Analytical Study of Connectivity in Wireless Ad hoc Networks with Random Beamforming

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Abstract— **Random beamforming is a technique in which each node in a wireless ad hoc network directs its main beam in a randomly chosen direction. This paper presents an analytical method for investigating the effect of random beamforming on the connectivity of wireless ad hoc networks. We derive analytically an expression for an effective beamforming gain, which we use to characterize the impact of random beamforming on the number of direct connections for an arbitrary node, i.e. the local connectivity. Our results show that for a path-loss propagation model, random beamforming improves the local connectivity for a path-loss exponent $\alpha < 3$, while it degrades the local connectivity for larger values of α. The analytical method is validated by comparison with simulation results.**

I. INTRODUCTION

Beamforming in ad hoc networks has been widely investigated in recent years. Beamforming has been extensively studied in cellular systems [1], [2] but its application in wireless ad hoc networks poses unique design challenges due to the inherent lack of wired infrastructure. Most of the work in this area is concerned with MAC layer protocols for use with beamforming antennas [3], [4] and routing using beamforming antennas [5]. Work has also been done with regard to neighbour discovery via beamforming [6] and using beamforming to improve network capacity [7].

Recently there has been a growing interest in the connectivity of wireless ad hoc networks with beamforming antennas. Different beamforming techniques have been proposed to improve the connectivity of ad hoc networks [8], [9]. The simplest beamforming technique is random beamforming, which allows each node in the network to direct its main beam in a direction from a uniform distribution on $[0, 2\pi)$. It does not require knowledge about location of neighbouring nodes and is appealing in terms of practical implementation. The connectivity of ad hoc networks for traditional omnidirectional antenna transmission in a path-loss and shadowing environment has been studied using a semi-analytical procedure in [10] and analytically using the concept of effective communication range in [11]. For the case of beamforming antennas, investigations are largely limited to simulation based studies. These have shown that while randomized beamforming can lead to an improvement in the overall connectivity, it also increases the number of isolated nodes in ad hoc networks [8], [9]. In [12], simplifying assumptions about gain patterns of beamforming antennas are made to analytically derive an expression for node distribution, which is then used to analyse the connectivity. It is shown that randomized beamforming can improve or degrade the overall connectivity of wireless ad hoc networks. However no insight is provided for its effect on the number of isolated nodes.

In this paper, we extend the analytical method in [11] for the case of random beamforming. We derive analytically an expression for effective beamforming gain, which we use to characterize the effect of random beamforming on local connectivity of an ad hoc network. We use the probability of node isolation as a metric for local connectivity and show that for a path-loss channel model, random beamforming can increase or decrease this probability depending on whether the path-loss exponent α is < 3 or > 3, respectively.

The rest of this paper is organized as follows. The antenna array and signal model are detailed in Section II. The proposed analytical method is discussed in Section III and is used to gain insight into the effect of beamforming on the local connectivity of an ad hoc network. Simulation results, which validate our analytical method, are given in Section IV. Finally, conclusions are drawn in Section V.

II. SYSTEM MODEL

A. Node Distribution Model

We consider that the nodes in the network are distributed in a two dimensional space. We use a homogeneous Poisson point process to model the spatial distribution of the nodes. The probability mass function of number of nodes X in an area A, is given by [13]

$$P(X = x) = \frac{\mu^x}{x!} e^{-\mu} \tag{1}$$

where $E[X] = \mu = \rho A$, ρ is the node density and $E[\cdot]$ denotes expectation. A homogeneous Poisson process can be regarded as the limiting form of a uniform distribution of X nodes on an area A, as x and A approach ∞ but their ratio $\rho = x/A$ remains constant.

B. Antenna Model

We consider a Uniform Circular Array (UCA) of N identical omnidirectional antenna elements, spaced in a circle of radius a in the xy-plane and located at the origin of a spherical coordinate system. Without loss of generality, we assume plane wave propagation. Beamforming is achieved by phase shifting each antenna element in the array such that its main beam
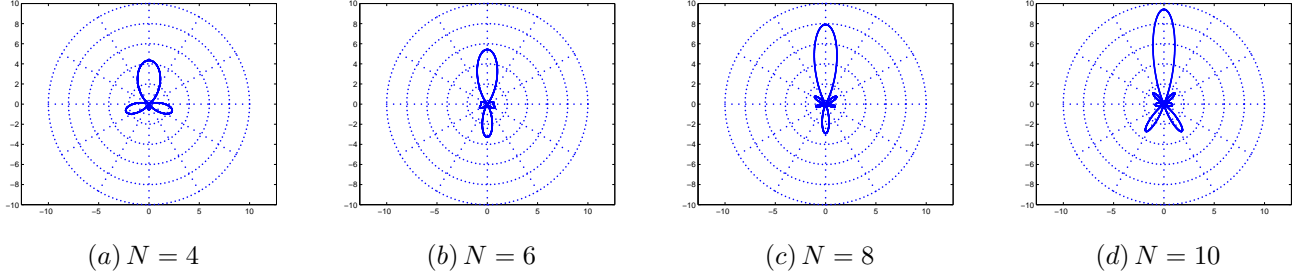
Fig. 1. Gain patterns of UCA for different number of antenna elements, where the main beam is 90 degrees, from equation (2).

