On the Performance of Golden Codes in Rayleigh Fading Channels with Doppler Spread

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***Abstract*—The Golden code has recently been proposed as a 2 x 2 space time block code that achieves the optimal diversity - multiplexing gain tradeoff for a multiple antenna system. In this paper we review the decoding methodology for the Golden code, followed by performance comparisons with the Alamouti code and V-BLAST in Rayleigh fading environments with Doppler spread. Simulation results show that the Golden Code outperforms both the Alamouti code and V-BLAST at high SNR levels. For a symbol error rate of 10^{-4} the Eb/N0 requirement for the Golden code is 5dB less than the Alamouti code and V-BLAST.**

I. Introduction

In recent years, multiple antenna systems (commonly referred to as multi-input multi-output or MIMO systems) have proven to be an effective method for realising high-rate reliable wireless communications. Research in MIMO systems has generally focused on providing either higher-rate or increased diversity over traditional single antenna (SISO) systems.

Foschini [1] introduced the layered space-time (BLAST) architecture where a high throughput rate is achieved by using multiple transmit antennas to transmit multiple independent data sub-streams in parallel. Multiple receive antennas and multi-user detection algorithms are used at the receiver end to separate and decode the individual sub-streams. Although providing high-rate, BLAST has the shortcoming that it does not provide diversity gain as each data symbol is only transmitted once from one antenna.

Alamouti [2] introduced a simple orthogonal space time block code (STBC) that provided diversity gain for 2×1 and 2×2 multi-antenna systems. This scheme was generalised and extended by Tarokh et. al. [3] to include higher-dimension MIMO systems, using real and complex orthogonal STBCs. Although providing diversity gain, orthogonal STBCs have the shortcoming that (with the exception of a few sporadic codes) the coding rate does not exceed $\frac{1}{2}$.

A generalised class of space-time codes that encompassed both orthogonal STBCs and BLAST architectures was proposed by Hassibi and Hochwald [4]. This generalised class of codes, which are known as linear dispersion (LD) codes, are defined as codes that break up the input data stream into sub-streams that are dispersed in linear combinations over space and time. Theoretically, LD codes can provide both diversity gain and high-rate. In general, LD codes can outperform their orthogonal STBC and BLAST sub-classes.

Sethuraman et. al. [5] proposed a methodology for designing full-diversity high-rate LD codes using cyclic division algebras. A division algebra is used to provide a structured set of invertible matrices to construct LD space-time codes. Using this technique, Belfiore et. al. [6] developed the Golden Code, a 2×2 LD code that provides both diversity gain and full-rate.

In this paper, we investigate the effect of Doppler spread on the performance of the 2×2 Golden Code. Doppler spread is a measure of spectral broadening caused by the relative motion between the transmitter and receiver antennas or by the movement of reflecting objects in the channel. Doppler spread is an important consideration in the design of mobile communication systems.

The paper is organised as follows. Section II presents an overview of the system model. Section III provides the definitions of LD codes. Section IV summarises the LD decoding algorithm used in the simulations. Section V describes the Golden code. Section VI presents the simulation results and conclusions are presented in section VII.

II. System Model

The system model for a multiple-antenna communications system with M transmit and N receive antennas is shown in fig. 1.

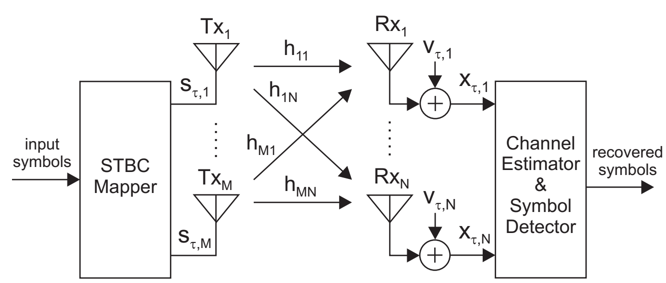

Fig. 1. MIMO communications system model

If we assume a narrow-band flat-fading wireless channel which is constant for at least T channel uses, then the transmitted and received signals are related by

