# Search with Constraints

COMP8620, S2 2008
Advanced Topics in A.I.

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# Overview

- What you will get from this lecture:
  - What is Constraint Programming
  - What it's good for
  - What is [Arc | Generalized-Arc | Path | Bound ] Consistency, and how they're useful
  - What drives efficient searches with constraints
  - How to play Sudoku!

# Introduction

- Constraints
  - Specifies what you can/can’t do!
  - Find a solution that conform to all the rules
- Search with constraints
  - Yes, you still have to search!
  - Once a variable assignment is made, your other choices are limited.
  - You can limit your search space.

# Search with Constraints

- Most typical example:

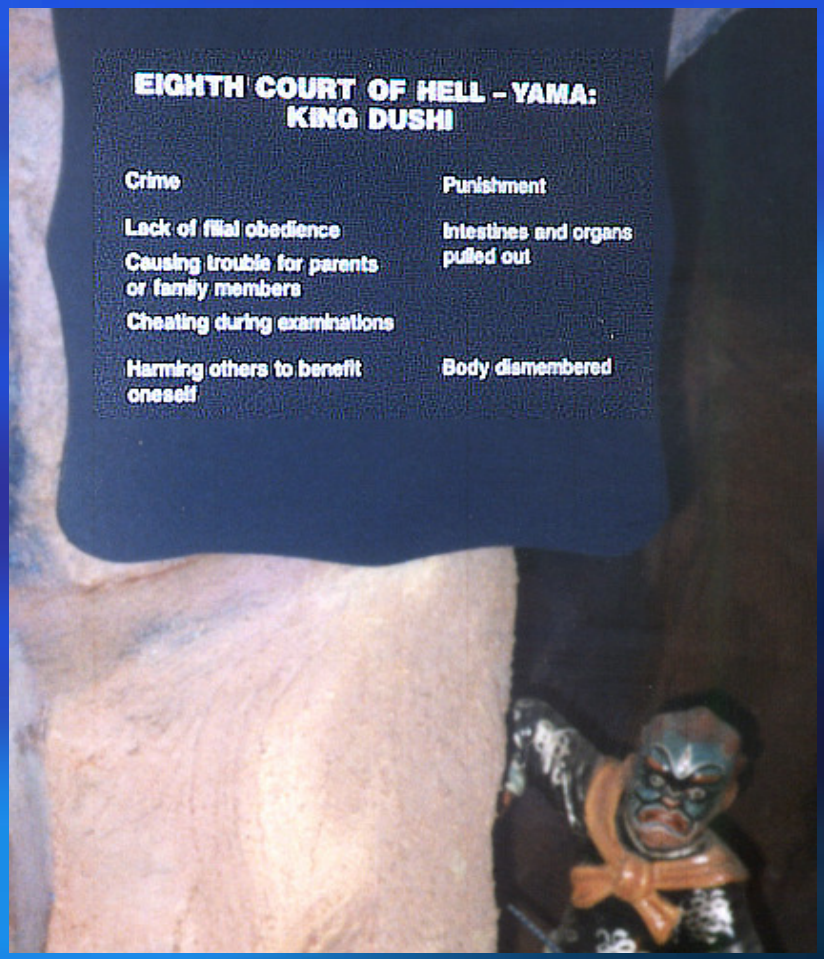
```
      S E N D
+     M O R E
-------------
    M O N E Y
```

# Holy Grail

- Dream of A.I. and declarative programming
  - Freuder, Walsh
  - User describes a problem
  - Computer comes up with a solution

# Constraints and Culture

- Prevalent in many East-Asian cultures
  - Think via constraints
  - You can't do this, can't do that…
  - Heavy punishments for social constraint violation
- Compare versions of hell:



# Real-world applications

- They are everywhere!
  - Delivery before 10am
  - Within 4km radius of Civic
  - Achieve a grade of at least 80
- Car-assembly lines
  - No more than 2 cars in every 5 are red

# Constraint Satisfaction

- In Constraint Satisfaction Problem (CSP), we have:
  - A set of variables (V)
  - Each variable has a set of values (D)
    - Usually finite
    - {true,false}, {red,green,blue}, {0,1,2,3,…}
  - Set of constraints (C)
    - Describe the relationship between the variables
- Goal:
  - Find assignment of each variable to a value such that all constraints are satisfied

# CSP Examples

- Sudoku
  - Variables
    - Each entry in the table $X_{row,col}$
  - Domain
    - Each variable between (1..9)
  - Constraints
    - Row: AllDifferent($X_{x,1},\ldots,X_{x,ncol}$)
    - Column: AllDifferent($X_{1,y},\ldots,X_{nrow,y}$)
    - Square: AllDifferent($X_{1,1},\ldots,X_{3,3}$), etc.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# CSP Examples

