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Numerical Methods for Solving Inverse Eigenvalue Problems for Nonnegative Matrices

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Abstract— **Presented are two related numerical methods, one for the inverse eigenvalue problem for nonnegative or stochastic matrices and another for the inverse eigenvalue problem for symmetric nonnegative matrices. The methods are iterative in nature and utilize alternating projection ideas. For the symmetric problem, the main computational component of each iteration is an eigenvalue-eigenvector decomposition, while for the other problem, it is a Schur matrix decomposition. Numerical results are presented demonstrating the effectiveness of the algorithms.**

Keywords— **Inverse eigenvalue problem, nonnegative matrices, stochastic matrices, alternating projections, Schur's decomposition.**

I. INTRODUCTION

A real $n \times n$ matrix is said to be nonnegative if each of its entries is nonnegative.

The *Nonnegative Inverse Eigenvalue Problem* (NIEP) is the following: given a list of n complex numbers $\lambda = \{\lambda_1, \dots, \lambda_n\}$, find a nonnegative $n \times n$ matrix with eigenvalues λ (if such a matrix exists).

A related problem is the *Symmetric Nonnegative Inverse Eigenvalue Problem* (SNIEP): given a list of n real numbers $\lambda = \{\lambda_1, \dots, \lambda_n\}$, find a symmetric nonnegative $n \times n$ matrix with eigenvalues λ (if such a matrix exists)[1].

Finding necessary and sufficient conditions for a list λ to be realizable as the eigenvalues of a nonnegative matrix has been a challenging area of research for over fifty years and this problem is still unsolved; see the recent survey paper [2]. As noted in [3, Section 6], while various necessary or sufficient conditions exist, the necessary conditions are usually too general while the sufficient conditions are too specific. Under a few special sufficient conditions, a nonnegative matrix with the desired spectrum can be constructed, however, in general, proofs of sufficient conditions are non-constructive.

In this paper we are interested in generally applicable numerical methods for solving NIEPs and SNIEPs. To the best of our knowledge, the only algorithms that have up to now appeared in the literature consist of [4] for the SNIEP and [5] for the NIEP. In this paper we present a numerical algorithm for the NIEP and another for the SNIEP. In both cases, the problems are posed as problems of finding a point in the intersection of two particular sets. Unlike the approaches in [4] and [5] which are based on gradient flows, our algorithms are iterative in nature. For the SNIEP, the solution methodology is based on an alternating projection scheme between the two sets in question. The solution methodology for the NIEP is also based on an alternating projection like scheme but is more involved, as we will shortly explain.

While alternating projections can often be a very effective means of finding a point in the intersection of two or more convex sets, for both the SNIEP and NIEP formulations, one set is nonconvex. Nonconvexity of one of the sets means that alternating projections may not converge to a solution. This is in contrast to the case where all sets are convex and convergence to a solution is guaranteed.

The nonconvex set for the NIEP is particularly complicated; it consists of all matrices with the desired spectrum. At least some of the members of this set will be nonsymmetric matrices and it is this that causes complications. In particular, though the set is closed and hence projections are well defined theoretically, how to calculate projections onto such sets is an unsolved difficult problem. We formulate an alternate method for mapping onto this set. Though the resulting points are not necessarily projected points, they are members of the set and share a number of other desirable properties. As will be shown, this alternate 'projection' is very effective in our context. Furthermore, we believe that it may also be quite effective for other inverse eigenvalue problems involving non-symmetric matrices[2].

It is also possible to use the NIEP algorithm to find stochastic matrices with a given spectrum. Suppose λ is the spectrum of a stochastic matrix and that A is a nonnegative matrix with this spectrum. If A satisfies a rather mild

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[1] The NIEP and SNIEP are different problems even if λ is restricted to contain only real entries; there exist lists of n real numbers λ for which the NIEP is solvable but the SNIEP is not, [1].

[2] Preliminary indications of this are given in [6] and [7] where this idea is applied to inverse eigenvalue type problems arising in control theory.

condition then it can be transformed to a stochastic matrix with the same spectrum via an easily constructed similarity transformation, [5].

General background material on nonnegative matrices, including inverse eigenvalue problems and applications, can be found in the texts [8] and [9].

