive range estimation is [13]

\[
\frac{\sigma}{\|x\|} \geq \left(\frac{\sqrt{\pi}}{2\pi}\right)\left(\frac{\lambda}{4R^2}\right)\text{SNR}^{-1/2}
\]

(44)

which suggests that the regions of uncertainty will grow linearly with the radius of the source. This is consistent with the partitioning of the reciprocal space introduced in Section IV-D.

The problem of distinct localization regions is particular to the way in which a receiver will view the electromagnetic environment in which it resides. Given a finite measurement resolution, it is apparent that there will be a fixed and finite number of distinct locations to which a source could be associated. Beyond some distance, it becomes impossible to determine the source range accurately.

VII. SUMMARY AND CONTRIBUTIONS

This work has detailed an attempt to analyse the number of distinct regions for a source that can be identified by a sensor array constrained to a finite volume. In essence, this
problem is one of mapping the world, as viewed by the sensor array, to a set of discrete observable regions. The problem was addressed in the context of analysing only the intensity information obtained from the field, with the incident field considered to have unit power at the sensor origin.

The following specific contributions were made in this paper:

- Demonstrated, through numerical analysis, that there will be a finite number of distinct location regions extending from outside the sensor array to an arbitrarily large distance.
- Presented an analysis of the sensor signal space and constructed a formal proof of the existence of a horizon radius beyond which all source locations will appear indistinguishable. This horizon is dependent on the radius of the sensor array and the detection threshold.
- Developed an analytic bound for the number of distinct locations that can be resolved. Since the field will be correlated over the sensor array, using an argument related to the number of distinct measurements without reference to the signal model produces a conservative bound for the number of distinct regions.
- Derived a tighter bound for the specific case of a uniform circular array based on a geometrical reflection argument and the regular tiling of a finite space. This bound is within an order of magnitude of the results obtained from the numerical investigations.
- Demonstrated that the addition of phase information provides a significant advantage in the ability to resolve both the direction of arrival and distance of a source.

Generally angular resolution is superior to range resolution. If intensity information is only available, beyond some radius, all sources will appear to be located in the same region of uncertainty. Where intensity and phase information is available, at a similar distance, range measurement becomes uncertain whilst the direction of arrival resolution remains effective.

The solutions and investigation of the problems posed was facilitated by considering a continuous model of the spatial field rather than by considering the signal vectors from a specific sensor array configuration.

REFERENCES