Minimizing Energy Functions on 4-Connected Lattices Using Elimination

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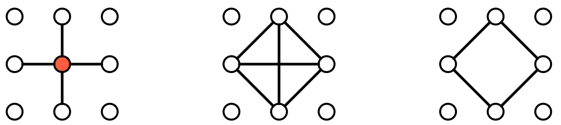
Background

- Many Markov Random Field (MRF) vision problems defined as 4-connected lattices. Few algorithms exploit this structure.
- Max-flow finds optimal labelling of submodular functions.
- **Key Idea**: Eliminate variables until function is submodular; solve using max-flow.
- Successive elimination possible using approximation.



Original Elimination Approximation

Elimination

- **Goal**: Find minimum of $f : \mathcal{B}^{n+1} \rightarrow \mathbb{R}$.
- Eliminate x_0 by specifying its optimal value $x_0^\star$ as function of remaining variables $x_1, \ldots, x_n$.

$$\min_{x_0,\ldots,x_n} f(x_0,\ldots,x_n) = \min_{x_1,\ldots,x_n} \min_{x_0} f(x_0,\ldots,x_n)$$
$$= \min_{x_1,\ldots,x_n} f(x_0^\star,\ldots,x_n)$$
$$= \min_{x_1,\ldots,x_n} f_1(x_1,\ldots,x_n)$$

- Continue until only $f_n(x_n)$ remains. Compute $x_n^\star$ and substitute into expression for $x_{n-1}^\star$. Repeat for other variables.

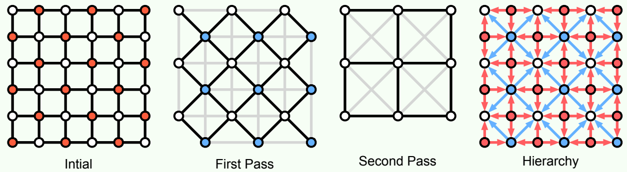
... on a 4-connected lattice

- Energy function of the form:

$$f(x_0,x_1,\ldots,x_n) = \sum_i a_i x_i + \sum_{i,j} b_{ij} x_i x_j$$

- $x_i^\star$ determined by sign of derivative $\nabla_i(\cdot)$ (see Example).
- If $x_i^\star \nabla_i(\cdot)$ does not preserve 4-connected lattice, successive eliminations involve more than four variables; problem becomes intractable.
- Preserve lattice by approximating $x_i^\star \nabla_i(\cdot)$ (see Example) and eliminating variables in checkerboard fashion (see Figure).

Recursive Checkerboard



Variables eliminated in a checkerboard fashion. Each pass rotates lattice by 45°. After second pass, lattice returns to original orientation with ¼ variables. Interactions ignored from approximation are shown in grey. Hierarchy (red→blue→white) illustrates how vertex and edge weights are passed down from eliminated variables. For example, in LazyElimination, a blue vertex must first eliminate its red predecessors.

LazyElimination

- Energy function submodular if every $b_{ij} \leq 0$.
- Only eliminate variables with positive incident edge weights.
- Recursive checkerboard defines hierarchy (see Figure). Predecessors must be eliminated first, since they determine edge weights of successors.

Example Elimination

$$f(x_i, x_u, \ldots, x_\ell) = -x_i + 4x_i x_u + 2x_i x_r - x_i x_d - 3x_i x_\ell$$

