Exact Error Probability Expressions for Arbitrary Two-Dimensional Signaling with I/Q Unbalances over Nakagami-*m* Fading Channels

Jaeyoon Lee, Dongweon Yoon, Sang Kyu Park, and Kyongkuk Cho
Dept. of Electronics and Computer Eng., Hanyang University, Seoul, Korea
E-mail: dwyoon@hanyang.ac.kr

***Abstract*—Recently, we provided closed-form expressions involving two-dimensional (2-D) joint Gaussian Q-function for the symbol error rate (SER) and bit error rate (BER) of an arbitrary 2-D signal with I/Q unbalances over an additive white Gaussian noise (AWGN) channel [1]. In this paper, we extend the expressions to Nakagami-*m* fading channels. Using Craig's representation of the 2-D Gaussian Q-function, we derive an exact and general expression for the error probabilities of arbitrary 2-D signaling with I/Q phase and amplitude unbalances over Nakagami-*m* fading channels.**

***Index Terms*—error probability, Nakagami-*m* fading, two-dimensional modulation, I/Q unbalance**

I. Introduction

In a practical coherent two-dimensional (2-D) modulation scheme, the performance of the receiver is less than ideal: imperfect components create I/Q phase and gain unbalances, and there are severe small- and large-scale variations in the received signal strength. The I/Q unbalances arise from an imperfect 90-degree phase shifter, and from mixers or filters with different losses, and the variations of the received signal strength are caused by multi-path and shadow fading.

A number of recent studies [1]-[2] have analyzed the effects of these impediments on the error performances of 2-D signaling. The exact expression for the error probability of arbitrary 2-D signaling with I/Q unbalances over an AWGN channel was reported in [1]. Error probabilities of 2-D M-ary signaling signal with I/Q balance over fading channels for Rayleigh, Nakagami-*m*, and Ricean distributions were presented in [2].

In this paper, as an extension of our previous work [1], we provide a new closed-form expression involving the 2-D Gaussian Q-function for the error probability of arbitrary 2-D signaling with I/Q phase and amplitude unbalances over a Nakagami-*m* fading channel. For this purpose, we first transform the 2-D Gaussian Q-function into Craig's form [3]. Then, using the moment generating function (MGF) of the Nakagami-*m* distribution, we obtain the error probability expression for a 2-D signal over the Nakagami-*m* fading channel. Finally, we verify the provided expression through comparison with the previous results of [2] and [4] in Nakagami-*m* fading, and analyze the effect of I/Q unbalances on the performance.

II. System Model

We assume that the received signal envelope A has a Nakagami-*m* distribution with the probability density function (pdf) given as [5]

$$f_A(a) = \frac{2m^m a^{2m-1}}{\Omega^m \Gamma(m)} \exp\left(-\frac{ma^2}{\Omega}\right),\ a \ge 0 \tag{1}$$

where $m = \Omega^2 / E\left[\left(A^2 - \Omega^2\right)^2\right]$ and $\Omega = E\left[A^2\right]$ are fading and power-scaling parameters, respectively. The PDF of the instantaneous signal-to-noise ratio (SNR), $\gamma = a^2 E_b / N_0$ can be expressed as

$$f_\gamma(\gamma) = \frac{m^m \gamma^{m-1}}{\bar{\gamma}^m \Gamma(m)} \exp\left(-\frac{m\gamma}{\bar{\gamma}}\right), \qquad \gamma \ge 0 \tag{2}$$

where $\bar{\gamma} = \Omega E_b / N_0$ is the average SNR per bit and $\Gamma(x)$ is the Gamma function.