The rest of the paper is structured as follows. Projections play a key part in the algorithms and Section II contains general properties of projections that are used throughout the paper. The SNIEP algorithm is presented first, in Section III, and then insights from this algorithm are used to address the more difficult NIEP in Section IV. Numerical results for both algorithms are presented in Section V.

II. PROJECTIONS

Projections play a key part in the algorithms. This section contains general properties of projections that will be used throughout the paper.

Let x be an element in a Hilbert space H and let C be a closed (possibly non-convex) subset of H. Any $c_0 \in C$ such that $\|x - c_0\| \leq \|x - c\|$ for all $c \in C$ will be called a *projection* of x onto C. In the cases of interest here, namely that H is a finite dimensional Hilbert space, there is always at least one such point for each x. If C is convex as well as closed then each x has exactly one such minimum distance point [10]. Where convenient, we will use $y = P_C(x)$ to denote that y is a projection of x onto C. We emphasize that $y = P_C(x)$ only says y is *a* projection of x onto C and does *not* make any statement regarding uniqueness.

All problems of interest in this paper are feasibility problems of the following abstract form.

Problem 1: Given closed sets $C_1, \ldots, C_N$ in a finite dimensional Hilbert space H, find a point in the intersection

$$\bigcap_{i=1}^{N} C_i$$

(assuming the intersection is non-empty). □

(In fact, we will solely be interested in the case $N = 2$.)

If all the C_i's in Problem 1 are convex, a classical method of solving Problem 1 is to alternatively project onto the C_i's. This method is often referred to as the Method of Alternating Projections (MAP). If the C_i's have non-empty intersection, the successive projections are guaranteed to asymptotically converge to an intersection point [11].

Theorem 2 (MAP): Let $C_1, \ldots, C_N$ be closed convex sets in a finite dimensional Hilbert space H. Suppose $\bigcap_{i=1}^{N} C_i$ is nonempty. Then starting from an arbitrary initial value x_0, the following sequence

$$x_{i+1} = P_{C_{\phi(i)}}(x_i), \text{ where } \phi(i) = (i \bmod N) + 1,$$

converges to an element in $\bigcap_{i=1}^{N} C_i$.

We remark that the usefulness of MAP for finding a point in the intersection of a number of sets is dependent on being able to compute projections onto each of the C_i's.

While MAP is not guaranteed to converge to a solution if one or more of the C_i's is non-convex, for alternating projections between two sets, the following distance reduction property always holds.

Theorem 3: Let C_1 and C_2 be closed (non-empty) sets in a finite dimensional Hilbert space H. For any initial value $y_0 \in C_2$, if

$$\begin{aligned} x_1 &= P_{C_1}(y_0), \\ y_1 &= P_{C_2}(x_1), \\ x_2 &= P_{C_1}(y_1), \end{aligned}$$

then

$$\|x_2 - y_1\| \leq \|x_1 - y_1\| \leq \|x_1 - y_0\|.$$

Proof: The second inequality holds as y_1 is a projection of x_1 onto C_2 and hence its distance to x_1 is less than or equal to the distance of x_1 to any other point in C_2 such as y_0. The first inequality holds by similar reasoning. ■

Corollary 4: If for $n = 0, 1, \ldots$,

$$x_{n+1} = P_{C_1}(y_n), \quad y_{n+1} = P_{C_2}(x_{n+1}),$$

that is, the x_n's and y_n's are successive projections between two closed sets, then $\|x_n - y_n\|$ is a non-increasing function of n.

Suppose one is interested in solving Problem 1 in the case of two sets, C_1 and C_2, when one or both sets are non-convex. If projections onto these sets are computable, a solution method is to alternately project onto C_1 and C_2. Corollary 4 ensures that the distance $\|x_n - y_n\|$ is non-increasing with n. While this is promising, there is, however, no guarantee that this distance goes to zero and hence that a solution to the problem will be found.

Most of the literature on alternating projection methods deals with the case of convex subsets of a (possibly infinite dimensional) Hilbert space; a survey of these results is contained in [12]. The text [13] is also recommended. There is much less available for the case of one or more nonconvex sets; see in particular [14].

III. THE SYMMETRIC PROBLEM

Our algorithm for solving the SNIEP consists of alternately projecting onto two particular sets. The details are given in this section.