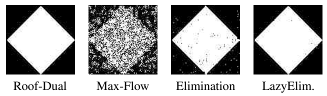
x_u	x_r	x_d	x_ℓ	$\nabla_i(\cdot)$	$x_i^\star$	$x_i^\star\nabla_i(\cdot)$	$\approx x_i^\star\nabla_i(\cdot)$
0	0	0	0	−1	1	−1	−1.375
0	0	0	1	−4	1	−4	−3.875
0	0	1	0	−2	1	−2	−1.875
0	0	1	1	−5	1	−5	−4.875
0	1	0	0	1	0	0	0.125
0	1	0	1	−2	1	−2	−2.375
0	1	1	0	0	1	0	0.125
0	1	1	1	−3	1	−3	−2.875
1	0	0	0	3	0	0	0.125
1	0	0	1	0	1	0	0.125
1	0	1	0	2	0	0	−0.375
1	0	1	1	−1	1	−1	−0.875
1	1	0	0	5	0	0	0.125
1	1	0	1	2	0	0	0.125
1	1	1	0	4	0	0	0.125
1	1	1	1	1	0	0	−0.375

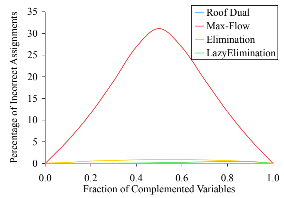
- $\nabla_i(\cdot) = -1 + 4x_u + 2x_r - x_d - 3x_\ell$
- *Enumerate sixteen combinations of* x_u, x_r, x_d *and* x_ℓ.
- $x_i^\star = \begin{cases} 0 & \text{if } \nabla_i(\cdot) > 0, \\ 1 & \text{otherwise.} \end{cases}$
- *True* $x_i^\star \nabla_i(\cdot)$ *has only non-positive terms, and is not representable as a 4-connected lattice.*
- *Approximate* $x_i^\star \nabla_i(\cdot)$ *by fitting a particular polynomial using least squares.*

$$x_i^\star \nabla_i(\cdot) = -1 + x_u + x_r - x_d - 3x_\ell - x_u x_r + x_r x_d + 3x_u x_\ell + x_r x_\ell$$
$$+ x_u x_d - x_u x_d x_r - x_u x_d x_\ell - x_u x_r x_\ell - x_r x_d x_\ell + 2x_u x_r x_d x_\ell$$
$$\approx -1.375 + 1.5x_u + 1.5x_r - 0.5x_d - 2.5x_\ell - 1.5x_u x_r$$
$$+ 0.5x_r x_d - 0.5x_d x_\ell + 2.5x_\ell x_u$$

Experiments

- Generate submodular pseudo-boolean function, and complement random subset of variables.
- Function is not submodular, but an optimal labelling can be found with roof-dual.



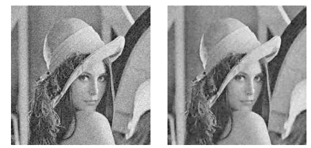


- Extended to real multilabel problems using α-expansion

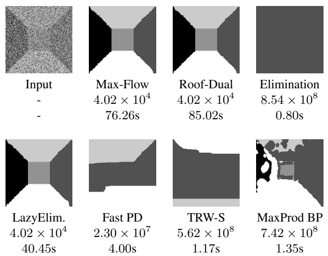
Denoising

- Add synthetic Gaussian noise to greyscale image. Truncated quadratic cost function for both data and smoothness terms.

Algorithm	Energy ($\times 10^5$)	Time (s)	RMS Error
Max-Flow	8.0006	10.70	2.37
Roof-Dual	8.0002	17.81	2.37
Elimination	8.0103	10.05	2.39
LazyElimination	8.0836	14.43	2.40
Fast PD	8.4940	8.23	2.64
TRW-S	7.9761	176.13	2.36
MaxProd BP	8.5962	152.18	2.62



Corridor Labelling



- Identify the floor and ceiling as well as the left, right and far walls.
- Incorporate spatial constraints into $E_{ij}(x_i, x_j)$ to avoid impossible solutions.
- The high cost assigned to an invalid labelling distorts the approximation step of the Elimination algorithm.

Conclusions

- LazyElimination achieves excellent results on arbitrary pseudo-boolean functions defined on a 4-connect lattice.
- It is able to obtain quality solutions to multilabel problems using α-expansion.
- It is faster (or on par) with all but FastPD.