$$\mathbf{x}_\tau = \sqrt{\frac{\rho}{M}}\mathbf{H}\mathbf{s}_\tau + \mathbf{v}_\tau, \quad \tau = 1, 2, \ldots, T \tag{1}$$

where τ is an individual channel use, and we define

$$\mathbf{x}_\tau = \begin{bmatrix} x_{\tau,1} \\ x_{\tau,2} \\ \vdots \\ x_{\tau,N} \end{bmatrix}, \mathbf{s}_\tau = \begin{bmatrix} s_{\tau,1} \\ s_{\tau,2} \\ \vdots \\ s_{\tau,M} \end{bmatrix}, \mathbf{v}_\tau = \begin{bmatrix} v_{\tau,1} \\ v_{\tau,2} \\ \vdots \\ v_{\tau,N} \end{bmatrix} \quad (2)$$

where $\mathbf{x}_\tau$ is the N-dimensional vector of complex received signals during channel use τ, $\mathbf{s}_\tau$ is the M-dimensional vector of complex transmitted signals, $\mathbf{H}$ is the $N \times M$ channel matrix, and $\mathbf{v}_\tau$ is the N-dimensional vector of additive complex-Gaussian noise (assumed to be zero-mean and unit-variance).

If we assume that $\mathbf{H}$, $\mathbf{s}_\tau$ and $\mathbf{v}_\tau$ are random and independent quantities, the signal power normalisation $\sqrt{\rho/M}$ ensures that ρ is the signal-to-noise ratio (SNR) at each receive antenna, independently of M. The channel matrix is assumed to be known to the receiver.

We define the matrices $\mathbf{X}$, $\mathbf{S}$, and $\mathbf{V}$ as:

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_T \end{bmatrix}^t \\ \mathbf{S} &= \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \dots & \mathbf{s}_T \end{bmatrix}^t \\ \mathbf{V} &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_T \end{bmatrix}^t \end{aligned} \quad (3)$$

where the superscript t denotes transpose. It is generally more convenient to write equation (1) in its transposed form

$$\mathbf{X} = \sqrt{\frac{\rho}{M}}\mathbf{S}\mathbf{H} + \mathbf{V} \quad (4)$$

where the transpose notation is omitted from $\mathbf{H}$, and the channel matrix is simply redefined to have dimension $M \times N$. $\mathbf{X}$ is the $T \times N$ received signal matrix, $\mathbf{S}$ is the $T \times M$ transmitted signal matrix, and $\mathbf{V}$ is the $T \times N$ additive noise matrix. In matrices $\mathbf{X}$, $\mathbf{S}$, and $\mathbf{V}$, time runs vertically and space runs horizontally.

III. Linear Dispersion Codes

A Linear Dispersion (LD) code is a general class of space time block code (STBC) that breaks up the input data stream into sub-streams that are dispersed in linear combinations over space and time. Specifically, a linear dispersion code is defined as:

$$\mathbf{S} = \sum_{q=1}^{Q} (s_q \mathbf{C}_q + s_q^* \mathbf{D}_q) \quad (5)$$

where the data sequence is broken up into Q sub-streams, $s_1, \dots, s_Q$ are complex symbols from an arbitrary constellation (typically r-PSK or r-QAM), and $\mathbf{C}_q$ and $\mathbf{D}_q$ are fixed $T \times M$ complex matrices. The code is completely determined by the set of dispersion matrices $\{\mathbf{C}_q, \mathbf{D}_q\}$.