The 2-D joint Gaussian Q-function is defined by [6]

$$Q(x_1, y_1; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{x_1}^{\infty} \int_{y_1}^{\infty} \exp\left\{-\frac{x^2 + y^2 - 2\rho xy}{2\left(1-\rho^2\right)}\right\} dxdy\ . \tag{3}$$

From the Craig representation in [7], (3) is rewritten as

$$\begin{aligned} Q(x_1, y_1; \rho) &= \frac{1}{2\pi} \int_0^{w_1} \exp\left(-\frac{x_1^2}{2\sin^2\theta}\right) d\theta \\ &+ \frac{1}{2\pi} \int_0^{w_2} \exp\left(-\frac{y_1^2}{2\sin^2\theta}\right) d\theta, \quad x_1 \ge 0,\ y_1 \ge 0 \end{aligned} \tag{4}$$

where

$$\begin{aligned} w_1 &= \tan^{-1}\left\{\left(\sqrt{1-\rho^2}\, x_1 / y_1\right) / \left(1 - \rho x_1 / y_1\right)\right\} \\ w_2 &= \tan^{-1}\left\{\left(\sqrt{1-\rho^2}\, y_1 / x_1\right) / \left(1 - \rho y_1 / x_1\right)\right\} \end{aligned} . \tag{5}$$

The error probability $P_{fading}(E)$ in a flat fading channel can be obtained as

$$
\begin{aligned}
&P_{fading}(E) \\
&= \int_0^\infty P_S(E \mid \gamma) p_\gamma(\gamma)\, d\gamma = \int_0^\infty Q\left(x_1\sqrt{\gamma}, x_2\sqrt{\gamma}; \rho\right) p_\gamma(\gamma)\, d\gamma \\
&= \frac{1}{2}\sum_{k=1}^{2} \frac{1}{\pi} \int_0^{w_k} M_\gamma\left(-\frac{x_k^2}{2\sin^2\theta}\right) d\theta
\end{aligned} \tag{6}
$$

where $P_s(E \mid \gamma)$ is the error rate expression which can be expressed as the 2-D joint Gaussian Q-function for AWGN, $p_\gamma(\gamma)$ is a pdf of the flat fading, and $M_\gamma(-s)$ is the MGF of the fading pdf, which is defined as

$$
M_\gamma(-s) \triangleq \int_0^\infty e^{-s\gamma} p_\gamma(\gamma)\, d\gamma . \tag{7}
$$

Then, the MGF of the Nakagami-*m* pdf of (2) can be obtained as follows:

$$
M_\gamma(-s) = \left(1 + \frac{\bar{\gamma}}{m} \cdot s\right)^{-m} . \tag{8}
$$

Substituting (8) into (6), and using [4, eq. (5A.24)], (6) becomes

$$
P_{Na\_fading}(E) = \frac{1}{2}\sum_{k=1}^{2} I\left[w_k, z_k\right] \tag{9}
$$

where

$$
\begin{aligned}
I\left[w_k, z_k\right] = \frac{w_k}{\pi} - \frac{1}{\pi}\beta_k \Bigg[&\left(\frac{\pi}{2} + \tan^{-1}\left(\frac{-\beta_k}{\tan(w_k)}\right)\right) \times \sum_{l=0}^{m-1} \frac{{}_{2l}\mathrm{C}_l}{\left\{4\left(1 + z_k^2\bar{\gamma}/2m\right)\right\}^l} \\
&+ \sin\left(\tan^{-1}\left(\frac{-\beta_k}{\tan(w_k)}\right)\right) \times \sum_{l=1}^{m-1}\sum_{i=1}^{l} \frac{T_{il}}{\left(1 + z_k^2\bar{\gamma}/2m\right)^l} \left\{\cos\left(\tan^{-1}\left(\frac{-\beta_k}{\tan(w_k)}\right)\right)\right\}^{2(l-i)+1} \Bigg],
\end{aligned}
$$

$$
-\pi \le w_k \le \pi \tag{10}
$$

in which the parameters, β_k and T_{il} are given as follows:

$$
\beta_k \triangleq \sqrt{\frac{\left(z_k^2\bar{\gamma}/2m\right)}{1 + \left(z_k^2\bar{\gamma}/2m\right)}}\,\mathrm{sgn}(w_k), \quad T_{il} \triangleq \frac{{}_{2l}\mathrm{C}_l}{{}_{2(l-i)}\mathrm{C}_{l-i} \cdot 4^i \left\{2(l-i)+1\right\}} . \tag{11}
$$