Given a list of real eigenvalues $\lambda = \{\lambda_1, \ldots, \lambda_n\}$, renumbering if necessary, suppose $\lambda_1 \geq \ldots \geq \lambda_n$. Let

$$\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n), \tag{1}$$

and let $\mathcal{M}$ denote the set of all real symmetric matrices with eigenvalues λ,

$$\begin{aligned} \mathcal{M} = \{A \in \mathcal{S}^n \mid A = V \Lambda V^T \\ \text{for some orthogonal } V\}. \end{aligned} \tag{2}$$

Let $\mathcal{N}$ denote the set of symmetric nonnegative matrices,

$$\mathcal{N} = \{A \in \mathcal{S}^n \mid A_{ij} \geq 0 \text{ for all } i, j\}. \tag{3}$$

The SNIEP can now be stated as the following particular case of Problem 1:

$$\text{Find } X \in \mathcal{M} \cap \mathcal{N}. \tag{4}$$

Our solution approach is to alternatively project between $\mathcal{M}$ and $\mathcal{N}$, and we next show that it is indeed possible to calculate projections onto these sets. First, in order for the term 'projection' to make sense, we need to define an appropriate Hilbert space and associated norm. From now on $\mathcal{S}^n$ will be viewed as a Hilbert space with inner product

$$\langle A, B\rangle = \operatorname{tr}(AB) = \sum_{i,j} a_{ij} b_{ij}. \tag{5}$$

The associated norm is the Frobenius norm $\|A\| = \langle A, A\rangle^{\frac{1}{2}}$.

The projection of $A \in \mathcal{S}^n$ onto $\mathcal{M}$ is given by Theorem 6 below. More precisely, it gives *a* projection of A onto $\mathcal{M}$. The reason for this is that the set $\mathcal{M}$ is non-convex[3] and hence projections onto this set are not guaranteed to be unique. Theorem 6 is based on the following result of Hoffman and Wielandt (see for example [15, Corollary 6.3.8]).

Lemma 5: Suppose $A, B \in \mathcal{S}^n$ have eigenvalue-eigenvector decompositions

$$\begin{aligned} A &= VDV^T, & D &= \operatorname{diag}(\lambda_1^A, \ldots, \lambda_n^A),\\ B &= WEW^T, & E &= \operatorname{diag}(\lambda_1^B, \ldots, \lambda_n^B), \end{aligned}$$

where $V, W \in \mathbb{R}^{n\times n}$ are orthogonal and $\lambda_1^A \geq \ldots \geq \lambda_n^A$ and $\lambda_1^B \geq \ldots \geq \lambda_n^B$. Then

$$\|D - E\| \leq \|A - B\|.$$

Theorem 6: Given $A \in \mathcal{S}^n$, let $A = V \operatorname{diag}(\mu_1, \ldots, \mu_n)V^T$ with V a real orthogonal matrix and $\mu_1 \geq \ldots \geq \mu_n$. If Λ is given by (1), then $V\Lambda V^T$ is a best approximant in $\mathcal{M}$ to A in the Frobenius norm.

Proof: A matrix $B \in \mathcal{M}$ is a projection of A onto $\mathcal{M}$ if $\|A - B\| \leq \|A - M\|$ for all $M \in \mathcal{M}$. $M \in \mathcal{M}$ if and only if $M = W\Lambda W^T$ for some real orthogonal matrix W. Hence Lemma 5 implies

$$\|\operatorname{diag}(\mu_1, \ldots, \mu_n) - \Lambda\| \leq \|A - M\| \quad \text{for all } M \in \mathcal{M}. \tag{6}$$

If $B = V\Lambda V^T$ then, using the fact that the Frobenius norm is orthogonally invariant, it follows that

$$\begin{aligned} \|A - B\| &= \|V(\operatorname{diag}(\mu_1, \ldots, \mu_n) - \Lambda)V^T\| \\ &= \|\operatorname{diag}(\mu_1, \ldots, \mu_n) - \Lambda\|. \end{aligned} \tag{7}$$

The result follows by combining (6) and (7). ■

Projection onto $\mathcal{N}$ is straightforward and is given by Theorem 7 below.

Theorem 7: Given $A \in \mathcal{S}^n$, define $A_+ \in \mathcal{S}^n$ by

$$(A_+)_{ij} = \max\{A_{ij}, 0\} \text{ for all } 1 \leq i, j \leq n. \tag{8}$$

A_+ is the best approximant in $\mathcal{N}$ to A in the Frobenius norm.