It is generally more convenient to decompose the complex scalar s_q into its real and imaginary components

$$s_q = \alpha_q + j\beta_q, \;\; q = 1, \dots, Q \quad (6)$$

The LD code can then be redefined in terms of real and imaginary components as follows:

$$\mathbf{S} = \sum_{q=1}^{Q} (\alpha_q \mathbf{A}_q + j\beta_q \mathbf{B}_q) \quad (7)$$

where $\mathbf{A}_q = \mathbf{C}_q + \mathbf{D}_q$ and $\mathbf{B}_q = \mathbf{C}_q - \mathbf{D}_q$. The dispersion matrices $\{\mathbf{A}_q, \mathbf{B}_q\}$ also completely specify the code.

LD codes include many commonly used ST codes including the Alamouti Scheme and V-BLAST (Vertical-encoding spatial multiplexing).

IV. Decoding of Linear Dispersion Codes

The simulations presented in this paper were performed using the LD decoding method proposed by Hassibi and Hochwald [4].

An important property of LD codes (7) is their linearity in the variables α_q, β_q, leading to efficient decoding schemes. To see this, we substitute the LD code equation (7) into the received signal equation (4) which forms the following block equation:

$$\mathbf{X} = \sqrt{\frac{\rho}{M}}\mathbf{S}\mathbf{H} + \mathbf{V} = \sqrt{\frac{\rho}{M}} \sum_{q=1}^{Q} (\alpha_q \mathbf{A}_q + j\beta_q \mathbf{B}_q)\mathbf{H} + \mathbf{V} \quad (8)$$

The matrices in (8) can be decomposed into their real and imaginary components to obtain:

$$\begin{aligned} \mathbf{X}_R &= \sqrt{\frac{\rho}{M}} \sum_{q=1}^{Q} [(\mathbf{A}_{R,q}\mathbf{H}_R - \mathbf{A}_{I,q}\mathbf{H}_I)\alpha_q \\ &\quad + (-\mathbf{B}_{I,q}\mathbf{H}_R - \mathbf{B}_{R,q}\mathbf{H}_I)\beta_q] + \mathbf{V}_R \end{aligned} \quad (9)$$

$$\begin{aligned} \mathbf{X}_I &= \sqrt{\frac{\rho}{M}} \sum_{q=1}^{Q} [(\mathbf{A}_{I,q}\mathbf{H}_R - \mathbf{A}_{R,q}\mathbf{H}_I)\alpha_q \\ &\quad + (\mathbf{B}_{R,q}\mathbf{H}_R - \mathbf{B}_{I,q}\mathbf{H}_I)\beta_q] + \mathbf{V}_I \end{aligned} \quad (10)$$

where $\mathbf{X}_R = Re(\mathbf{X})$, $\mathbf{X}_I = Im(\mathbf{X})$, $\mathbf{H}_R = Re(\mathbf{H})$ and $\mathbf{H}_I = Im(\mathbf{H})$. (Where $Re(z)$ and $Im(z)$ denote the real and imaginary parts respectively of the complex value z).

We denote the columns of $\mathbf{X}_R$, $\mathbf{X}_I$, $\mathbf{H}_R$, $\mathbf{H}_I$, $\mathbf{V}_R$ and $\mathbf{V}_I$ by $\mathbf{x}_{R,n}$, $\mathbf{x}_{I,n}$, $\mathbf{h}_{R,n}$, $\mathbf{h}_{I,n}$, $\mathbf{v}_{R,n}$ and $\mathbf{v}_{I,n}$ respectively and define:

$$\mathcal{A}_q = \begin{bmatrix} \mathbf{A}_{R,q} & -\mathbf{A}_{I,q} \\ \mathbf{A}_{I,q} & \mathbf{A}_{R,q} \end{bmatrix}$$

$$\mathcal{B}_q = \begin{bmatrix} -\mathbf{B}_{I,q} & -\mathbf{B}_{R,q} \\ \mathbf{B}_{R,q} & -\mathbf{B}_{I,q} \end{bmatrix}, \underline{\mathbf{h}}_n = \begin{bmatrix} \mathbf{h}_{R,n} \\ \mathbf{h}_{I,n} \end{bmatrix} \quad (11)$$

