# Internal Stability Issues in Output Stabilization of a Class of Nonlinear Control Systems

Robert J Orsi* Robert E Mahony†

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**Abstract**

Fundamental in the design of practical stabilizing control laws is the internal stability of the closed loop system. In this paper we consider the issue of output stabilization for a continuous time affine system using a static state output linearising control law. The system considered exhibits no drift for zero output, each of its outputs has relative degree one and its input-output decoupling matrix is full rank. The control considered exponentially stabilizes the system output provided the closed loop system remains internally stable. In general it is too much to hope for globally well defined closed loop dynamics, however, in this paper we show that there exists a neighbourhood of the zero output level set for which the closed loop system is internally stable.

## 1 Introduction

Fundamental in the design of practical stabilizing control laws is whether the state remains bounded and well defined for all time, that is, whether the closed loop system remains *internally stable*.

In general it is too much to hope that a nonlinear system will have globally well defined closed loop dynamics (Sussmann 1990, Sussmann & Kokotovic 1991). In the case where the zero dynamics are asymptotically stable (at a point $x^*$) it can be shown (Byrnes & Isidori 1991, Lemma 4.2) that there exists a control law and a local neighbourhood around $x^*$ for which the closed loop system is asymptotically stable to $x^*$. In general this neighbourhood does not contain the entire zero output level set and as a consequence this result has limited applicability to output stabilization.

*Department of Systems Engineering, Research School of Information Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia. E-mail: Robert.Orsi@anu.edu.au IEEE membership number: 40122318

†Cooperative Research Centre for Robust and Adaptive Systems, Australian National University, Canberra ACT 0200, Australia. E-mail: Robert.Mahony@anu.edu.au

By assuming that there exists some form of energy structure related to zeroing the output (Lin, Sontag & Wang 1994) one can hope to use Lyapunov theory to obtain results valid in a neighbourhood of the zero output level set. An energy based approach is also used in (Byrnes, Isidori & Willems 1991) for full state stabilization and is closely related to recent work on input to state stability (Sontag 1989). If the system is derived as a Hamiltonian system then a natural energy structure exists which can be exploited for output or state stabilization (Nijmeijer & van der Schaft 1990, Ch. 10).

Alternately it is possible to get similar results by assuming certain structure of the system equations. For example for systems with no drift and linearly appearing input, there exists a control and a neighbourhood of the zero output level set for which the closed loop system is internally stable (Mahony, Mareels, Bastin & Campion 1995).

In this paper we take a similar approach to (Mahony et al. 1995) and extend their result to include a class of system with non-trivial drift term. We consider the issue of output stabilization for a continuous time affine system using a static state output linearising control law. The system considered exhibits no drift for zero output, each of its outputs has relative degree one and its input-output decoupling matrix is full rank. We show that the control considered exponentially stabilizes the system output provided the closed loop system remains internally stable. We then show that there exists a neighbourhood of the zero output level set for which the closed loop system is internally stable.

The paper consists of this introductory section, a problem formulation section and a main result section. In the problem formulation section we introduce the candidate control law. The main result section contains the main result of the paper.

## 2 Problem Formulation

Consider a system of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u, \qquad x(0) = x_0 \\ y(t) &= h(x(t)) \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are smooth functions. Let $D$ denote the derivative operator[1] and assume the system satisfies the following properties:

[1]For a differentiable map $f : \mathbb{R}^n \to \mathbb{R}^m$,

$$Df(x) := \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(x) & \cdots & \frac{\partial f^1}{\partial x^n}(x) \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1}(x) & \cdots & \frac{\partial f^m}{\partial x^n}(x) \end{pmatrix}.$$

System Properties

(P1) The system is input-output square[2].

(P2) $Dh(x)g(x)$ is full rank for all $x \in \mathbb{R}^n$.

**Remark 2.1** Property (P2) is equivalent to the charactistic number of each output being zero and the input-output decoupling matrix being full rank for all $x \in \mathbb{R}^n$ (Nijmeijer & van der Schaft 1990, Section 8.1) or alternatively the system having vector relative degree $\{1, \ldots, 1\}$ for all $x \in \mathbb{R}^n$ (Isidori 1995, pg. 220). □

Consider the evolution of the output $y(t)$ of such a system. If $x(t)$ satisfies (2.1), taking the time derivative of $y(t)$ yields

$$\begin{aligned} \dot{y}(t) &= Dh(x(t))\dot{x}(t) \\ &= Dh(x(t))f(x(t)) + Dh(x(t))g(x(t))u. \end{aligned}$$

For a system of the type described, the static state feedback control law

$$u(x) = -\big(Dh(x)g(x)\big)^{-1}\big(h(x) + Dh(x)f(x)\big) \tag{2.2}$$

is well defined for all $x \in \mathbb{R}^n$. Substituting this control into the dynamics for $y(t)$ gives

$$\dot{y}(t) = -y(t),$$

provided the closed loop system remains internally stable. If the system does remain internally stable

$$y(t) = y(0)e^{-t}$$

and the output converges exponentially to zero.

