CLASSIFICATION WITH MIXTURES OF CURVED MAHALANOBIS METRICS
Supplemental information of #2228

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Equivalent definitions of the hyperbolic Cayley-Klein distances of constant curvature $\kappa < 0$:

$$D_S(p,q) = -\frac{1}{2}\kappa \log \frac{S_{p,q} + \sqrt{S_{p,q}^2 - S_{p,p}S_{q,q}}}{S_{p,q} - \sqrt{S_{p,q}^2 - S_{p,p}S_{q,q}}} \tag{1}$$

$$D_S(p,q) = -\kappa \times \mathrm{arctanh}\sqrt{1 - \frac{S_{p,p}S_{q,q}}{S_{p,q}^2}} \tag{2}$$

$$D_S(p,q) = -\kappa \times \mathrm{arccosh}\frac{-S_{p,q}}{\sqrt{S_{p,p}S_{q,q}}}, \tag{3}$$

with $\mathrm{arccosh}(x) = \log(x + \sqrt{x^2 - 1})$ for $x \geq 1$, $\mathrm{arctanh}(x) = \frac{1}{2}\log\frac{1+x}{1-x}$, and:

$$\langle p, q \rangle_S = \left[\begin{array}{cc} p^\top & 1 \end{array} \right]^\top S \left[\begin{array}{c} q \\ 1 \end{array} \right] = \tilde{p} S \tilde{q} = S_{p,q} = S_{q,p}.$$

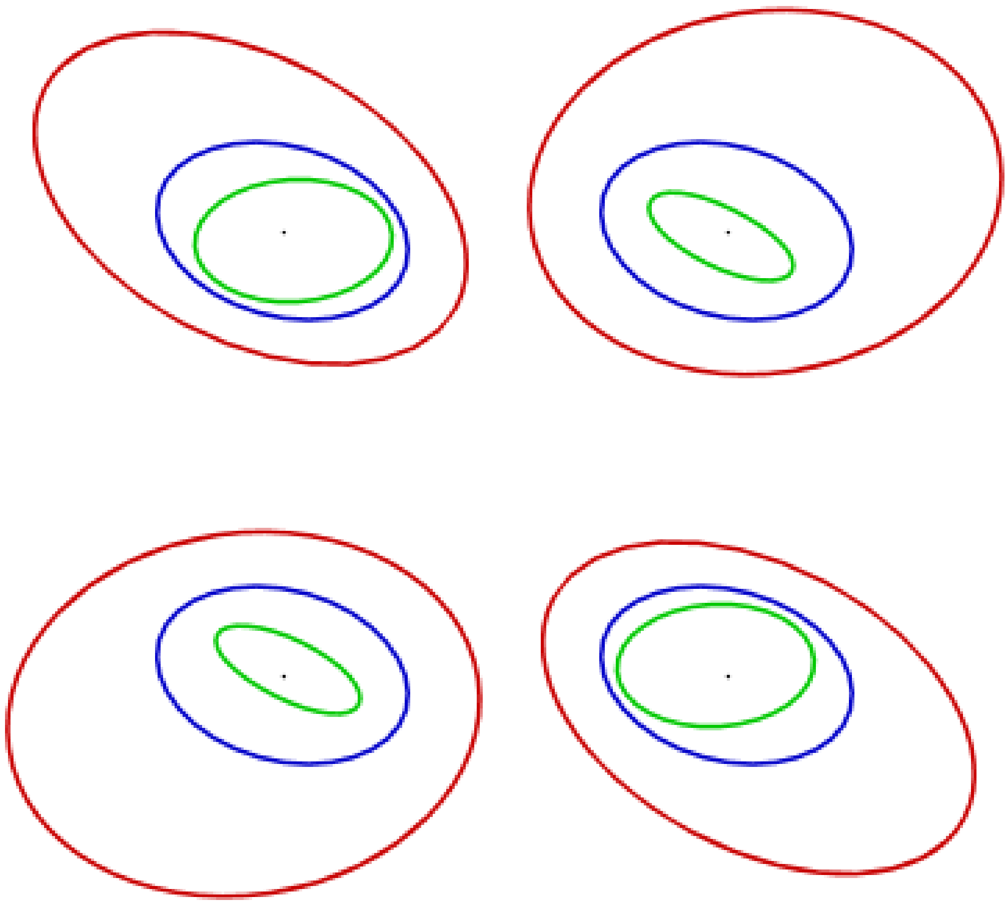


Figure 1: Riemannian metric tensors induced by the flat Euclidean Mahalanobis distance (blue, constant), the negatively-curved hyperbolic Mahalanobis distance (green), and the positively-curved elliptical Mahalanobis distance (red).