where $n = 1, \dots, N$. The equations in $\mathbf{X}_R$ and $\mathbf{X}_I$ can be assembled to form the single real system of equations

$$\begin{bmatrix} \mathbf{x}_{R,1} \\ \mathbf{x}_{I,1} \\ \vdots \\ \mathbf{x}_{R,N} \\ \mathbf{x}_{I,N} \end{bmatrix} = \sqrt{\frac{\rho}{M}}\mathcal{H} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_Q \\ \beta_Q \end{bmatrix} + \begin{bmatrix} \mathbf{v}_{R,1} \\ \mathbf{v}_{I,1} \\ \vdots \\ \mathbf{v}_{R,N} \\ \mathbf{v}_{I,N} \end{bmatrix} \quad (12)$$

where the equivalent $2NT \times 2Q$ real channel matrix is given by:

$$\mathcal{H} = \begin{bmatrix} \mathcal{A}_1\underline{\mathbf{h}}_1 & \mathcal{B}_1\underline{\mathbf{h}}_1 & \dots & \mathcal{A}_Q\underline{\mathbf{h}}_1 & \mathcal{B}_Q\underline{\mathbf{h}}_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{A}_1\underline{\mathbf{h}}_N & \mathcal{B}_1\underline{\mathbf{h}}_N & \dots & \mathcal{A}_Q\underline{\mathbf{h}}_N & \mathcal{B}_Q\underline{\mathbf{h}}_N \end{bmatrix} \tag{13}$$

We now have a linear relation between the input and output vectors $\mathbf{s}$ and $\mathbf{x}$

$$\mathbf{x} = \sqrt{\frac{\rho}{M}}\mathcal{H}\mathbf{s} + \mathbf{v} \tag{14}$$

where the equivalent channel $\mathcal{H}$ is known to the receiver because the original channel $\mathbf{H}$, and the dispersion matrices are all known to the receiver. The receiver uses (13) to find the equivalent channel. The system of equations between the transmitter and receiver is not underdetermined as long as $Q \leq NT$.

Any decoding scheme that can solve a well-conditioned system of linear equation can be used for decoding of LD codes. Suitable decoding techniques include successive nulling and canceling (as used for V-BLAST), and sphere decoding.

V. Golden Code

Sethuraman et. al. [5] proposed a methodology for designing full-diversity high-rate LD codes using cyclic division algebras. A division algebra is used to provide a structured set of invertible matrices to construct LD space-time codes. In general, LD codes derived from cyclic division algebra have been found to provide better performance than LD codes derived using the original information theoretic approach proposed by Hassibi and Hochwald [4].

The Golden Code is a full-rate 2×2 LD code and is defined as subset of the cyclic division algebra $(\mathbb{Q}(i,\sqrt{5}), i)$ with centre $\mathbb{Q}(i)$ [6]. The 2×2 Golden Code has the structure:

$$\mathbf{S} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha(s_1 + s_2\theta) & \alpha(s_3 + s_4\theta) \\ i\overline{\alpha}(s_3 + s_4\overline{\theta}) & \overline{\alpha}(s_1 + s_2\overline{\theta}) \end{bmatrix} \tag{15}$$

where $\theta = \frac{1+\sqrt{5}}{2}$, $\overline{\theta} = \frac{1-\sqrt{5}}{2} = (1-\theta)$, $\alpha = i(1-\theta)$, and $\overline{\alpha} = 1 + i(1-\overline{\theta})$

In [7], Tarokh et. al. defined the rank criterion and determinant criterion for designing ST codes. Oggier et. al. [8] extended this design criteria to include: (a) full rate; (b) full diversity; (c) non-vanishing determinant for increasing spectral efficiency; (d) good shaping of the constellation; and (e) uniform average transmitted energy per antenna. ST Codes that meet all of these criteria are termed "perfect" space-time block codes. The Golden Code has been found to be the best "perfect" code for MIMO systems with 2 transmit and 2 or more receive antennas.