- Olympic games scheduling
  - Variables for each event
    - 50mFreeStyleMen, 100mFreeStyleWomen, 10mDivingMen, etc…
  - Domain is the time for the event
    - Monday9am, Tuesday3pm, etc…
  - Constraints:
    - 50mFreeStyleMen != Monday9am
    - Venue: AllDifferent([50mFreeStyleMen, 100mFreeStyleWomen, … ])
    - Capacity: AtMost(3, [50mFreeStyleMen, 100mFreeStyleWomen, …], Tuesday12pm)

# Constraints

- A constraint consists of
  - A list of *n* variables
  - A relation with *n* columns
- Example: a * b = 6
  - (a, b)
  - Relations: see table:

| a | b |
|---|---|
| 1 | 6 |
| 2 | 3 |
| 3 | 2 |
| 6 | 1 |

# Binary/Non-Binary Constraints

- Binary
  - Scope over 2 variables
    - Not Equal: a != b
    - Ordering: a < b
    - Topological: a is disconnected to b
- Non-Binary
  - More than 2 variables
    - AllDifferent(x1,x2,x3,…)
    - $x^2 + 2y^2 - z^2 = 0$

# Non-Binary Constraints

- Most non-binary constraints can be reduced to binary constraints
  - AllDifferent(a,b,c) $\Leftrightarrow$ a!=b, b!=c, a!=c
- Advantages of non-binary constraints:
  - Polynomial algorithms – efficient solving
  - More on this later on!

# Arc-Consistency

- A binary constraint rel(X1,X2) is arc-consistent (AC) iff
  - For every value *v* for X1, there is a consistent value *v'* for X2, and vice versa
  - In this case, *v'* is called the **support** of *v*
- Example:
  - Both x, y are prime numbers, x is less than 10
  - For GreaterThan(x, y):
    - 2,3,5 are all supports of x = 7
    - But 7 is NOT!

# Enforcing Arc-Consistency

- We enforce arc-consistency by deleting domain values that cannot have support
  - As they cannot participate in the solution
- It may remove support for other values
- Complexity: $O(nD^2)$
  - n: number of constraints
  - D: domain size

# Enforcing Arc-Consistency

- Example:
  - X: { 1,2,3,4,5,6,7,8,9,10 }
  - Y: { 3,4,5,6,7 }
  - Z: { 6 }
  - Constraints:
    - X < Y
    - Y >= Z

# Enforcing Arc-Consistency

- Example: (Enforcing Y >= Z)
  - X: { 1,2,3,4,5,6,7,8,9,10 }
  - Y: { 3,4,5,6,7 }
  - Z: { 6 }
  - Constraints:
    - X < Y
    - Y >= Z

# Enforcing Arc-Consistency

- Example: (Enforcing X < Y)
  - X: { 1,2,3,4,5,6,7,8,9,10 }
  - Y: { 3,4,5,6,7 }
  - Z: { 6 }
  - Constraints:
    - X < Y
    - Y >= Z

# Generalized Arc-Consistency

- For CSP with non-binary constraints, it is Generalised Arc-Consistent (GAC) iff:
  - For every variable x in V
  - For every constraint C(x, y1, … , yn)
  - For every value d in D(x)
    - There are values d1, … , dn in D(y1), … , D(yn)
    - Such that C(d, d1, … , dn) is true.
- GAC = AC for binary constraints

# GAC in Action… Sudoku!

- Look at Col 1:
- Enforce: AllDifferent(col_1)
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {8}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {4}

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# GAC in Action… Sudoku!

- Look at Col 1:
- Enforce: AllDifferent(row_1)
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {8}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {4}

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|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# GAC in Action… Sudoku!

- Look at Col 1:
- Enforce: AllDifferent(row_2)
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {8}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {4}

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# GAC in Action… Sudoku!

- Look at Col 1:
- Enforce: AllDifferent(sq_1)
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {8}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {4}

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# GAC in Action… Sudoku!

- Look at Col 1:
- Enforce: AllDifferent(row_4)
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {8}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {4}

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# GAC in Action… Sudoku!

- Look at Col 1:
- Enforce: AllDifferent(row_5)
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {8}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {4}

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# GAC in Action… Sudoku!

- Look at Col 1:
- Enforce: AllDifferent(row_6)
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {8}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {4}

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# GAC in Action… Sudoku!