Proof: The projection of $x \in \mathbb{R}$ onto the non-negative real numbers equals $\max\{x, 0\}$. The general result follows by noting that if $B \in \mathcal{S}^n$, and in particular if $B \in \mathcal{N}$, then

$$\|A - B\| = \left(\sum_{i,j} |A_{ij} - B_{ij}|^2\right)^{\frac{1}{2}}$$

and hence that the problem reduces to n^2 decoupled scalar problems. ■

Our proposed algorithm for solving the SNIEP is the following.

SNIEP algorithm:

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Problem Data. List of desired real eigenvalues $\lambda = \{\lambda_1, \ldots, \lambda_n\}$, $\lambda_1 \geq \ldots \geq \lambda_n$.
Initialization. Choose a randomly generated symmetric nonnegative matrix $Y \in \mathbb{R}^{n\times n}$.
repeat
    1. Calculate an eigenvalue-eigenvector decomposition of $Y$: $Y = V \operatorname{diag}(\mu_1, \ldots, \mu_n)V^T$, $\mu_1 \geq \ldots \geq \mu_n$.
    2. $X := V \operatorname{diag}(\lambda_1, \ldots, \lambda_n)V^T$.
    3. $Y := X_+$.
until $\|X - Y\| < \epsilon$.
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In the above algorithm, X_+ is given by (8).

Note that at each iteration of the algorithm, X has the desired spectrum λ and Y is nonnegative. If ϵ is small, say $\epsilon = 10^{-14}$, termination of the loop ensures X equals Y (approximately) and hence that Y solves the SNIEP.

Due to small numerical inaccuracy, X from Step 2 of the algorithm may not be perfectly symmetric. To counter this, immediately after Step 2, X should be replaced by $(X + X^T)/2$.

Of course, while Corollary 4 ensures $\|X - Y\|$ is non-increasing from one iteration to the next, the set $\mathcal{M}$ is non-convex and hence there is no guarantee that the algorithm will terminate.

IV. THE GENERAL PROBLEM

Throughout this section, $\mathbb{C}^{n\times n}$ will be viewed as a Hilbert space with inner product

$$\langle A, B\rangle = \operatorname{tr}(AB^*) = \sum_{i,j} a_{ij} \bar{b}_{ij}.$$

The associated norm is the Frobenius norm $\|A\| = \langle A, A\rangle^{\frac{1}{2}}$.

Recall Schur's result that any matrix $A \in \mathbb{C}^{n\times n}$ is unitarily equivalent to an upper triangular matrix.

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[3] $\mathcal{M}$ is non-convex if its defining λ contains a pair of non-equal eigenvalues. For example, if $n = 2$, consider

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}.$$

If $\lambda_1 \neq \lambda_2$ then the convex combination $(A + B)/2$ does not have the same spectrum as A and B.

Theorem 8: Given $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\mu_1, \dots, \mu_n$ in any prescribed order, there is a unitary matrix $U \in \mathbb{C}^{n\times n}$ and an upper triangular matrix $T \in \mathbb{C}^{n\times n}$ such that

$$A = UTU^* \tag{9}$$

and $T_{ii} = \mu_i$, $i = 1, \dots, n$.

Proof: See for example [15, Theorem 2.3.1]. ■

We now re-define some terms from the prior section.

Let $\lambda = \{\lambda_1, \dots, \lambda_n\}$ be a given list of complex eigenvalues. Define

$$\mathcal{T} = \{T \in \mathbb{C}^{n\times n} \mid T \text{ is upper triangular with spectrum } \lambda\}. \tag{10}$$

Theorem 8 implies that the set of all complex matrices with spectrum λ is given by the following set $\mathcal{M}$, which in this section is defined as,

$$\mathcal{M} = \{A \in \mathbb{C}^{n\times n} \mid A = UTU^* \text{ for some unitary } U \text{ and some } T \in \mathcal{T}\}. \tag{11}$$

Let $\mathcal{N}$ denote the set of (not necessarily symmetric) nonnegative matrices,

$$\mathcal{N} = \{A \in \mathbb{R}^{n\times n} \mid A_{ij} \geq 0 \text{ for all } i, j\}. \tag{12}$$