Elia et. al. [9] have shown that the Golden code achieves the optimal diversity-multiplexing tradeoff for a 2×2 MIMO system. Zheng and Tse [10] developed a simple characterisation of the optimal tradeoff between diversity and degrees of freedom (multiplexing gain), and then used it to evaluate the performance of existing multiple antenna schemes. The concept is that for a given MIMO channel, both diversity and multiplexing gain can be simultaneously obtained, but there is a fundamental tradeoff between how much of each type of gain any coding scheme achieve. For example, for a particular coding scheme, increased spatial multiplexing gain comes at the cost of reduced diversity gain. Fig. 2 uses Zheng's and Tse's method to compare the Alamouti STBC, V-BLAST and the Golden Code STBC.

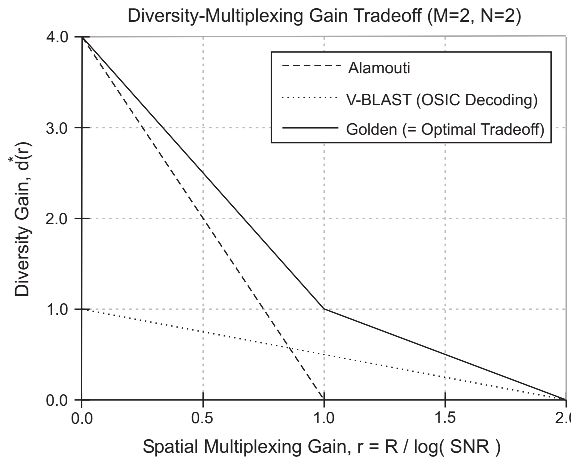

Fig. 2. Diversity-Multiplexing Gain Tradeoff (M=2, N=2) [10][11]

From Fig. 2 we see that neither the Alamouti STBC nor V-BLAST are optimal. The Alamouti STBC does not provide full spatial-multiplexing gain, while V-BLAST does not provide full diversity gain. The Golden Code however provides both the full spatial-muliplexing gain and the full diversity gain available for a 2×2 system.

VI. Simulation Results

The simulations assume the receiver has perfect channel knowledge. The individual channels in the channel matrix are uncorrelated, and the system does not use error correction coding. The constellations of each of the coding schemes has been chosen to ensure a common spectral efficiency of 8-bits per channel use. The V-BLAST and Golden code simulations both use 16-QAM constellations, while the Alamouti code simulations use 256-QAM (the higher-order constellation is required to compensate for the absence of spatial multiplexing gain).

In Fig. 3 we compare the performance of the Golden Code STBC against the Alamouti code and V-BLAST in a Rayleigh flat-fading environment. The figure shows the superior performance of the Golden code, particularly at higher SNR values. For a symbol error rate of 10^{-4} the Eb/N0 requirement for the Golden code is 5dB less than the Alamouti code and V-BLAST.

Fig. 4 compares the performance of the Golden code over a range of Doppler frequencies that would be typical in mobile communications scenarios. We observe a performance degradation of approximately 2dB for every 5Hz increase in Doppler frequency.

Fig. 5 compares the Golden code performance against the Alamouti code at selected Doppler frequencies. The per-

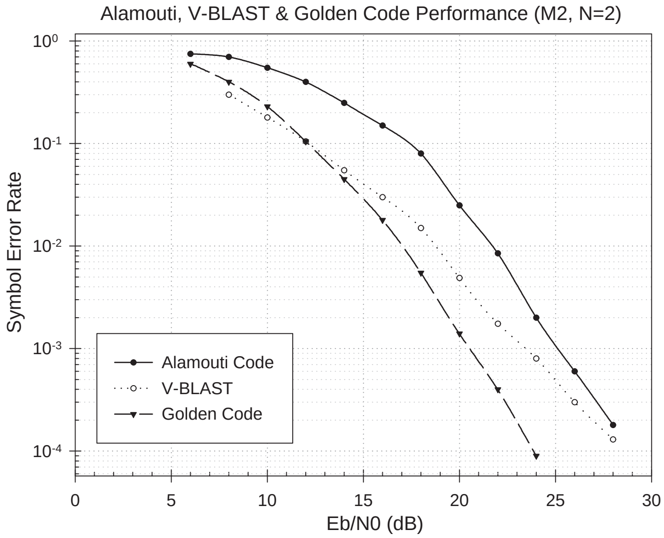

Fig. 3. Alamouti, V-BLAST and Golden Code Performance (M=2, N=2)

