Update-based Maximum Column Distance Coding Scheme for Index Coding Problem

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Abstract—In this paper, we propose a new scalar linear coding scheme for the index coding problem called update-based maximum column distance (UMCD) coding scheme. The central idea in each transmission is to code messages such that one of the receivers with the minimum size of side information is instantaneously eliminated from unsatisfied receivers. One main contribution of the paper is to prove that the other satisfied receivers can be identified after each transmission, using a polynomial-time algorithm solving the well-known maximum cardinality matching problem in graph theory. This leads to determining the total number of transmissions without knowing the coding coefficients. Once this number and what messages to transmit in each round is found, we then propose a method to determine all coding coefficients from a sufficiently large finite field. We provide concrete instances where the proposed UMCD scheme has a better broadcast performance compared to the most efficient existing linear coding schemes, including the recursive scheme (Arbabjolfaei and Kim, 2014) and the interlinked-cycle cover scheme (Thapa et al., 2017).

I. INTRODUCTION

In this paper, we consider an efficient communication system where a single server broadcasts $m$ messages to $n$ receivers through a noiseless channel. Each receiver requests one specific message from the server and may have prior knowledge of some other messages as its side information. The side information of the receivers can be exploited by the server to decrease the number of coded messages in order to communicate the messages. This problem was first introduced by Birk and Kol [1] in the context of satellite communication and is known as the index coding problem. The main objective is to minimize the overall number of transmissions so that all receivers can decode their desired message. Index coding problem provides a simple, yet rich model for other communication scenarios, such as network coding [2], [3], coded caching [4], [5], distributed storage [6], and topological interference management [7], [8].

Index coding problem has so far been extensively studied in the literature and various coding schemes have been proposed to solve this problem. Although the min-rank optimization scheme [8] can determine the optimal linear code for an index coding problem, other efficient linear coding methods have also attracted considerable attention due to their lower computational complexity compared to the min-rank scheme.

In the partial-clique-cover (PCC) scheme [1], the problem is partitioned into subproblems and then the maximum distance separable (MDS) code is employed to solve each subproblem. Time-sharing over the solution of each subproblem brings about a vector version of the scheme, called fractional-partial-clique cover (FPCC) scheme [1], which can offer a lower broadcast rate than the PCC scheme for many index coding instances. In the recursive coding scheme [9], the problem is recursively partitioned into subproblems and local time-sharing is applied at every stage, which strictly improves upon the FPCC scheme. The interlinked-cycle cover (ICC) scheme [10] decomposes the problem into subproblems possessing an interlinked-cycle structure, where each subproblem is solved using an interlinked-cycle code. Fractional ICC can outperform the recursive code for some instances. However, neither the recursive coding scheme nor the ICC scheme outperforms each other in general.

In this paper, we tackle the index coding problem from a different perspective. In the beginning, receivers are sorted according to the size of their side information. A linear combination of the messages is designed to instantaneously satisfy one of the receivers with the minimum size of side information. Then, the problem is updated by removing all receivers which can decode their requested message using their side information along with the coded messages received so far. This process is repeated until all receivers can successfully decode their requested message.

To design the update-based scheme, the following two questions must be addressed at each step of the transmission. How to design the coding coefficients of the messages? And how to determine whether a receiver can decode its requested message from the information available to it?

Similar to the MDS-based index coding schemes, the proposed update-based code can be employed as a modular code for solving subproblems. Thus, for the former question, the coefficients of the messages are designed so that the encoder matrix can achieve the maximum distance feature of the MDS generator matrix to the extent possible. In other words, the coefficients are constructed so as to make the columns of the encoder matrix linearly independent of the linear space spanned by other columns as much as possible. That is why our proposed coding is called update-based maximum column distance (UMCD) scheme. For the latter question, this linear independence property is used to prove that the problem of identifying the receivers who are able to decode their requested message at each stage of transmission is equivalent to a well-known problem in graph theory called the maximum cardinality matching (MCM) problem which can be solved in polynomial-time using the Hopcroft-Karp algorithm [11]. This leads to determining the broadcast rate of the proposed UMCD scheme independent of knowing the exact coefficients of the encoder matrix. This, in turn, results in reducing the computational complexity of the scheme, which is especially important when the UMCD code is applied to subproblems.
A. Summary of Contributions

1) We propose a new scalar linear coding scheme, namely the UMCD scheme in Algorithm 1. We show concrete instances where it outperforms the recursive and ICC coding schemes.