Having re-defined $\mathcal{M}$ and $\mathcal{N}$, the NIEP can now be stated as the following particular case of Problem 1:

$$\text{Find } X \in \mathcal{M} \cap \mathcal{N}. \tag{13}$$

A difficulty now occurs. We would like to use alternating projections to solve the NIEP. However, to the best of our knowledge, how to calculate projections onto $\mathcal{M}$ is an unsolved problem. Suppose instead we could find a mapping that was in some sense a reasonable substitute for a projection map for $\mathcal{M}$. Using this substitute mapping and the projection map for $\mathcal{N}$ in an alternating projection like scheme may still produce a viable algorithm. Indeed, we now propose the following function $P_{\mathcal{M}}$ as a substitute for a true projection map onto $\mathcal{M}$. (The notation $P_{\mathcal{M}}$ is used as it is suggestive, however, recall that we have already used $y = P_C(x)$ to denote that y is a projection of x onto a set C. The two different uses of the notation should be clear from their context and should not cause confusion.)

Definition 9: Suppose $U \in \mathbb{C}^{n\times n}$ is unitary and $T \in \mathbb{C}^{n\times n}$ is upper triangular. Let $\hat{\lambda}_i$, $i = 1, \dots, n$, be a permutation of the list of eigenvalues λ such that, amongst all possible permutations, it minimizes

$$\sum_{i=1}^{n} |\hat{\lambda}_i - T_{ii}|^2. \tag{14}$$

Define

$$P_{\mathcal{M}}(U, T) = U\hat{T}U^* \tag{15}$$

where $\hat{T} \in \mathcal{T}$ is given by

$$\hat{T}_{ij} = \begin{cases} \hat{\lambda}_i, & \text{if } i = j, \\ T_{ij}, & \text{otherwise.} \end{cases}$$

□

Note that $P_{\mathcal{M}}$ maps into the set $\mathcal{M}$.

A given $A \in \mathbb{C}^{n\times n}$ may have a non-unique Schur decomposition and we now show that $A = U_1T_1U_1^* = U_2T_2U_2^*$ does *not* imply $P_{\mathcal{M}}(U_1, T_1) = P_{\mathcal{M}}(U_2, T_2)$. The non-uniqueness of Schur's decomposition is demonstrated by the following example [15]:

$$T_1 = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} 2 & -1 & 3\sqrt{2} \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}$$

are unitarily equivalent via

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

as $UT_1U^* = T_2$. If $\lambda = \{0, 0, 0\}$, it is readily verified that $P_{\mathcal{M}}(U, T_1) \neq P_{\mathcal{M}}(I, T_2)$. Hence $P_{\mathcal{M}}$ is decomposition dependent.

It turns out that the fact that $P_{\mathcal{M}}$ may give different points for different Schur decompositions of the same matrix is not particularly important. The following result shows that different Schur decompositions lead to points in $\mathcal{M}$ of equal distance from the original matrix.

Theorem 10: Suppose $A = U_1T_1U_1^* = U_2T_2U_2^*$ where $U_1, U_2 \in \mathbb{C}^{n\times n}$ are unitary and $T_1, T_2 \in \mathbb{C}^{n\times n}$ are upper triangular. Then

$$\|P_{\mathcal{M}}(U_1, T_1) - A\| = \|P_{\mathcal{M}}(U_2, T_2) - A\|.$$

Proof: Suppose $A = UTU^*$ where U is unitary and T is upper triangular. If $\hat{T}$ is the matrix given in Definition 9, then by the unitary invariance of the Frobenius norm,

$$\|P_{\mathcal{M}}(U, T) - A\| = \|\hat{T} - T\|.$$

As $\|\hat{T} - T\|^2$ equals the quantity in (14), $\|P_{\mathcal{M}}(U, T) - A\|$ depends only on λ and $T_{11}, \dots, T_{nn}$. The result now follows by noting that the T_{ii}'s are the eigenvalues of A and that (14) does not depend on the ordering of the T_{ii}'s. ■

The next theorem shows that given $A = UTU^*$, if we restrict attention to matrices of the form $U\tilde{T}U^*$, $\tilde{T} \in \mathcal{T}$, then $P_{\mathcal{M}}(U, T)$ is a point in $\mathcal{M}$ closest to A.