We term the static control law (2.2) the *output linearising control* for the system. Due to its simplicity and the strong convergence of $y(t)$ it induces, this control strategy is a desirable candidate for output stabilization of systems of the type described. However, without a more explicit knowledge of (2.1), internal stability of the closed loop is in doubt.

# 3 Main Result

In this section we show that that there exists a neighbourhood of the set $\{x \in \mathbb{R}^n \mid h(x) = 0\}$ for which the closed loop system, consisting of the system (2.1) and control (2.2), is internally stable.

[2] A system is input-output square if it possesses the same number of inputs as outputs, that is, if $p = m$.

Consider a system of the form (2.1) which satisfies properties (P1) and (P2). Given an input $u$ and an initial condition $x_0$, let $x(t; x_0)$ denote the state of the system at time $t$ for the given initial condition and input. Define

$$G(x) := g(x)\big(Dh(x)g(x)\big)^{-1} \qquad \text{and} \tag{3.1}$$

$$F(x) := \big(I - G(x)Dh(x)\big)f(x). \tag{3.2}$$

The closed loop response of the system (2.1) to the control input (2.2) is given by

$$x(t; x_0) = x_0 + \int_0^t \Big(F\big(x(\tau; x_0)\big) - G\big(x(\tau; x_0)\big)h\big(x(\tau; x_0)\big)\Big)\, d\tau. \tag{3.3}$$

In the sequel we only consider systems satisfying the following additional property:

System Property

(P3) The drift term $f(x) = 0$ whenever the output $h(x) = 0$.

**Remark 3.1** Systems for which the output is related to the energy of the system tend to satisfy (P3). For example, if the system output consists of kinetic plus potential energy, with potential energy normalised such that it is greater than or equal to zero, zero output corresponds to zero energy in the system. As long as the applied control is zero then the system is naturally at rest for zero output and hence $f(x) = 0$ when $h(x) = 0$. □

**Lemma 3.2** *Consider a system of the form (2.1) which satisfies properties (P1), (P2) and (P3). Let the input $u$ be given by (2.2) and let $F(x)$ be defined as in (3.2). Then given $x^* \in \{x \in \mathbb{R}^n \mid h(x) = 0\}$ and $r > 0$, there exist constants $M_{x^*,r} > 0$ and $r'$, $0 < r' \leq r$, such that*

$$|F(x)| \leq M_{x^*,r}|h(x)| \qquad \forall\, x \in B_{x^*}(r').$$

**Proof.** As $Dh(x)g(x)$ is full rank, $Dh(x)$ must also be full rank. This implies that there exists a function $z : \mathbb{R}^n \to \mathbb{R}^{n-m}$ and a constant $r'$, $0 < r' \leq r$, such that the function

$$\phi(x) := \begin{pmatrix} y \\ z \end{pmatrix}$$

is a diffeomorphism on the ball $B_{x^*}(r')$ (Nijmeijer & van der Schaft 1990, Proposition 2.18, pg. 35).

In the ball $B_{x^*}(r')$ rewrite $f$ and $F$ as functions of the new coordinates $y$ and $z$,

$$\tilde{f}(y, z) := f\left(\phi^{-1}\begin{pmatrix} y \\ z \end{pmatrix}\right) \qquad \text{and}$$

$$\tilde{F}(y, z) := F\left(\phi^{-1}\begin{pmatrix} y \\ z \end{pmatrix}\right).$$

For a given $z$, if the set $\{y \mid \phi^{-1}((y^T\ z^T)^T) \in B_{x^*}(r')\}$ is non-empty, let

$$M_z := \sup_{\{y \ \mid \ \phi^{-1}((y^T\ z^T)^T) \in B_{x^*}(r')\}} \frac{\left|\tilde{F}(y,z) - \tilde{F}(0,z)\right|}{\left|(y^T\ z^T)^T - (0\ z^T)^T\right|}.$$

Otherwise, let $M_z := 0$. Define

$$M_{x^*,r} := \sup_z M_z.$$

Let $x \in B_{x^*}(r')$ and $(y^T\ z^T)^T = \phi^{-1}(x)$. There exist two possibilities, $y = 0$ and $y \neq 0$. If $y = 0$ then by assumption $f(x) = 0$. This in turn implies that $F(x) = 0$ and the result follows. If $y \neq 0$ then noting that $\tilde{F}(0, z) = 0$,

$$\begin{aligned} |F(x)| &= \frac{\left|\tilde{F}(y,z) - \tilde{F}(0,z)\right|}{\left|(y^T\ z^T)^T - (0\ z^T)^T\right|}\left|(y^T\ z^T)^T - (0\ z^T)^T\right| \\ &= \frac{\left|\tilde{F}(y,z) - \tilde{F}(0,z)\right|}{\left|(y^T\ z^T)^T - (0\ z^T)^T\right|}|y| \\ &\leq M_{x^*,r}|y|. \end{aligned}$$

■

**Theorem 3.3** *Consider a system of the form (2.1) which satisfies properties (P1), (P2) and (P3). Then, for the closed loop system using the output linearizing control law (2.2), there exists an open neighbourhood $\Omega \subset \mathbb{R}^n$ of the set $\{x \in \mathbb{R}^n \mid h(x) = 0\}$ such that for any initial condition $x_0 \in \Omega$ the solution of the system $x(t; x_0)$ is well defined and bounded for all time and the output $h(x(t)) \to 0$ as $t \to \infty$.*