points towards the desired direction. The gain of the UCA antenna is [9]

$$G = \frac{|E(\theta,\phi)|^2}{\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}|E(\theta,\phi)|^2 \sin(\theta)\,d\theta\,d\phi} \quad (2)$$

where $\phi \in [0, 2\pi)$ is the angle from the x-axis, $\theta \in [0, \pi)$ is the angle from the z-axis and $E(\theta, \phi)$ is the electric field given by

$$E(\theta,\phi) = \sum_{n=1}^{N} E_0 \exp[jka\sin(\theta)\cos(\phi - \phi_n) + j\alpha_n] \quad (3)$$

where E_0 is the electric field pattern of the omnidirectional antenna, $k = 2\pi/\lambda$, $\phi_n = 2\pi n/N$, and α_n is the phase shift of the nth element. For the conventional cophasal excitation [14]

$$\alpha_n = -ka\sin(\Theta_0)\cos(\Phi_0 - \phi_n) \quad (4)$$

where (Θ_0, Φ_0) are the desired angles of the main beam. For two dimensional space, i.e. the xy-plane, $\Theta_0 = \frac{\pi}{2}$. Substituting (4) and (3) into (2), we can calculate the antenna gain for a UCA for any azimuthal angle ϕ and main beam direction Φ_0.

The gain pattern of a UCA with different numbers of antenna elements is shown in Fig. 1. The main beam direction in all plots is set to $\Phi_0 = 90°$. We can see that the antenna gain in the main beam direction increases linearly with the number of antenna elements N. The maximum gain always stays around N. However the average gain in other directions does not increase with increasing N and the shape of the side lobes changes significantly.

C. Signal Model

We consider the large scale path loss model to determine whether or not there is a connection between two given nodes. Suppose that a node transmits a signal with power P_T. The received signal power at a distance d is given by [15]

$$P_R = C\frac{1}{d^\alpha}G_T G_R P_T \quad (5)$$

where $C = \left(\frac{\lambda}{4\pi}\right)^2$, G_T and G_R are the antenna gains of transmitting node and receiving node, given by (2).

Suppressing the constant C, the overall power attenuation can be expressed as

$$\beta(d) = \frac{P_T}{P_R} = \frac{d^\alpha}{G_T G_R} \quad (6)$$

In this work, we neglect the impact of interference, i.e. we assume effective MAC layer protocols. In this case, we can define a threshold power attenuation, β_{th}, above which there is no direct connection between the transmitting node and the receiving node. Therefore the probability of having no direct connection, with node separation d, is given by

$$
\begin{aligned}
P(\beta(d) \geq \beta_{\text{th}}) &= P\left(\frac{d^\alpha}{G_T G_R} \geq \beta_{th}\right) \\
&= P\left(d^\alpha \geq (\beta_{th})(G_T G_R)\right) \\
&= P((\beta_{\text{th}})^{\frac{1}{\alpha}}(G_T G_R)^{\frac{1}{\alpha}} \leq d) \\
&= P(R \leq d) \quad (7)
\end{aligned}
$$

where the random variable $R = (\beta_{\text{th}})^{\frac{1}{\alpha}}(G_T G_R)^{\frac{1}{\alpha}}$ is referred to as the effective communication range [11].

For the case of the deterministic path loss model considered in this work, the effective communication range indicates the maximum separation that a pair of nodes can have with the ability to communicate with each other. The effective coverage area of a node can thus be considered as a disk with radius $E[R^2]$, centered at the node [11].

D. Local Connectivity

The local connectivity of the network can be measured by the node degree D, which is the number of direct links that a node establishes. It has been shown that the node degree follows a Poisson distribution with parameter $\mu = \rho\pi E[R^2]$ [11]. Hence using the property of the Poisson distribution, the expected value of the node degree is given by [13]

$$E[D] = \rho\pi E[R^2]. \quad (8)$$

An important metric for measurement of local connectivity is probability of isolation. It is defined as the probability that a randomly chosen node does not have a connection to any other node. For a homogenous Poisson process, the probability of isolation is given by

$$
\begin{aligned}
P_I &= \exp\{-E[D]\} \\
&= \exp\{-\rho\pi E[R^2]\} \\
&= \exp\left\{-\rho\pi(\beta_{\text{th}})^{\frac{2}{\alpha}}E\left[(G_T G_R)^{\frac{2}{\alpha}}\right]\right\}. \quad (9)
\end{aligned}
$$

From (9), we see that the effect of beamforming can be expressed as a multiplicative factor, $E\left[(G_T G_R)^{\frac{2}{\alpha}}\right]$. We call it the *effective beamforming gain*.