2) We prove that the satisfied receivers in each transmission can be identified using a polynomial-time algorithm solving maximum cardinality matching (MCM) problem without any knowledge of coding coefficients. This requires each column of the encoder matrix to be linearly independent of the space spanned by other columns as much as possible.

3) In Algorithm 2, which we call the maximum column distance (MCD) algorithm, we propose a new method to generate the elements of the encoder matrix so that it meets the aforementioned linear independence requirement.

The rest of the paper is organized as follows. In Section II, we review the system model and scalar linear index code definition. In Section III-B, we propose the UMCD coding scheme and provide some instances to show its efficiency over the recursive and ICC coding schemes in terms of the broadcast rate. In Section IV, we prove the equivalence of identifying the satisfied receivers problem and the MCM problem. Section V presents the MCD algorithm. Finally, Section VI concludes the paper.

II. SYSTEM MODEL AND BACKGROUND

A. Notation

Scalar small letters such as $n$ denote an integer number where $[n] := \{1, ..., n\}$. Scalar capital letters such as $L$ denote a set, whose cardinality is denoted by $|L|$ and power set is denoted by $\mathcal{P}(L)$. Symbols in bold face such as $\mathbf{L}$ and $\mathbf{L}$ denote a vector and a matrix, respectively, with $|L|$ and $\mathbf{L}^T$ denoting the determinant and transpose of matrix $\mathbf{L}$, respectively. A calligraphic symbol such as $\mathcal{L}$ is used to denote a set whose elements are sets.

We use $\mathbb{F}_q$ to denote a finite field of size $q$ and write $\mathbb{F}_q^{n \times m}$ to denote the vector space of all $n \times m$ matrices over the field $\mathbb{F}_q$. Given a matrix $\mathbf{L} \in \mathbb{F}_q^{n \times m}$ with elements $l_{i,j} \in \mathbb{F}_q$, the $i$th row and $j$th column are denoted by $l_i$ and $l_j$, respectively. We write $l_{i,j} \leftrightarrow l_{i,j}$ to denote swapping the rows $i_1$ and $i_2$ and $j_1$ and $j_2$. For any subset $S \subset [n]$ and $V \subset [m]$, we use $\mathbf{L}_S$ and $\mathbf{L}^V$ to represent the $|S| \times m$ and $n \times |V|$ submatrices of $\mathbf{L}$ comprised of the rows of $\mathbf{L}$ indexed by $S$ and the columns of $\mathbf{L}$ indexed by $V$, respectively. More generally, $\mathbf{L}^S_V$ denotes the submatrix of size $|S| \times |V|$ of $\mathbf{L}$ comprised of the rows and columns indexed by $S$ and $V$, respectively.

B. System Model

Consider a broadcast communication system in which a server transmits the $m$ messages $X = \{x_1, x_2, ..., x_m\}$, $x_i \in \mathbb{F}_q$ to a number of receivers $U = \{u_1, u_2, ..., u_m\}$ via a noiseless broadcast channel. Each receiver $u_i, i \in [m]$ requests the specific message $x_i$, and may already know a subset of the messages $X_i := \{x_j : j \in A_i\}, A_i \subseteq [m] \setminus \{i\}$, which is referred to as its side information set. The main goal is to minimize the number of coded messages that the server requires to broadcast to the receivers so that each receiver is able to decode its requested message. Throughout the paper, an index coding problem is represented by $(1|A_1), ..., (m|A_m)$ and the set of interfering messages for receiver $u_i$ is denoted by $B_i = [m] \setminus \{\{i\} \cup A_i\}$.

Definition 1 (Scalar Linear Index Code). Given an index coding problem, a scalar linear index code is defined as $\mathcal{C} = (\mathbf{H}, \{\psi_i\})$, where

- $\mathbf{H} : \mathbb{F}_q^{m \times 1} \rightarrow \mathbb{F}_q^{r \times 1}$ is the encoder matrix which maps the message vector $x := [x_1, x_2, ..., x_m]^T \in \mathbb{F}_q^{m \times 1}$ to a coded message vector $y = [y_1, ..., y_r]^T \in \mathbb{F}_q^{r \times 1}$ as follows

$$y = \mathbf{H}x.$$

- $\psi_i$ represents the decoder function for receiver $u_i, i \in [m]$, where $\psi_i(y, X_i) = x_i$, which is able to correctly decode message $x_i$.