- Look at Col 1:
- Enforce: AllDifferent(sq_4)
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {8}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {4}

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# GAC in Action… Sudoku!

- Look at Col 1:
- Enforce: AllDifferent(row_7)
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {8}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {4}

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# GAC in Action… Sudoku!

- Look at Col 1:
- Enforce: AllDifferent(row_8)
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {8}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {1,2,3,5,6,7,9}
  - {4}

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | 6 | | 7 | | | 4 |
| | 4 | 2 | | | | | | |
| 8 | 7 | | 3 | | | 6 | | |
| | 8 | | | 7 | | | 2 | |
| | | | 8 | 9 | 3 | | | |
| | 3 | | | 6 | | | 1 | |
| | | 8 | | | 6 | | 4 | 5 |
| | | | | | | 1 | 7 | |
| 4 | | | 9 | | 8 | | 6 | |

# Path-Consistency

- Problem: sometimes arc-consistency is not sufficient
  - Usually involves transitivity of relations
- Example (Arc-consistent problem):
  - A != B, B != C, C != A
  - A: {1,2}, B: {2,3}, C: {2,3}
  - For A=2, there's no solution, but domains are arc consistent
  - Hence, we need path-consistency
    - Also known as (2,1)-consistency

# Path-Consistency

- A binary CSP is Path-Consistent iff:
  - For every pair of variable x, y in V
    - with constraint C(x,y)
  - And every other variable z in V
    - With constraint C(x,z), C(y,z)
  - For every pair d1 in D(x), d2 in D(y)
    - Such that C(d1,d2) is true
  - There is a value d in D(z)
    - Such that C(d1,d) C(d2, d) is true.

# Path-Consistency in Action!

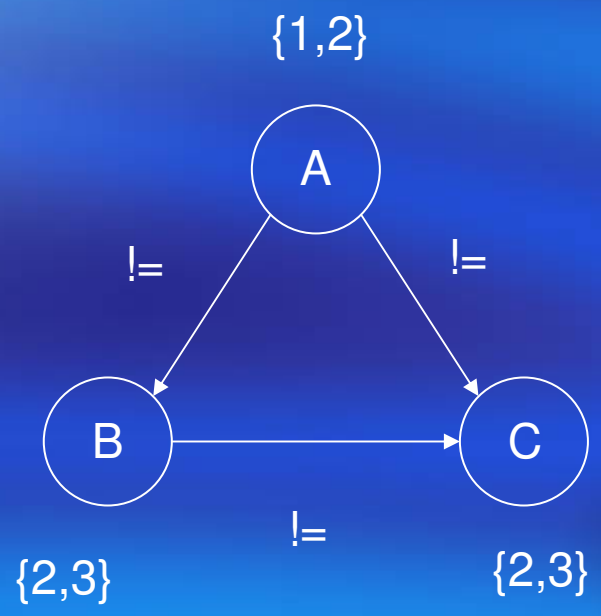
- Consider the problem again:
  - A != B, B != C, C != A
  - A: {1,2}, B: {2,3}, C: {2,3}
- Arc-Consistency:
  - A=2 gets support: B=3, C=3
  - However, this violates path-consistency!

# Path-Consistency

- Path-consistency enforces every constraint work with every other constraint
  - May work with binary constraint network with infinite domains (spatial-temporal reasoning)
- It may still be insufficient, where sometimes 3 constraints must be checked together
  - i,j-consistency

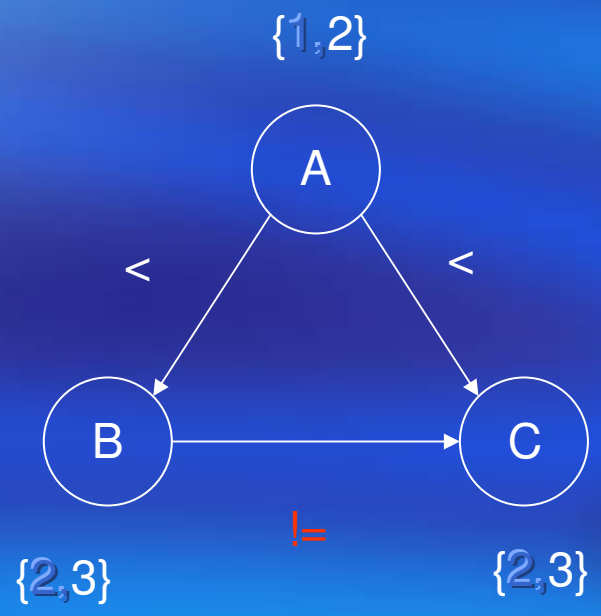
# Path Consistency in Action!