III. CLOSED-FORM EXPRESSION FOR THE SER OF AN ARBITRARY 2-D SIGNAL WITH I/Q UNBALANCES OVER NAKAGAMI-M FADING CHANNELS

Typical decision regions for arbitrary 2-D signaling are of two types: closed and open [8]. The error probabilities for the two types of regions were presented in [1], where the SER and BER for the arbitrary 2-D signals with I/Q unbalances were provided as a linear combination of the error probabilities for the closed and open regions. In this section, using the results obtained in section Ⅱ and in [1], we present a closed-form expression for the SER of an arbitrary 2-D signal in Nakagami-*m* fading channels, where *a priori* probabilities of all the signal points are supposed to be equally likely.

From [1], when a signal point $S_{t_1}^c$ that has a closed decision region $\left(R_{t_1\_n}^c\right)$ with *n*-sided polygonal shape is transmitted, we can find that the SER for the signal point $S_{t_1}^c$ is

$$
\begin{aligned}
&P_S^c(E \mid \gamma) = 1 - P\left\{S_r \in R_{t_1\_n}^c \mid S_t = S_{t_1}^c\right\} \cdot P\left\{S_{t_1}^c\right\} \\
&= \sum_{i=1}^{n}\left\{Q\left(\frac{E[Y_i]}{\sqrt{Var[Y_i]}}, 0; 1\right) - Q\left(\frac{E[Y_i]}{\sqrt{Var[Y_i]}}, \frac{E[Y_{i+1}]}{\sqrt{Var[Y_{i+1}]}}; \rho_{Y_iY_{i+1}}\right)\right\} . \\
&= \sum_{i=1}^{n}\left\{Q\left(z_i\sqrt{\gamma}, 0; 1\right) - Q\left(z_i\sqrt{\gamma}, z_{i+1}\sqrt{\gamma}; \rho_{Y_iY_{i+1}}\right)\right\}
\end{aligned} \tag{12}
$$

Similarly, the SER for a signal point $S_{t_1}^o$ that has an open decision region $\left(R_{t_1\_q}^o\right)$ with *q* sides is

$$
\begin{aligned}
&P_S^o(E \mid \gamma) = 1 - P\left\{S_r \in R_{t_1\_q}^o \mid S_t = S_{t_1}^o\right\} \cdot P\left\{S_{t_1}^o\right\} \\
&= \sum_{i=1}^{q} Q\left(\frac{E[Y_i]}{\sqrt{Var[Y_i]}}, 0; 1\right) - \sum_{i=1}^{q-1} Q\left(\frac{E[Y_i]}{\sqrt{Var[Y_i]}}, \frac{E[Y_{i+1}]}{\sqrt{Var[Y_{i+1}]}}; \rho_{Y_iY_{i+1}}\right) . \\
&= \sum_{i=1}^{q} Q\left(z_i\sqrt{\gamma}, 0; 1\right) - \sum_{i=1}^{q-1} Q\left(z_i\sqrt{\gamma}, z_{i+1}\sqrt{\gamma}; \rho_{Y_iY_{i+1}}\right)
\end{aligned} \tag{13}
$$

In (12) and (13), $z_i = E[Y_i]/\sqrt{Var[Y_i] \cdot \gamma}$ and Y_i is a random variable on the perpendicular axis to the decision boundary of the closed or open regions. Then, Y_i and Y_{i+1} have the joint Gaussian distribution with