Proposition 1. The necessary and sufficient condition for decoder $\psi_i, \forall i \in [m]$ to be able to correctly decode message $x_i$ is

$$\text{rank } \mathbf{H}^{(1) \cup B_i} = \text{rank } \mathbf{H}^{B_i} + 1. \quad (1)$$

Proof. For the if condition, we suppose that (1) holds. Now, we show that there is a decoder function $\phi_i(y, X_i) = x_i$, which is able to correctly decode $x_i$ as follows

$$y - \sum_{j \in A_i} h^i_j x_j = \sum_{j \in [m]} h^i_j x_j - \sum_{j \in A_i} h^i_j x_j = h^i_j x_i + \sum_{j \in B_i} h^i_j x_j. \quad (2)$$

Let rank $\mathbf{H}^{B_i} = s_i$, then we partition $B_i = S_i \cup (B_i \setminus S_i)$ such that $|S_i| = s_i$, and $\{h^i_j : j \in S_i\}$ represents the set of columns of $\mathbf{H}$ which are linearly independent. Then, for all $l \in (B_i \setminus S_i)$, we have

$$h^l = \sum_{j \in S_i} p_{l,j} h^j,$$

where, $p_{l,j} \in \mathbb{F}_q, \forall j \in S_i$. So, (2) will be equal to

$$h^l x_i + \sum_{j \in S_i} h^l (x_j + \sum_{l \in B_i \setminus S_i} p_{l,j} x_l) \quad (3)$$

Since (1) holds, then $\{h^l\} \cap \{h^j : j \in S_i\}$ are the column space basis for $\mathbf{H}^{(1) \cup S_i}$. Hence, the set of messages $\{x_l\} \cup \{x_j \cup \sum_{l \in B_i \setminus S_i} p_{l,j} x_l : j \in S_i\}$ can be decoded by receiver $u_i$, which completes the proof for the if condition.

Conversely, according to the polymatroidal bound [12], in order to decode all the messages correctly for the described system model, the following constraint must be met by any polymatroidal function $f : 2^{[m]} \rightarrow r$

$$f(\{i\} \cup B_i) - f(B_i) \geq 1, \quad \forall i \in [m].$$

Now, by setting $f(L) = \text{rank } (\mathbf{H}^L)$, we must have

$$\text{rank } \mathbf{H}^{(1) \cup B_i} \geq \text{rank } \mathbf{H}^{B_i} + 1,$$

which will complete the proof because the equality is the sufficient condition as well. \qed
Definition 2 (Broadcast Rate). The broadcast rate of scalar linear code $C$ satisfying the decoding condition in (1) for all receivers is defined as $\beta(C) = r$.

III. MAIN RESULT

A. A Brief Discussion of The MCM problem and the MCD Algorithm

Consider a matrix $H$ such that some of its elements are fixed to zero, while other elements can be selected arbitrarily. A binary matrix $G$ can be associated to $H$ as follows

$$g_{h,j} = \begin{cases} 0 & \text{if } h_{i,j} = 0, \\ 1 & \text{if } h_{i,j} \neq 0. \end{cases}$$

(4)

Now, consider the following optimization problem

$$\max_{\forall h_{i,j} \neq 0} \text{rank } H,$$

(5)

which gives the maximum rank of $H$ over all possible values for its nonzero elements.

In Section IV, we prove that the solution of (5) is equal to the MCM of the bipartite graph associated with the binary matrix $G$, denoted by $\text{mcm}(G)$. In other words,

$$\text{mcm}(G) = \max_{\forall h_{i,j} \neq 0} \text{rank } H.$$ 

The MCM problem can be solved by the polynomial-time Hopcroft-Karp algorithm where its complexity for $G_{k \times m}$ is at most $O(k^2 m)$.

In Section V, we propose the MCD algorithm, which designs the nonzero elements of $H$ such that

$$\text{rank}(H^L_{[k]}) = \text{mcm}(G^L_{[k]}), \quad \forall L \in \mathcal{P}([m]).$$

Thus, the condition in (1) will be equivalent to

$$\text{mcm}(G^{(1)}_{[k]} \cup B_1) = \text{mcm}(G^{B_1}_{[k]}) + 1,$$

(6)

which means that, the satisfied receivers in each transmission can be determined just by having access to the binary matrix $G$. This leads to achieving the broadcast rate of the UMCD scheme without knowing the exact value of the nonzero elements of $H$. This results in reducing the computational complexity, especially when the UMCD is used as a basic code for solving index coding subproblems.