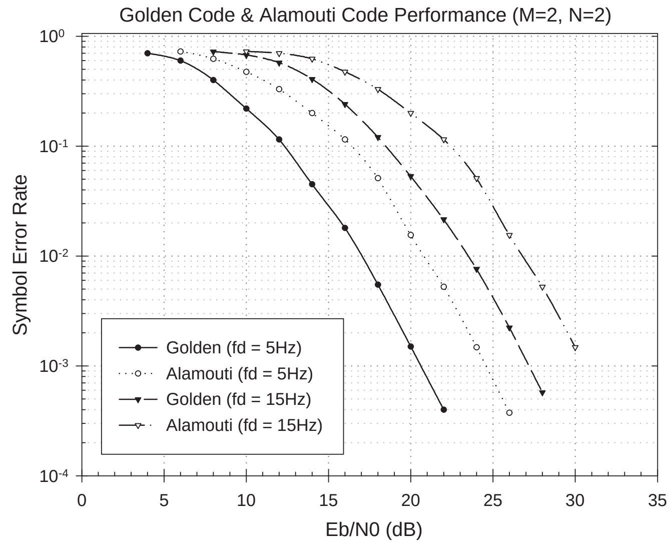

Fig. 5. Golden Code and Alamouti Code Performance Comparison

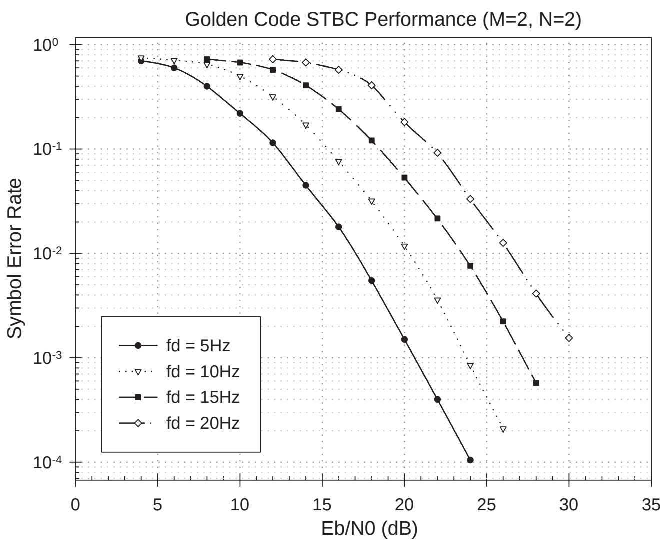

Fig. 4. Golden Code Performance at Various Doppler Frequencies

formance degradation of approximately 2dB for every 5Hz increase in Doppler frequency previously observed with the Golden code is also observe with the Alamouti code. The 5dB performance advantage at high SNR levels of the Golden code compared to the Alamouti code is maintained over the range of Doppler frequencies investigated.

VII. Conclusion

The performance of the Golden Code has been presented and compared with common multiple antenna systems, namely the Alamouti code and V-BLAST (spatial multiplexing). The Golden Code has been shown to provide superior performance at high SNR levels while using the same low-complexity linear dispersion code decoding schemes typically used to decode Alamouti and V-BLAST schemes.

Simulation results of the Golden Code performance at various Doppler frequencies were presented. These results showed that the Golden Code maintains its superior performance when compared to the Alamouti scheme over the range of Doppler frequencies that would typically be encountered in a mobile communications system.

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