**Proof**. Let $F(x)$ be defined as in (3.2) and let $x^* \in \{x \in \mathbb{R}^n \mid h(x) = 0\}$ and $r > 0$. By Lemma 3.2 there exist constants $M_{x^*,r} > 0$ and $r'$, $0 < r' \leq r$, such that

$$|F(x)| \leq M_{x^*,r}|h(x)| \qquad \forall\ x \in B_{x^*}(r').$$

Let

$$\Omega_{x^*,r} := \left\{x \in B_{x^*}(r'/4) \ \middle|\ \left(M_{x^*,r} + |G(x)|\right)|h(x)|e^{\lambda|h(x)|} < \frac{r'}{2}\right\}$$

where $G(x)$ is given by (3.1) and $\lambda = \lambda(x^*, r')$ and is given by

$$\lambda(x^*, r') := \sup_{x,y \in B_{x^*}(r')} \frac{|G(x) - G(y)|}{|x - y|}.$$

Further, define the set

$$\Omega := \cup_{x^* \in \{x \ \mid \ h(x)=0\}} \cup_{r>0} \Omega_{x^*,r}.$$

Note that each set $\Omega_{x^*,r}$ is non-empty (as $x^* \in \Omega_{x^*,r}$) and that each of these sets is an open subset of $\mathbb{R}^n$. Hence $\Omega$ is open and $\{x \in \mathbb{R}^n \mid h(x) = 0\} \subset \Omega$.

We now proceed to show that $x_0 \in \Omega$ is sufficient for the state to remain well defined for all time.

Let $x_0 \in \Omega$. Then $x_0 \in \Omega_{x^*,r}$ for some $x^*$ and $r$. For this particular $x^*$ and $r$, let $M_{x^*,r}$ and $r'$ be the corresponding constants defined in Lemma 3.2. As $f(x)$, $g(x)$ and $h(x)$ are smooth and $Dh(x)g(x)$ is full rank there exists a unique local solution to the system, $x(t;x_0)$, that is well defined on some maximal interval $[0,t^*)$.

The proof proceeds by contradiction. Assume that there exists a time for which $|x(t;x_0) - x_0)| \geq 3r'/4$. Define $t_1 < t^*$ as

$$t_1 = \inf_{t>0}\{t \mid |x(t;x_0) - x_0| \geq 3r'/4\}.$$

Note that $t_1$ is defined in such a way that for the given $x_0 \in \Omega_{x^*,r}$, $x(t;x_0) \in B_{x^*}(r')$ for $t \in [0,t_1)$.

The closed loop system response (3.3) may be rewritten as

$$x(t;x_0) - x_0 = \int_0^t \Big( F\big(x(\tau;x_0)\big) - G\big(x(\tau;x_0)\big) h\big(x(\tau;x_0)\big) \Big)\ d\tau.$$

Computing the norm of this expression and approximating the integral for $t \in [0,t_1)$ one obtains

$$\begin{aligned}
|x(t;x_0) - x_0| &\leq \int_0^{t_1} \Big( \big|F\big(x(\tau;x_0)\big)\big| + \big|G\big(x(\tau;x_0)\big)\big|\,\big|h\big(x(\tau;x_0)\big)\big| \Big)\ d\tau \\
&\leq \int_0^{t_1} (M_{x^*,r} + |G(x_0)| + \lambda|x(\tau;x_0) - x_0|)|h(x_0)|e^{-\tau}\ d\tau \\
&= \Big(M_{x^*,r} + |G(x_0)|\Big)|h(x_0)|\left(1 - e^{-t_1}\right) + \int_0^{t_1} \lambda|x(\tau;x_0) - x_0||h(x_0)|e^{-\tau}\ d\tau \\
&\leq \Big(M_{x^*,r} + |G(x_0)|\Big)|h(x_0)| + \int_0^{t_1} \lambda|x(\tau;x_0) - x_0||h(x_0)|e^{-\tau}\ d\tau.
\end{aligned}$$

The above inequality is in a form in which the Bellman-Gronwall Lemma (Sontag 1990, Lemma C.3.1, pg. 346) may be applied. Application of the lemma yields that

$$|x(t;x_0) - x_0| \leq \big(M_{x^*,r} + |G(x_0)|\big)|h(x_0)|e^{\lambda|h(x_0)|\left(1-e^{-t_1}\right)}.$$

Hence,

$$|x(t;x_0) - x_0| \leq \big(M_{x^*,r} + |G(x_0)|\big)|h(x_0)|e^{\lambda|h(x_0)|}$$

and by construction of $\Omega_{x^*,r}$ one has that

$$|x(t;x_0) - x_0| < r'/2.$$

This contradicts our starting assumption, namely the existence of a time at which $|x(t;x_0) - x_0)| \geq 3r'/4$. Hence, the state must remain in a ball about $x^*$ of radius $3r'/4$ and is well defined for all time. The remainder of the theorm follows directly from the nature of the output function. ■

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