$$
\begin{cases}
E[Y_i] = \sqrt{E_s}\left(\beta\zeta_1 \cos\theta_i \sin(\psi_1 + \phi_R) - \alpha\zeta_1 \sin\theta_i \cos\psi_1\right) - d_i \\
Var[Y_i] = \sigma^2\left(\alpha^2 \sin^2(\theta_i) + \beta^2\cos^2(\theta_i) - \alpha\beta\sin\phi_R \sin(2\theta_i)\right) \\
\rho_{Y_iY_{i+1}} = \dfrac{COV[Y_iY_{i+1}]}{\sqrt{Var[Y_i]}\sqrt{Var[Y_{i+1}]}} \\
= \dfrac{\alpha^2\sin\theta_i\sin\theta_{i+1} + \beta^2\cos\theta_i\cos\theta_{i+1} - \alpha\beta\sin\phi_R\sin(\theta_i + \theta_{i+1})}{\sqrt{Var[Y_i]}\sqrt{Var[Y_{i+1}]}}
\end{cases} \tag{14}
$$

where α and β are the gains of filters or mixers which represent the amplitude unbalance; ζ_i is a scale factor which varies with the position of the signal point; ψ_1 is the phase of the transmitted signal; d_i is a distance between the origin and the *i*-th decision boundary of the closed or open regions; θ_i is the slope of the *i*-th decision boundary, respectively [1].

By combining (12) and (13), the average SER of an arbitrary 2-D signaling with I/Q unbalances can be obtained [1]:

$$P_S(E|\gamma) = P_S^c(E|\gamma) + P_S^o(E|\gamma)$$
$$= \sum_{h=1}^{U}\left[\left(1 - P\left\{S_r \in R^c_{t_h\_n} \mid S_t = S^c_{t_h}\right\}\right)\cdot P\left\{S^c_{t_h}\right\}\right]$$
$$+ \sum_{j=1}^{V}\left[\left(1 - P\left\{S_r \in R^o_{t_j\_q} \mid S_t = S^o_{t_j}\right\}\right)\cdot P\left\{S^o_{t_j}\right\}\right] \quad (15)$$

where U is the number of the closed regions with n-sided polygonal shape and V is the number of the open regions with q-sided polygonal shape.

The result of (15) is expressed as a linear combination of the 2-D joint Gaussian Q-functions. Thus, we can obtain the average SER of an arbitrary 2-D signaling with I/Q phase and amplitude unbalances over Nakagami-*m* fading channels from (6), (9), and (15):

$$P_{Na\_fading}(E) = \int_0^\infty P_S(E|\gamma)p_\gamma(\gamma)d\gamma$$
$$= \frac{1}{U+V}\cdot\frac{1}{2}\left\{\sum_{h=1}^{U}\sum_{i=1}^{n}\left(I[\pi, z_i] - I\left[w_{z_i,z_{i+1}}, z_i\right] - I\left[w_{z_{i+1},z_i}, z_{i+1}\right]\right)\right.$$
$$\left. + \sum_{j=1}^{V}\left(\sum_{i=1}^{q}\left(I[\pi, z_i]\right) - \sum_{i=1}^{q-1}\left(I\left[w_{z_i,z_{i+1}}, z_i\right] + I\left[w_{z_{i+1},z_i}, z_{i+1}\right]\right)\right)\right\} \quad (16)$$

where

$$\begin{aligned} w_{z_i,z_{i+1}} &= \tan^{-1}\left\{\left(\sqrt{1-\rho^2}\, z_i / z_{i+1}\right)/\left(1-\rho\, z_i / z_{i+1}\right)\right\} \\ w_{z_{i+1},z_i} &= \tan^{-1}\left\{\left(\sqrt{1-\rho^2}\, z_{i+1} / z_i\right)/\left(1-\rho\, z_{i+1} / z_i\right)\right\} \end{aligned}. \quad (17)$$

Note that for $\alpha = \beta = 1$, $\phi_R = 0°$ and $m = 1$, (16) reduces to the SER of an I/Q balanced MPSK signal in Rayleigh fading, which is equal to [4, eq. (8.113)].