III. Theoretical Analysis

In this section, we discuss our proposed method for calculating the effective beamforming gain for random beamforming. This then allows us to investigate how the use of random beamforming improves the local connectivity of a randomly chosen node in the network.

In the random beamforming scheme each node in a wireless ad hoc network directs its main beam towards a randomly chosen direction. Figure 2 shows a pair of transmitting (TX) and receiving (RX) nodes in a random beamforming scenario. The arrows indicate the main beam directions and all angles are measured with respect to the x-axis. The model parameters shown in the figure are defined as follows:-

d = distance between the TX and RX nodes;

ϕ = relative angle of RX from TX, with respect to the x-axis;

Φ_T = main beam direction of TX node;

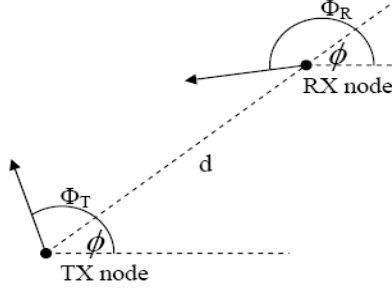Φ_R = main beam direction of RX node;

Fig. 2. Relative positions of a transmitting and receiving node pair in random beamforming scenario.

In this scenario, the antenna gains depend on the main beam directions and the relative location of the transmitting and receiving node pair. Since each node randomly chooses a main beam angle from a uniform distribution, the probability density function (PDF) of the main beam angle has a uniform distribution. For uniformly distributed nodes, the direction of any other node from a chosen node has a uniform distribution as well. Therefore the angles ϕ, Φ_T and Φ_R have the same (but mutually independent) PDF, being

$$f(\gamma) = \frac{1}{2\pi}; \qquad \gamma \in [0, 2\pi) \qquad (10)$$

Using the above assumptions, the effective beamforming gain can be expressed as

$$E\left[(G_T G_R)^{\frac{2}{\alpha}}\right] = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi}$$
$$(G(\phi, \Phi_T)G(\pi+\phi, \Phi_R))^{\frac{2}{\alpha}} d\Phi_R d\Phi_T d\phi$$
$$(11)$$

where G_T and G_R are the antenna gains of the transmitting node and the receiving node, given by (2). From (2), (3)

and (11), we can see that the effective beamforming gain for random beamforming only depends on the number of antenna elements N and the path-loss exponent α.

There is no closed-form solution for (11), so we evaluate the effective beamforming gain numerically using Matlab. Table I summarizes the values of the effective beamforming gain from (11) for different N and α.

TABLE I
EFFECTIVE BEAMFORMING GAIN FOR RANDOM BEAMFORMING
(COMPUTED FROM (11))

Path-loss α	No. of antennas N	Beamforming Gain $E\left[(G_T G_R)^{\frac{2}{\alpha}}\right]$
2	4	1.48
2	6	1.51
2	8	1.60
2	10	1.84
2.5	4	1.12
2.5	6	1.12
2.5	8	1.14
2.5	10	1.24
3	4	0.95
3	6	0.96
3	8	0.96
3	10	1.00
3.5	4	0.87
3.5	6	0.89
3.5	8	0.87
3.5	10	0.90
4	4	0.82
4	6	0.85
4	8	0.82
4	10	0.85

From (9), it can be seen that the use of beamforming will improve the local connectivity (i.e. reduce probability of isolation) if the effective beamforming gain exceeds 1. Table I shows that the effective beamforming gain decreases as α increases. For example for $\alpha = 2$ (i.e. free space propagation environment), the gain for random beamforming with 4 antenna elements is 1.48 but decreases to a value less than 1 for $\alpha > 3$ (i.e. an urban propagation environment). This suggests that random beamforming will lead to higher probability of isolation in urban areas (with $\alpha > 3$), compared to the case of a single omnidirectional antenna (gain =1).