- A different perspective:



# Path-Consistency in Action!

- A different perspective:

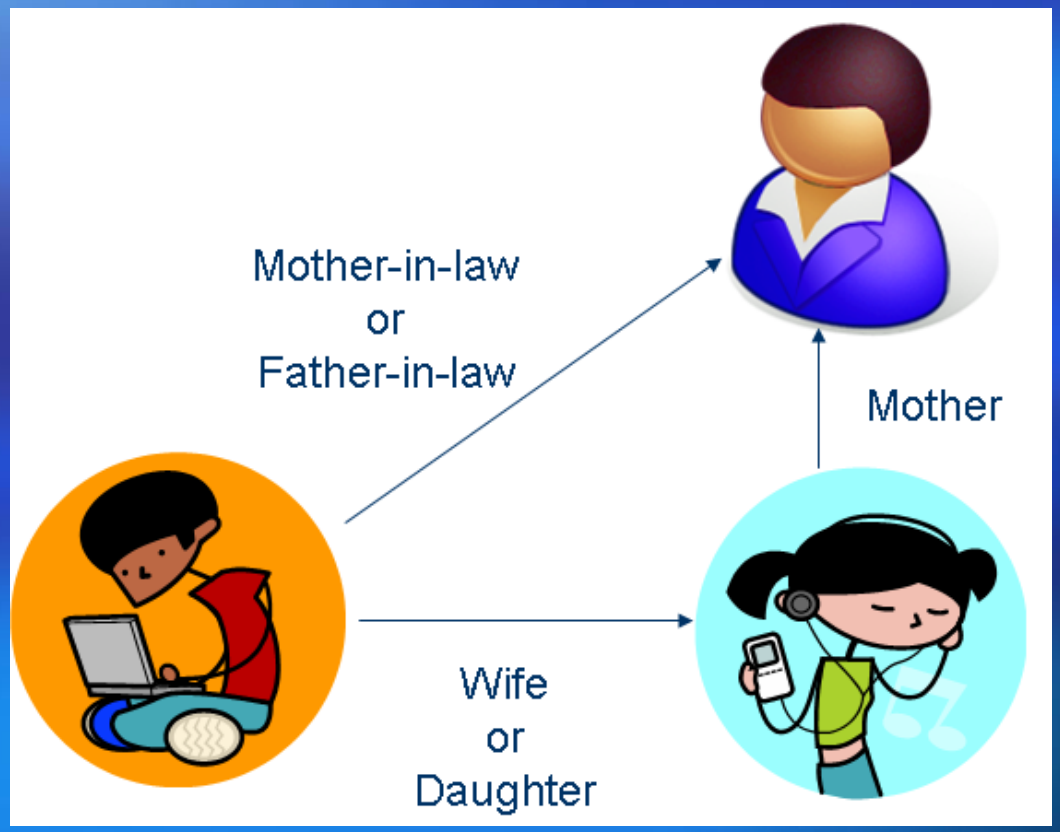


# Infinite Domains

- Path-consistency can also work with constraint networks with infinite domains (Montanari 78, Van Beek, 92)
  - Reason about relations between the entities
  - Complexity is $O(n^3)$, if path-consistency implies consistency
    - Any path-consistent CSP has a realization