IV. SER of MPSK and 16-Star-QAM with I/Q Unbalances over Nakagami-m Fading Channels

To verify the validity of the derived result, we consider MPSK and 16-star-QAM. The decision regions for MPSK have only open regions with two-sided polygonal shape. Hence, from the result [9] of the previous research, the average symbol error probability for MPSK over Nakagami-*m* fading channels is obtained as follows:

$$P^{MPSK}_{Na\_fading}(E) = \int_0^\infty P_S(E|\gamma)p_\gamma(\gamma)d\gamma$$
$$= \frac{1}{2M}\left\{\sum_{j=1}^{M}\left(\sum_{i=1}^{2}\left(I[\pi, z_i]\right) - \left(I\left[w_{z_1,z_2}, z_1\right] + I\left[w_{z_2,z_1}, z_2\right]\right)\right)\right\} \quad (18)$$

For 16-star-QAM, the decision regions have eight open regions and eight closed regions with three-sided polygonal shape. Similarly with MPSK, from the result in [1], the average symbol error probability of 16-star-QAM with I/Q unbalances over Nakagami-*m* fading channels is rewritten as

$$P^{16-star-QAM}_{Na\_fading}(E) = \int_0^\infty P_S(E|\gamma)p_\gamma(\gamma)d\gamma$$
$$= \frac{1}{2M}\left\{\begin{aligned}&\sum_{h=1}^{8}\sum_{i=1}^{3}\left(I[\pi, z_i] - I\left[w_{z_i,z_{i+1}}, z_i\right] - I\left[w_{z_{i+1},z_i}, z_{i+1}\right]\right) \\ &+\sum_{j=1}^{8}\left(\sum_{i=1}^{3}\left(I[\pi, z_i]\right) - \sum_{i=1}^{2}\left(I\left[w_{z_i,z_{i+1}}, z_i\right] + I\left[w_{z_{i+1},z_i}, z_{i+1}\right]\right)\right)\end{aligned}\right\} \quad (19)$$

To compare the results in this paper with the results in previous researches [2] and [4], we assume 8-PSK and 16-star-QAM in Nakagami-m fading channel, and show the effect of I/Q unbalances on the error performance. For this end, the values of parameters, $\phi_R = 0^\circ, 5^\circ$ and α or $\beta = 1, 1.1$, are used since typical values achievable with careful design are $\phi_R = 5^\circ$ and α or β =1.1 [10].

In Fig. 1 and Fig. 2, we have plotted the average SER for 8-PSK and 16-star-QAM with I/Q unbalances corresponding to several values of the fading parameter m. As shown in Fig. 1 and Fig. 2, the results for $\alpha = \beta = 1$ and $\phi_R = 0°$ in eq. (18) and (19) of this paper are exactly the same as the results of [4, Fig. 8.4] and [2, eq.(12)], respectively. Fig. 3 shows the effects of I/Q unbalances on the SER performance of MPSK, and in Fig 4, for $m = 1$, $\alpha = 1$, $\beta = 1.1$ and $\phi_R = 5°$ the SER of several 16-APSK modulation schemes is depicted.

As shown in Fig. 1 and Fig. 2, as m increases, we can confirm that the severity of fading decreases, but the effect of I/Q unbalances on the SER performance becomes serious. And, in Fig. 3, we can see that the SER performances are more sensitive to the effect of I/Q unbalances according to increasing M. Also, we can observe that the 1+5+10 APSK outperform the other 16-APSK modulation schemes through Fig. 4. Consequently, for fading channel, one of the dominant causes of performance degradation is multi-path fading rather than I/Q unbalances of the components.

V. Conclusions

In this paper, we have provided a new closed-form expression involving the 2-D Gaussian Q-function for the SER of arbitrary 2-D signaling with I/Q unbalances over a Nakagami-*m* fading channel. The BER is also easily obtained from the provided result by using [1, eq. (14)]. We first transformed the 2-D joint Gaussian Q-function into Craig's form. Then, using the MGF of the Nakagami-*m* distribution we provided the error probability of arbitrary 2-D signaling with I/Q unbalances over a Nakagami-*m* fading channel. Finally, from the provided result, we analyzed that one of the dominant causes of performance degradation is multi-path fading rather than I/Q unbalances of the components. The result can be readily applied to numerical evaluation for various cases of practical interest involving unbalanced I/Q modulation systems operating in a wide range of fading environments.