It can also be observed that the effective beamforming gain is relatively constant with increasing N for $\alpha \geq 3$. This suggests that for $\alpha > 3$, adding extra antenna elements in the antenna array will not lead to proportional improvement in the local connectivity. This can be intuitively explained as follows. From Fig. 1 it can be seen that the antenna gain increases linearly with number of antennas, but doesn't change much in width after about $N = 6$. So, as N increases, the maximum possible distance between a communicating node pair increases as well. As a result, the number of direct links to any chosen node in the direction of its main beam would increase. However, the attenuation is exponential with increasing path-loss exponent. So, after $\alpha = 3$, the attenuation dominates, making any increase in communication distance due to antenna gain ineffective.
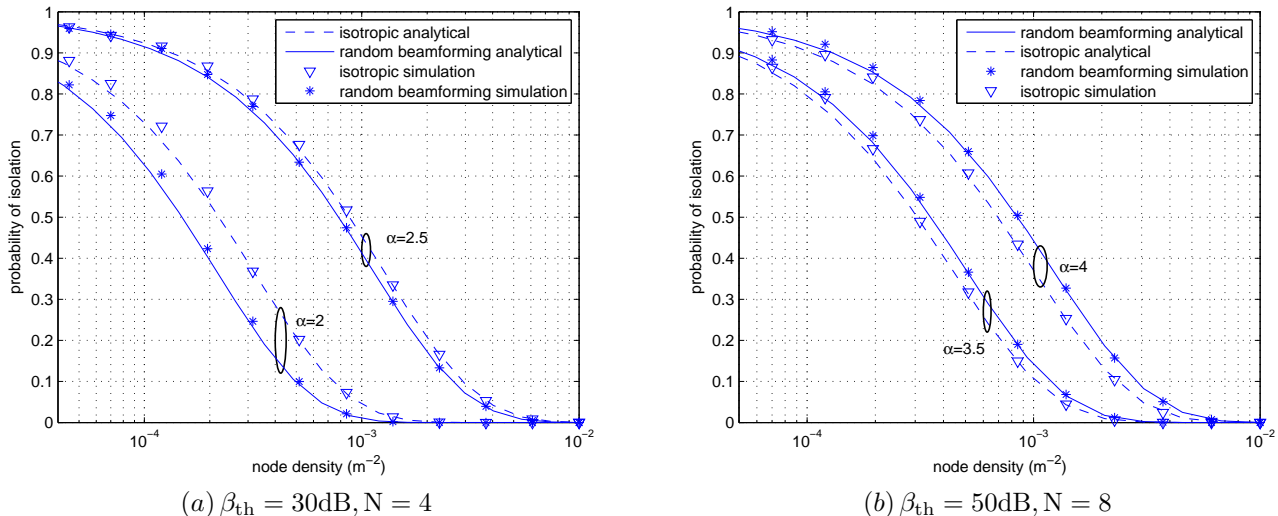
Fig. 3. Probability of node isolation in ad hoc networks with and without random beamforming. Analytical results for the random beamforming case are from (9) and Table I

IV. RESULTS

A. Model Validation

Simulations are carried out in Matlab in order to verify the theoretical results. In the simulations we distribute nodes uniformly on a square of area 200,000 m^2. To eliminate border effects, we compute the local connectivity for nodes located on an inner square of 125,000 m^2. The probability of isolation is calculated as the statistical average of fraction of isolated nodes in the subnetwork as

$$P_I = E\left[\frac{\text{No. of isolated nodes}}{\text{No. of nodes}}\right]. \quad (12)$$

Fig. 3 shows the results for the probability of isolation for both random beamforming and (reference) single omnidirectional antenna scenarios. The theoretical results for probability of isolation are calculated by substituting the values of effective beamforming gain from Table I in (9). Different numbers of antenna elements and thresholds are used in the scenarios shown in Fig. 3(a) where $N = 4$, $\beta_{\text{th}} = 30$dB and Fig. 3(b) where $N = 8$, $\beta_{\text{th}} = 50$dB. We can see that the simulation results are in excellent agreement with the theoretical results, which validates the proposed model.

B. Effect of Random beamforming

In Fig. 3(a), the use of beamforming reduces the number of isolated nodes for small path-loss exponents α. The improvement is noticeable when $\alpha = 2$. For example, the probability of isolation in the omnidirectional scenario is 0.2 at a node density of 0.0005, whereas the probability of isolation in the random beamforming case is only half of this, i.e. 0.1 at the same density. The opposite effect is shown in Fig. 3(b), where $\alpha > 3$, e.g. for $\alpha = 4$ the increase in value of probability of isolation by utilizing beamforming is about 0.05. These trends are in perfect agreement with the analytical insights provided by the proposed theoretical model discussed in the last section.

V. CONCLUSION

In this paper, we have proposed a novel theoretical model for analysis of random beamforming in wireless ad hoc networks. We have defined an effective beamforming gain to characterize the effect of random beamforming on the local connectivity. The calculated values of the effective beamforming gain provide insights into the effects of random beamforming on local connectivity of ad hoc networks. It has been shown that random beamforming improves the local connectivity for $\alpha < 3$, while it degrades the local connectivity for $\alpha > 3$. In addition for $\alpha > 3$, increasing the number of antenna elements does not lead to any significant improvement in the local connectivity. Simulation results have also been presented which validate the proposed theoretical model.

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