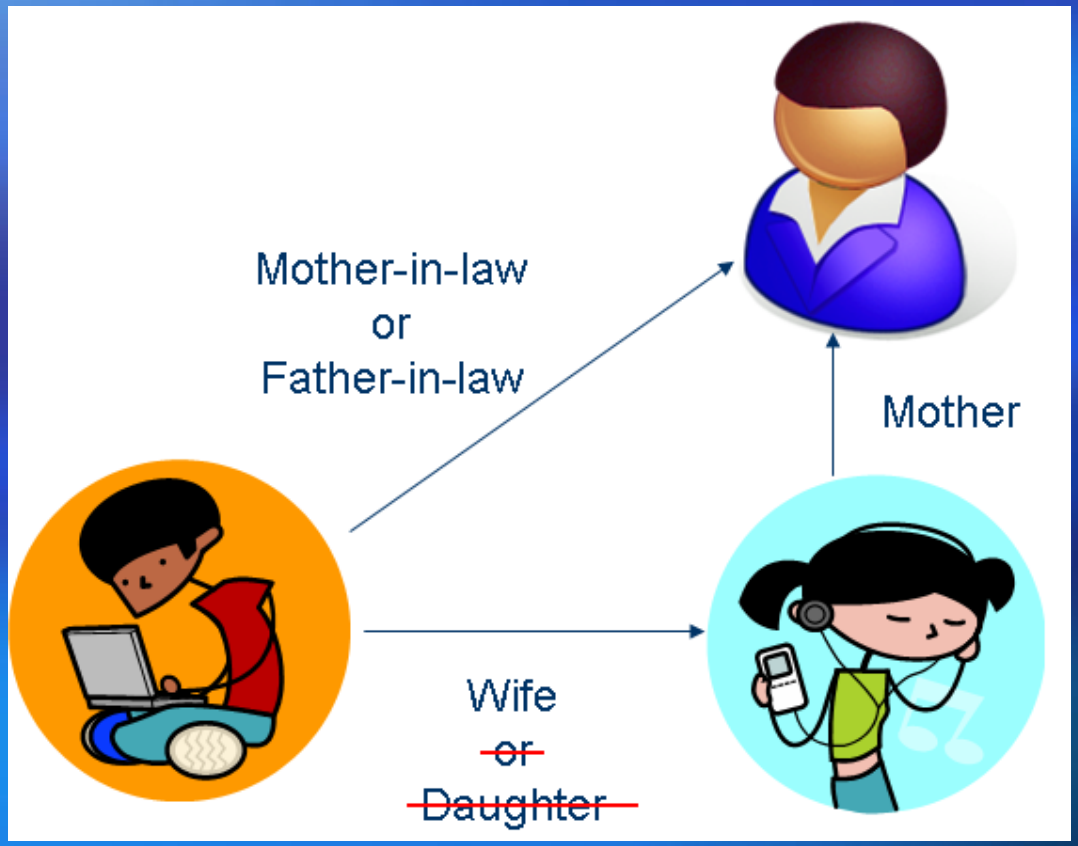
# Path-Consistency

- A confusion of relationships:



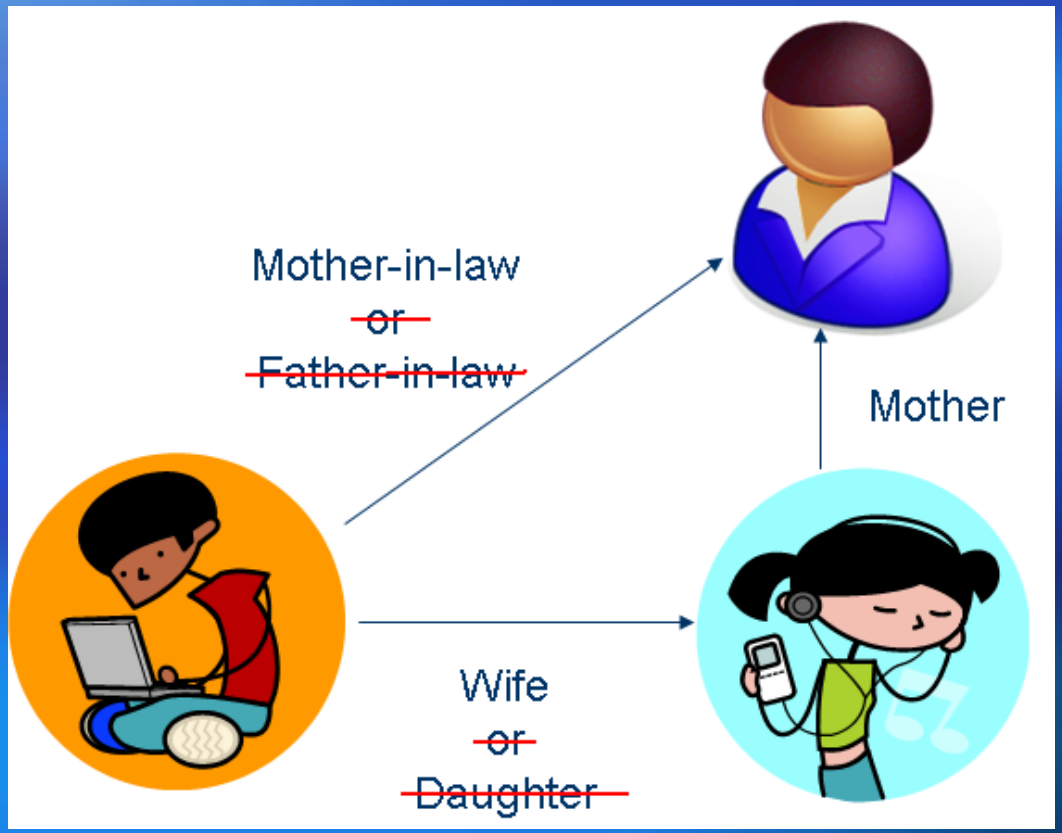
# Path-Consistency

- Applying path-consistency:



# Path-Consistency

- Applying path-consistency one more time:



# Bound-Consistency

- Sometimes, there are a lot of possibilities in a large number of ordered values
  - It's inefficient to check all cases
  - Only the bounds are interesting
- When to use bound consistency?
  - Domain is ordered
  - Only necessary to enforce Arc-Consistency on the max/min elements

# Bound-Consistency

- A CSP is Bound-Consistent if
  - For every x in V and constraint C(x,y1,…,yn)
  - For both d = min(D(x)) and d = max(D(x))
  - There are values d1,…,dn in D(y1),…,D(yn), such that C(d,d1,…,dn) is true.

# Bound-Consistency in Action!

- Linear Inequalities:
  - Constraint: $2X + 3Y <= 10$
  - X: {1,2,3,4,5,6,7,8,9,10}
  - Y: {1,2,3,4,5,6,7,8,9,10}
- Bound-consistent constraints:
  - $\text{UpperBound}(X) <= \lfloor (10 - 3*\text{LowerBound}(Y))/2 \rfloor$
  - $\text{UpperBound}(Y) <= \lfloor (10 - 2*\text{LowerBound}(X))/3 \rfloor$

# Bound-Consistency in Action!

- Linear Inequalities:
  - Constraint: $2X + 3Y <= 10$
- Enforcing
  - $UpperBound(X) <= \lfloor (10 - 3*LowerBound(Y))/2 \rfloor$
  - X: {1,2,3,4,5,6,7,8,9,10}
  - Y: {1,2,3,4,5,6,7,8,9,10}

# Bound-Consistency in Action!

- Linear Inequalities:
  - Constraint: $2X + 3Y <= 10$
- Enforcing
  - $UpperBound(Y) <= \lfloor (10 - 2*LowerBound(X))/3 \rfloor$
  - X: {1,2,3,4,5,6,7,8,9,10}
  - Y: {1,2,3,4,5,6,7,8,9,10}
- Note only 2 bounds needs checking

# Maintaining local-consistency

- Tree Search
  - Assign variable to value
  - Enforce consistency
    - Remove future values / add constraints
  - If no possible values can be assigned, backtrack
- Local search
  - Generate a complete assignment
  - Make changes in violated constraints

# What makes constraints fast

- How the problem is modelled:
- Heuristic-guided search
- Efficient propagation of constraints
  - When enforcing constraints, prune as much as possible, but not at too greater costs

# Trade offs

- Too strong consistency:
  - Too much overhead at each node of the search tree
- Too weak consistency:
  - Not pruning enough of the search space
- Unfortunately, this can only be determined empirically
  - Carry out experiments, see which one does better

# Trade offs

- Propagation vs. Search
  - In general, we don’t want to spend more time enforcing consistency than doing search
  - Problem dependent

# In Summary

- Inference with constraints prunes potential search-space
- A CSP <V,D,C> is consisted of
  - Variables
  - Domains
  - Constraints

# In Summary

- Arc-Consistency removes impossible values for each binary constraint
- Generalized-Arc-Consistency removes impossible values for non-binary constraints
- Path-Consistency removes impossible values between constraints
- Bound-Consistency checks AC for upper and lower bounds in ordered domains

# Search with Constraints

Part 2

# Overview

- What are Global Constraints?
  - Why we use them
  - AllDifferent constraint
  - The Marriage Theorem and Hall Intervals
  - Puget's Algorithm
- What is Symmetry?
  - How to break them

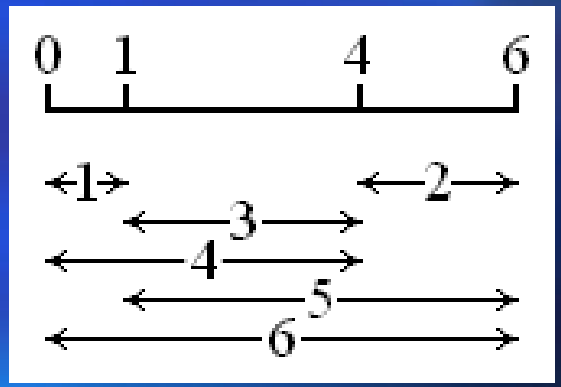
# Global Constraints

- A constraint involving a arbitrary number of variables
  - AllDifferent
  - LexOrder
- Can be modelled with binary constraints
  - E.g. AllDifferent(X1,X2,X3)