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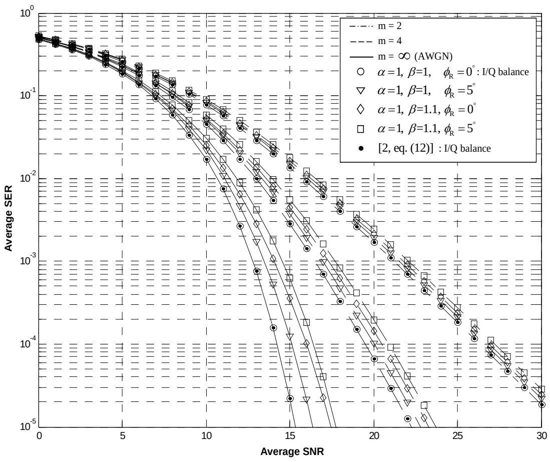

Fig. 1 Effect of I/Q unbalances on the SER of 8-PSK in fading channels

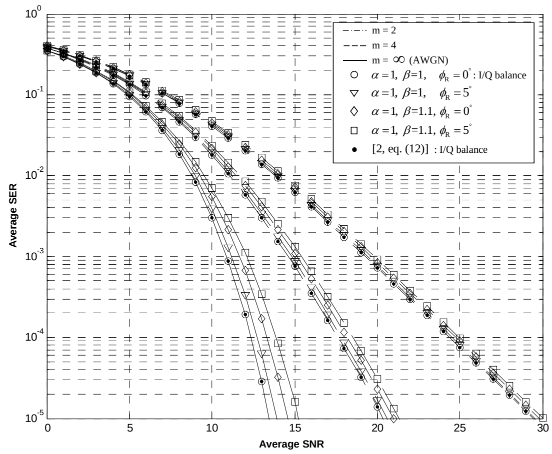

Fig. 2 Effect of I/Q unbalances on the SER of 16-star-QAM in fading channels

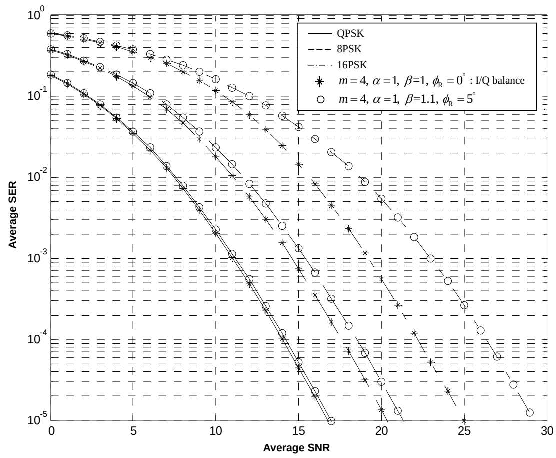

Fig. 3 Effect of I/Q unbalances on the SER of MPSK in fading channels

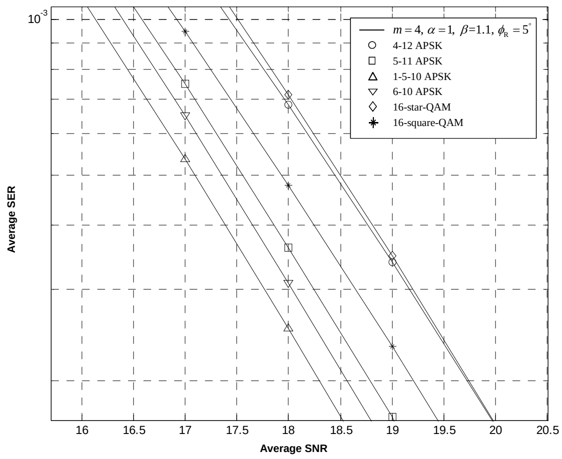

Fig. 4 Effect of I/Q unbalances on the SER of 16-APSK in fading channels