B. The Proposed UMCD Coding Scheme

In this section, we describe the proposed UMCD coding scheme, which is a scalar linear code and is based on updating the problem after each transmission. In this scheme, in the $k$-th transmission, the server designs a linear combination of the messages indexed by subset $G_k \subseteq [m]$ (i.e., $y_k = \sum_{i \in G_k} h_{k,i} \cdot y_i$, $h_{k,j} \in \mathbb{F}_q$), and then removes from the problem those receivers which are capable of decoding their desired message from $y_1, \ldots, y_k$ using their side information. Note that, a binary matrix $G$ can be built by setting its $k$-th row as $y_{k,j} = 1, \forall j \in G_k$ and $y_{k,j} = 0$, otherwise.

To characterize the UMCD scheme, the following three questions need to be addressed. (i) Which messages should we include in the coded message $y_k$? That is, how should we determine the subset $G_k$? (ii) How can we efficiently identify receivers that can decode their desired message upon receiving the first $k$ transmissions? (iii) How should we determine the message coefficients $h_{k,i}, \forall i \in G_k$?

- Regarding question (i), in transmission $k$, the UMCD scheme aims at satisfying one of the receivers with the minimum size of side information by sending a linear combination of its desired message and the messages in its side information. This will give set $G_k$ which will, in turn, determine the $k$-th row of $G$.
- Regarding question (ii), assuming that the nonzero elements of $H$ will be suitably designed by the MCD algorithm, the satisfied receivers can be determined by checking the condition in (6) using the polynomial-time Hopcroft-Karp algorithm which solves the MCM problem.
- Finally, for question (iii), the nonzero elements of the encoder matrix $H$ will be determined by the proposed MCD algorithm.

C. Description of the UMCD Algorithm

In the UMCD algorithm, which is provided in Algorithm 1, $N$ denotes the set of unsatisfied receivers which is updated after each transmission. At the beginning, all the receivers are considered to be unsatisfied and $N = [m]$. In transmission $k$, let $W = \arg\min_{i \in N} |A_i|$ represent the indices of the unsatisfied receivers with the minimum size of side information. Then, one element, denoted by $w$ is chosen randomly from $W$. Now, to satisfy receiver $u_w$, UMCD designs a linear combination of the messages indexed by $G_k = \{w\} \cup A_w$, which determines the $k$-th row of the binary matrix. Then, using the Hopcroft-Karp algorithm, the receivers satisfying the condition in (6) are identified and removed from $N$. Note that at each iteration, receiver $u_w$ is guaranteed to be removed from $N$, since $G_k$ only contains what $u_w$ wants to decode and what it knows. The instant decodability for one of the remaining receivers with the minimum size of side information is a key to reducing the overall broadcast rate in an adaptive manner. This process continues until all the receivers are satisfied. Finally, having obtained the binary matrix $G$, the encoder matrix $H$ is constructed using the MCD algorithm. It is worth noting that the broadcast rate $\beta_{\text{UMCD}}$ is obtained independent of knowing the exact value of the elements of the encoder matrix $H$.

D. UMCD Scheme Can Outperform Recursive and ICC Schemes

Example 1. Consider the instance of index coding problem $(1,3), (2,4), (3,2,5), (4,1,5), (5,3,4)$, where $N = [5]$. In the first round $k = 1$, UMCD begins with one of the receivers indexed by $W = \{1, 2\}$ which have the minimum size of side information. Let $w = 1 \in W$ be chosen. Then, the first transmission is a linear combination of the messages indexed by $G_1 = \{1\} \cup A_1 = \{1, 3\}$, which means

$$G_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$ 

Using the Hopcroft-Karp algorithm, (6) holds only for receiver 1, and so $N = [5] \setminus \{1\}$. Then, for the second round $k = 2$, we have $W = \{2\}, w = 2, G_2 = \{2\} \cup A_2 = \{2, 4\}$, and
Algorithm 1: UMCD Coding Scheme

Input: $A_1, \ldots, A_n$
Output: $\beta_{\text{UMCD}}$ and $H$

initialization:
$k = 0;$

$N = |m|;$

set $G$ as a full-zero matrix of size $m \times m;$

while $N \neq \emptyset$ do

$k \leftarrow k + 1;$

$W = \arg \min_{i \in N} |A_i|;$

choose an element $w \in W$ at random;

set $G_k = \{w\} \cup A_w;$

set $g_{k,j} = 1, \forall j \in G_k;$

for $i \in N$ do

if mcm($G_{[k]}^{(i)} \cup B_i$) = mcm($G_{[k]}^{B_i}$) + 1 then

$N \leftarrow N \setminus \{i\};$

end

end

$\beta_{\text{UMCD}} = k$: Broadcast rate;