Theorem 11: Suppose $A = UTU^* \in \mathbb{C}^{n\times n}$ with U a unitary matrix and T upper triangular. Then $P_{\mathcal{M}}(U, T)$ satisfies

$$\|P_{\mathcal{M}}(U, T) - A\| \leq \|U\tilde{T}U^* - A\| \text{ for all } \tilde{T} \in \mathcal{T}.$$

Proof: Let $\tilde{T}$ be a matrix in $\mathcal{T}$. The unitary invariance of the Frobenius norm implies the result will be established if we can show

$$\|\hat{T} - T\| \leq \|\tilde{T} - T\|$$

where $\hat{T}$ is the matrix given in Definition 9. As $\tilde{T}$ and T are upper triangular and $\tilde{T}$ has spectrum λ, it follows that

$$\|\tilde{T} - T\|^2 = \sum_{i=1}^{n} |\tilde{T}_{ii} - T_{ii}|^2 + \sum_{i<j} |\tilde{T}_{ij} - T_{ij}|^2 \tag{16}$$

and that the $\tilde{T}_{ii}$'s are some permutation of the list of eigenvalues λ. The result follows by noting that $\|\hat{T}-T\|^2$ equals the quantity in (14) and that this value must be less than or equal to the first summation on the right hand side of the equality in (16). ■

For completeness, we note that, given $A = UTU^*$, $P_{\mathcal{M}}(U,T)$ may not satisfy

$$\|P_{\mathcal{M}}(U,T) - A\| \le \|M - A\| \text{ for all } M \in \mathcal{M}.$$

For example, if

$$U = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix},$$

and

$$\tilde{U} = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} \text{ and } \tilde{T} = \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix},$$

then, if $\lambda = \{0,0\}$, one can readily verify that

$$\|P_{\mathcal{M}}(U,T) - UTU^*\| \nleq \|\tilde{U}\tilde{T}\tilde{U}^* - UTU^*\|.$$

As for the symmetric case, projection onto $\mathcal{N}$ is straightforward.

Theorem 12: Given $A \in \mathbb{C}^{n\times n}$, define $A_+ \in \mathbb{R}^{n\times n}$ by

$$(A_+)_{ij} = \max\{\mathrm{Re}(A_{ij}), 0\} \text{ for all } 1 \le i,j \le n. \quad (17)$$

A_+ is the best approximant in $\mathcal{N}$ to A in the Frobenius norm.

Proof: The projection of $z \in \mathbb{C}$ onto the nonnegative real numbers equals $\max\{\mathrm{Re}(z), 0\}$. The remainder of the proof follows by exactly the same reasoning used in the proof of Theorem 7. ■

Our proposed algorithm for solving the NIEP is the following.

NIEP algorithm:

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Problem Data. List of desired complex eigenvalues $\lambda = \{\lambda_1, \ldots, \lambda_n\}$.
Initialization. Choose a randomly generated nonnegative matrix $Y \in \mathbb{R}^{n\times n}$.
repeat
  1. Calculate a Schur decomposition of $Y$: $Y = UTU^*$.
  2. $X := P_{\mathcal{M}}(U,T)$.
  3. $Y := X_+$.
until $\|X - Y\| < \epsilon$.
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In the above algorithm, $P_{\mathcal{M}}(U,T)$ is given by Definition 9 and X_+ is given by (17).

As for the SNIEP algorithm, at each iteration of the NIEP algorithm, X has the desired spectrum λ and Y is nonnegative. If ϵ is small, say $\epsilon = 10^{-14}$, termination of the loop ensures X equals Y (approximately) and hence that Y solves the NIEP.

Remark 13: If each of the members of λ are real and we seek a symmetric nonnegative matrix with spectrum λ, then the NIEP algorithm reduces to the SNIEP algorithm. More precisely, this is true if the members of λ are real, if the initial condition Y is a symmetric nonnegative matrix, and, for Schur decompositions used in the NIEP algorithm, U is restricted to be real. This can be shown by comparing the steps in each algorithm. (Details are omitted.) □

TABLE I

SNIEP: A COMPARISON OF PERFORMANCE FOR DIFFERENT PROBLEM SIZES n. i DENOTES THE AVERAGE NUMBER OF ITERATIONS AND T DENOTES THE AVERAGE CONVERGENCE TIME IN CPU SECONDS.

n	i	T	% solved
5	19	0.0016	100
10	18	0.0030	100
20	17	0.0075	100
100	12	0.15	100

We close this section by noting that unlike the SNIEP algorithm, for the NIEP algorithm there is no guarantee that $\|X - Y\|$ is non-increasing from one iteration to the next. (In fact, numerical experiments show that occasional (small) increases in this quantity may occur in some problems.)

