Tsallis Regularized Optimal Transport and Ecological Inference
Boris Muzellec, Richard Nock, Giorgio Patrini and Frank Nielsen
Ecole polytechnique and Data61

Overview
A new framework for optimal transport which
- Unifies Monge-Kantorovich and Sinkhorn-Cuturi
- Interpolates a broad class of distance distortions
- Has metric properties
- Can be efficiently solved for algorithmically
- Can be successfully applied to ecological inference

Optimal Transport
A powerful framework for computing distances between probability distributions. Two approaches:
- Monge-Kantorovich (unregularized)
  \[ d_{MK}(r, c) = \frac{1}{\lambda} \sum \phi(M_{ij}) \] (OT)
- Sinkhorn-Cuturi (entropic regularization)
  \[ d_{SC}(r, c) = \frac{1}{\lambda} \sum \psi(M_{ij}) \] (SC)
- Fast convergence with Sinkhorn’s algorithm!

Tsallis Entropy
- Generalization of Shannon’s entropy:
  \[ H_q(P) := \sum \phi(M_{ij}) \]
- Generalized logarithm \( \ln_q(z) := \frac{1}{q-1} (z^{1-q} - 1) \rightarrow \ln(z) \)

TROT
Add a Tsallis regularizer to OT:
\[ d_{TROT}(r, c) = \min_{P \in \Delta(c,r)} (P, M) - \frac{1}{\lambda} H_q(P) \] (TROT)

TROT Properties
- Has a unique solution
- Interpolates Monge-Kantorovich \((q \rightarrow 0)\) and Sinkhorn-Cuturi \((q \rightarrow 1)\)
- Unrecoverability \((q, \lambda) \neq (q', \lambda') \Rightarrow \text{TROT}(q, \lambda) \neq \text{TROT}(q', \lambda')\)
- Efficient solving by approximate matrix balancing
- Metric properties \((q \geq 1)\) e.g. triangle inequality

Economic Inference
- Def: Reconstruction of joint distributions from marginal distributions
  - e.g., find the vote share within social groups
- Ill-posed problem: zero or infinite number of solutions
- Usual approach: use an information theoretic criterion
- TROT: information theory + encode a priori information in the cost matrix \(M\)

Ecological Inference with TROT
- Example: Afro-American
- Additional data (census, …)
- (unknown)
- (known)
- (unknown)
- Fast convergence with Sinkhorn’s algorithm!

Experiments
- On 2012 Florida US vote data. Two approaches:
  - Demographic information only: \( m_{\text{DV}} = \sqrt{2 - 2 \exp(-q |x - y|)} \)
  - Survey statistics: \( m_{\text{SV}} = 1 - p_{ij}^{\text{SV}} \)

TROT vs Sinkhorn inferred probabilities and ground truth for all Florida counties. Parameters noted with \( \dagger \) are not cross-validated but defined by the algorithm.

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References
Tsallis regularized optimal transport and ecological inference.
Code repository: https://github.com/BorisMuzellec/TROT