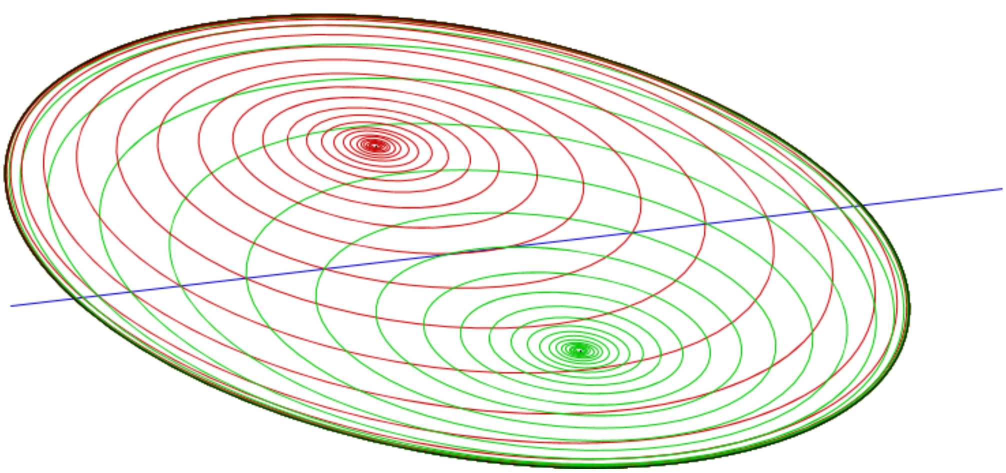


Figure 2: Bisector for the negatively-curved Mahalanobis distance. The hyperbolic spheres are converted to equivalent flat Mahalanobis spheres for rasterization. The spheres become tangent to the fundamental conic as the radius tend to infinity.