V. NUMERICAL EXPERIMENTS

This section contains some numerical results for both the SNIEP and NIEP algorithms.

All computational results were obtained using a 3 GHz Pentium 4 machine. The algorithms were coded using Matlab 7.0.

Throughout this section, when we say a matrix is 'randomly generated' we mean each entry of that matrix is randomly drawn from the uniform distribution on the interval $[0,1]$. When dealing with the SNIEP algorithm, all randomly generated matrices are chosen symmetric.

For both algorithms, the initial starting Y is always randomly generated. For both algorithms, the convergence tolerance ϵ is set to 10^{-14}.

A. SNIEP

This subsection starts with some results for randomly generated SNIEPs. To ensure each problem is feasible, each desired spectrum is taken from a randomly generated matrix.

Results for various problem sizes n are given in Table I. For each value of n, 1000 problems were considered. The table contains the average number of iterations required to find a solution, the average time required to find a solution, and the success rate. As can be seen, the algorithm performed extremely well and was able to solve every problem. In all cases, both the average number of iterations and the average solution time was very small.

It is interesting to note that T increases with n, as would be expected, while i decreases. Without going into details, this is to a certain extent not so surprising as $\mathcal{M}$ is a manifold and its dimension relative to the dimension of $\mathcal{S}^n$ can be shown to be an increasing function of n.

TABLE II

SNIEP: A problem with repeated eigenvalues, $\lambda = \{3-t, 1+t, -1, -1, -1, -1\}$. i denotes the average number of iterations and T denotes the average convergence time in CPU seconds. i and T do not include the attempts that had not converged after 5000 iterations.

t	i	T	% solved
0.25	480	0.061	100
0.5	470	0.061	97
0.75	340	0.050	65
0.95	310	0.046	59

Convergence is linear.

Randomly generated problems have properties that are not shared by all SNIEPs. For example, randomly generated problems have distinct eigenvalues. We next consider a problem with repeated eigenvalues, namely $\lambda = \{3-t, 1+t, -1, -1, -1, -1\}$ for $0 < t < 1$. The $t = 1/2$ version of the problem is also considered in [4] were a numerical solution is sought via the gradient flow approach of that paper. An analytic solution to this problem is given in [16].

Notice that for any value of t the desired eigenvalues sum to zero and hence there exist arbitrarily small perturbations of the spectrum which lead to a non-feasible SNIEP. In particular this problem cannot have any solutions in the interior of $\mathcal{N}$. We have tried the SNIEP algorithm on a number of other problems with repeated eigenvalues with excellent results. This is the hardest problem we have encountered so far.

The results of applying the algorithm to the problem for various values of t are given in Table II. They are based on running the algorithm a 100 times for each value of t.

Firstly, the results indicate that the SNIEP algorithm is not always successful in finding a solution. However they also show that the algorithm can still be quite successful if a number of initial conditions are tried. It is interesting to note that the algorithm becomes more sensitive to the choice of the initial condition the larger t is. It is unclear why this is the case.

Aside: Regarding initial conditions, as noted before, both the SNIEP and NIEP algorithms use a nonnegative initial starting point. This is important and in fact the performance of neither algorithm is as good if non-nonnegative initial condition are used.

B. NIEP

This subsection presents some results for randomly generated NIEPs. Again, to ensure each problem is feasible, each desired spectrum is taken from a randomly generated matrix. Results are given in Table III.

TABLE III

NIEP: A comparison of performance for different problem sizes n. i denotes the average number of iterations and T denotes the average convergence time in CPU seconds. i and T do not include the problems that had not converged after 5000 iterations.

n	i	T	% solved
5	26	0.011	99.7
10	44	0.045	99.8
20	48	0.12	99.8
100	200	12	96.6

As can be seen, the results are again very good with almost all problems being solved.

The results indicate that NIEPs are harder to solve than SNIEPs. Also, the number of iterations, time, and time per iteration is greater. Part of the reason for an increase in time per iteration will be the extra computation required to calculate the least squares matching component of each $P_{\mathcal{M}}(U,T)$ calculation; see (14). (For SNIEPs the corresponding step is easy: the eigenvalues are real and just need to be sorted in decreasing order.)

For the NIEPs, both i and T increased with n.

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