    $\Leftrightarrow$ X1!=X2, X2!=X3, X1!=X3
- In practice, they can be more efficiently solved without this decomposition

# Golomb Ruler

- Marking ticks on a ruler
- Unique distance between any two ticks
- Applications
  - X-Ray Crystallography
  - Radio Astronomy
- Problem 006 in CSPLib



# Golomb Ruler

- Naive solution: exponentially long ruler
  - Ticks at 0, 1, 3, 7, 15, 31, 63, 127, 255, etc…
- Key is to find a ruler of minimal length
  - Known for up to 23 ticks
  - Distributed internet project for larger lengths

# Golomb Ruler as CSP

- Explicit Representation
- Variable: $X_i$ for each tick
  - Auxillary Variables: $D_{ij} = |X_i - X_j|$
- Constraints:
  - $X_i < X_j$ for all $i < j$
  - AllDifferent($D_{11}$, $D_{12}$, $D_{13}$, … )
  - Minimize($X_n$)

# AllDifferent Constraint

- One of the oldest global constraints
  - ALICE: [Lauriere 78]
- They are everywhere!
  - Golomb Ruler: AllDifferent($D_{11}$, $D_{12}$, $D_{13}$, … )
  - Standard constraint
  - Incorporated by all constraint solvers

# AllDifferent Constraint

- Can be modelled with binary constraints
  - AllDifferent(X1,X2,X3)

    $\Leftrightarrow$ X1!=X2, X2!=X3, X1!=X3
- However, this may be done more efficiently
  - X1: {1,2}, X2: {1,2}, X3: {1,2,3}
  - X3 can never be 1 or 2
  - How can we exploit this?
- Efficient algorithms:
  - [Puget AAAI'98] Bound Consistency Algorithm runs in O(n log n)

# Bound Consistency with AllDifferent

- Uses Hall's Theorem
  - Also termed "Marriage Theorem"
  - Given *k* sets
  - There is an unique and distinct element in each set iff
    - For 0<j<=k
    - Any union of j of the sets has at least j elements
- Example:
  - X1: {1,2}, X2: {1,2} is okay
  - X1: {1,2}, X2: {1,2}, X3: {1,2} will not work.

# Marriage Theorem

- There are n men and women in a town…
  - Each man is happy to be married to any woman
  - Each woman has some preferred men to marry (subset of all men)
- Given *j* women…
  - The number of men they wish to marry must be *j* or more!

# Following Hall's (Marriage) Theorem

- Hall Interval
  - Interval of domain values that has just as many variables as domain values
  - E.g. X1:{1,2}, X2:{1,2}
    - Two variables in the interval {1,2}
- AllDifferent is Bound-Consistent iff:
  - Each interval in the domain do not cover more variables than its length (Hall Interval)
  - A variable with possible domain value outside a Hall Interval do not have value within it

# Bound-Consistency in Action!

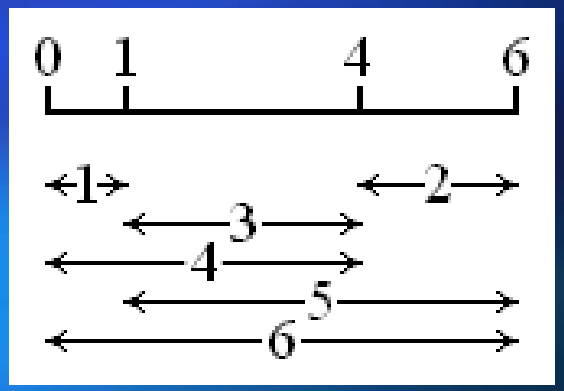
- Consider our old example:
  - X1: {1,2}, X2: {1,2}, X3: {1,2,3}
  - AllDifferent(X1,X2,X3)
- Obviously:
  - [1…2] is a Hall Interval covering X1, X2
  - X3 has a value outside a Hall Interval, therefore we prune it
  - Result is Bound-Consistent:
    - X1:{1,2}, X2: {1,2}, X3: {3}

# Puget's Algorithm

- Naive implementation consider $O(n^2)$ intervals
- Puget order the intervals in $O(n \log n)$ time
  - Then go through them in order
- Best BC-Algorithm for AllDifferent
  - Still, in some problems GAC can do better
    - Problems closer to combinatorial problems

# Further problem in Golomb Ruler

- Problem: there are trivial repetitions of the same solution in the search space.
  - Ruler 1: Ticks at: 0, 1, 4, 6
  - Ruler 2: Ticks at: 0, 2, 5, 6
- Is there any fundamental difference between the above two rulers?