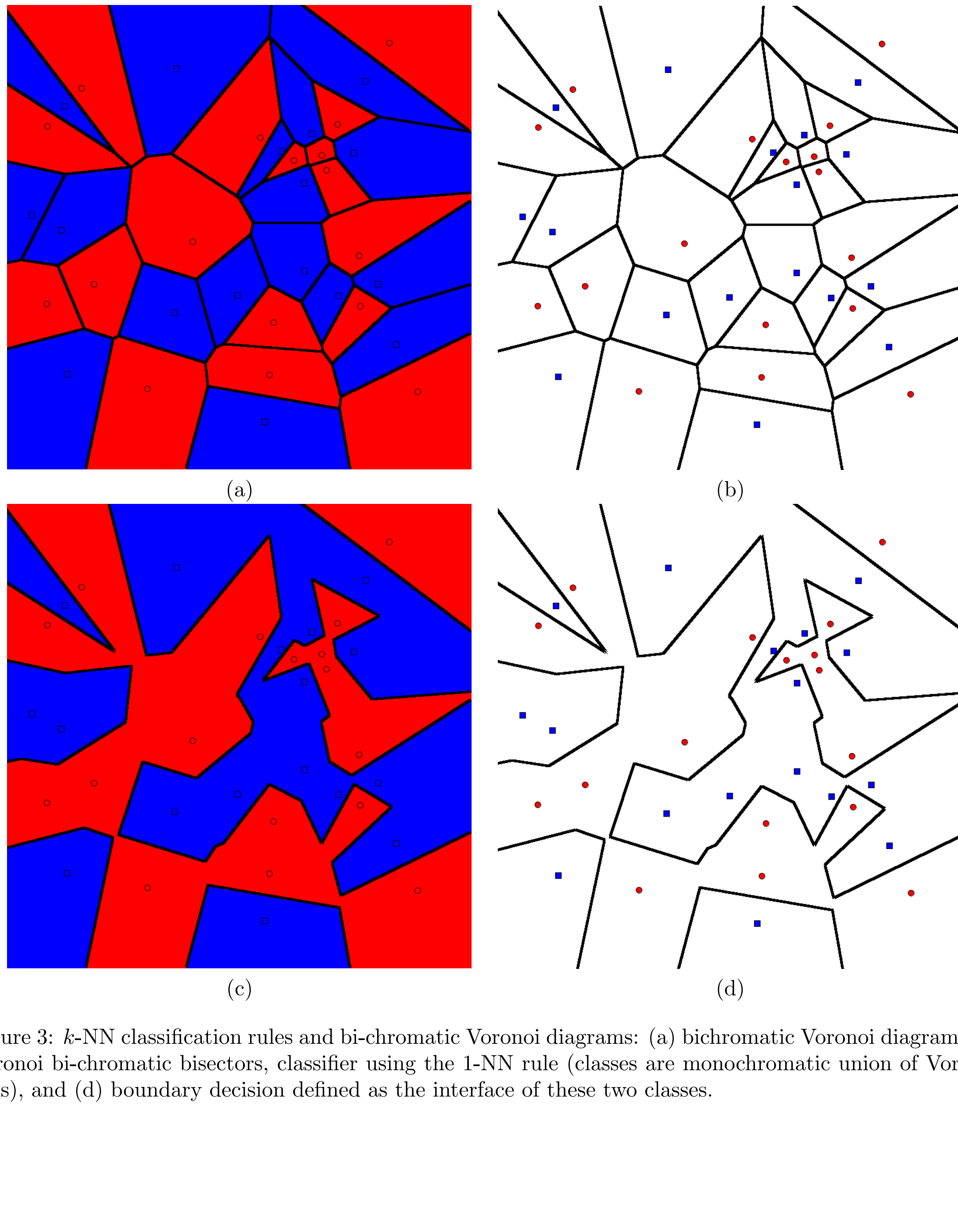
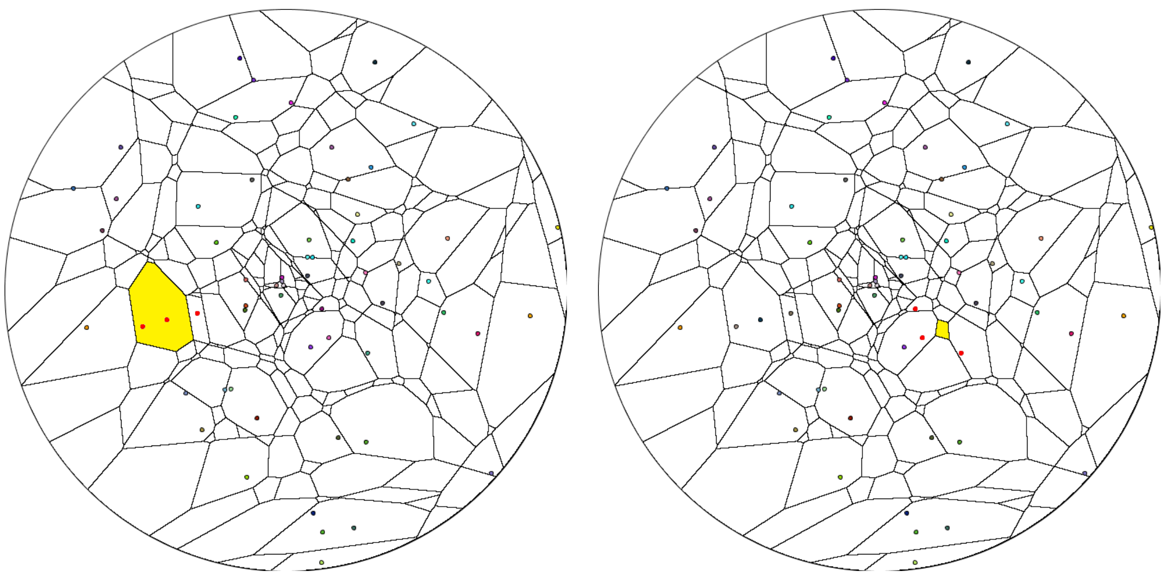


Figure 3: k-NN classification rules and bi-chromatic Voronoi diagrams: (a) bichromatic Voronoi diagram, (b) Voronoi bi-chromatic bisectors, classifier using the 1-NN rule (classes are monochromatic union of Voronoi cells), and (d) boundary decision defined as the interface of these two classes.


(a)

(b)

Figure 4: 3-order hyperbolic Voronoi diagram: The three NNs of points in the yellow cell are depicted by three large red dots. Therefore the 3-NN classifiers have piecewise-linear boundaries.