# Symmetry

- Symmetries occur frequently in Constraint Programming and Search
  - Any permutations of rows or columns of a table
  - Real-world scheduling problems
- It's a very active area of CP research

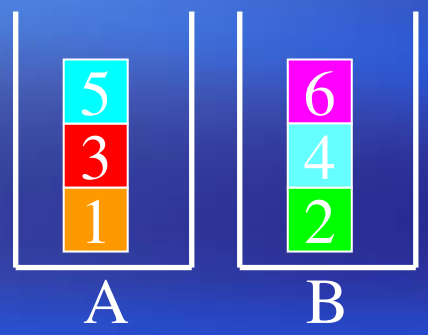
# Breaking Symmetry

- In Golomb Ruler, we ensure the ruler cannot be reversed.
- Easiest way to break symmetry is to add additional constraints:
  - $D_{12} < D_{n-1,n}$
- Another symmetry: preventing permutations in the ticks:
  - $X_1 < X_2 < \ldots X_n$

# Another Example of Symmetry

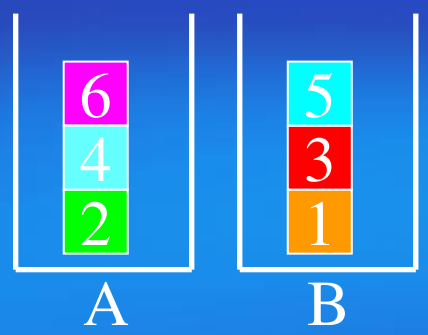
- Consider two bins:
  - Example of row-symmetry (Walsh)

a)




| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 1 | 0 | 1 | 0 | 1 | 0 |
| B | 0 | 1 | 0 | 1 | 0 | 1 |

b)

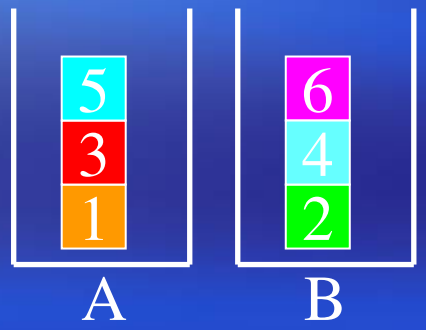



| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 0 | 1 | 0 | 1 | 0 | 1 |
| B | 1 | 0 | 1 | 0 | 1 | 0 |

# Another Example of Symmetry

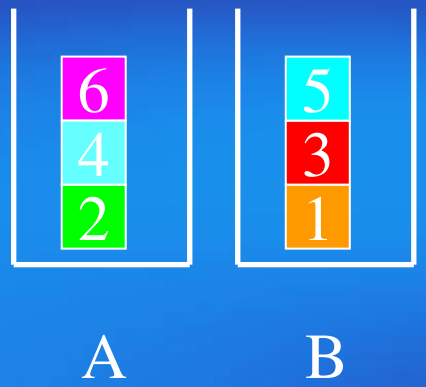
- Such symmetry can be broken by lexicographical constraints
  - Row(A) <=LEX Row(B)

a)




| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 1 | 0 | 1 | 0 | 1 | 0 |
| B | 0 | 1 | 0 | 1 | 0 | 1 |

b)




| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 0 | 1 | 0 | 1 | 0 | 1 |
| B | 1 | 0 | 1 | 0 | 1 | 0 |

# Breaking Symmetry

- Symmetries hide in all kind of problems
- They are expensive!
  - Therefore breaking them is essential in making CP/Search efficient
- General methods may add exponential number of constraints! [Crawford, Ginsberg and Luks]
- Other way of breaking symmetry
  - Ignore them in search strategy

# Breaking Symmetry

- Two major strategies in modifying search strategy:
  - During Search:
    - After exploring a branch, implicitly add constraints preventing exploring other symmetric branches
  - Dominance Detection:
    - Before exploring a branch, check if it is dominated by a previously visited branch

# Summary

- Global Constraints are efficient with clever algorithms
  - AllDifferent and Marriage Theorem
- Symmetries can surface in many problems
  - They can be broken by adding constraints
    - Most times lexicographical constraints
  - They can also be broken by modifying search strategies

# Get your hands dirty!

- Most constraint solvers have two layers
  - High level: non-expert user input
  - Low level: for computer/expert user
- They come with easy examples!
- Try some yourself!
  - MiniZinc/FlatZinc:
    - http://www.g12.csse.unimelb.edu.au/minizinc/
  - Tailor/Minion:
    - http://www.dcs.st-and.ac.uk/~andrea/tailor/index.html
  - GeCode:
    - http://www.gecode.org/