$H = \text{MCD}(G_{[k]})$: Encoder matrix;

$G_{[2]} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$

Using the Hopcroft-Karp algorithm, (6) holds only for receiver 2, and so $N = \{5\} \setminus \{1, 2\}$. For the third round $k = 3$, we have $W = \{3, 4, 5\}$, let $w = 5 \in W$, $G_3 = \{5\} \cup A_5 = \{3, 4, 5\}$, and

$G_{[3]} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$

It can be verified using the Hopcroft-Karp algorithm that for all the remaining receivers, (6) will be satisfied, which leads to $\beta_{\text{UMCD}} = 3 < \beta_{\text{Recursive}} = 3.5$. Finally, the encoder matrix will be determined by the MCD algorithm in the binary field $F_2$.

Hence, $H_{[3]} = G_{[3]}$. Note that for the other possible random selections of $w_k$, it can be checked that the broadcast rate will be the same.

Example 2. For the index coding problem ($1|2, 3, 5), (2|1, 3, 5), (3|2, 4, 5), (4|2, 3, 5), (5|1, 4)$ it can be verified that $\beta_{\text{UMCD}} = 2 < \beta_{\text{ICC}} = 2.5$.

IV. THE EQUIVALENCE OF IDENTIFYING SATISFIED RECEIVERS PROBLEM AND MAXIMUM CARDINALITY MATCHING PROBLEM

In this section, we give a description of the relation between finding the satisfied receivers and the MCM problem which will be based on Theorem 1. In the following, we assume that some of the elements of $H$ are predetermined to zero, but other elements can be selected arbitrarily, and the binary matrix $G$ is associated to $H$ according to (4).

Theorem 1. Matrix $H_{k \times k}$ can be designed to be full-rank, if and only if for its associated binary matrix $G$, there exists a permutation of columns which can result in $g_{i,i} = 1, \forall i \in [k]$.

To prove the converse of Theorem 1, first, we provide Lemma 1 as below.

Lemma 1. Given $G_{k \times k}$, if $g_{i,k} = 1$ for some $i \in [k]$, and there exists a permutation of columns in $F = G_{[k]}^{[k-1]}$, which results in $f_{i,l} = 1, \forall l \in [k-1]$, then there will also exist a permutation of columns in $G$ which will lead to $g_{i,l} = 1, \forall l \in [k]$.

Proof. It can be verified that the relation between the elements of $G$ and $F = G_{[k]}^{[k-1]}$ is as follows

$$\begin{cases} g_{i,j} = f_{i,j}, & \forall l < i, \forall j \in [k-1] \\ g_{i+1,j} = f_{l,j}, & \forall l \geq i, \forall j \in [k-1] \end{cases}$$

So, if there exists a permutation of columns in $F$ to make $f_{i,l} = 1, \forall l \in [k-1]$, the same permutation of columns in $G$ will result in $g_{i,l} = 1, \forall l \in [k]$. For the induction hypothesis, we assume that the necessary condition holds for $H$ of size $(k-1) \times (k-1)$. Now, we need to prove that the necessary condition must also hold for $H_{k \times k}$. Let rank($H$) = $k$. The determinant of $H$ can be obtained using the Laplace expansion along the last column $k$, as follows

$$|H| = \sum_{i \in [k]} (-1)^{i+k} h_{i,k} |H_{[k]}^{[k-1]-1}_{[k]-1}|.$$ (7)

Thus, for having $|H| \neq 0$, there must exist at least one $i \in [k]$ such that $h_{i,k} |H_{[k]}^{[k-1]-1}_{[k]-1}| \neq 0$. So, we must have $h_{i,k} \neq 0$, which requires $g_{i,k} = 1$, and $|H_{[k]}^{[k-1]-1}_{[k]-1}| \neq 0$. This is guaranteed by the induction hypothesis only if for $F = G_{[k]}^{[k-1]}$ there exists a permutation of its columns which leads to $f_{i,l} = 1, \forall l \in [k]$, which completes the proof according to Lemma 1.

Remark 1. It can be easily concluded from Theorem 1 that the maximum rank of $H$ in (5) will be equal to the maximum number of ones, which can appear on the main diagonal of $G$ after doing a proper permutation of columns. Therefore, this will be equal to the maximum number of elements with value of 1, which are positioned in distinct rows and columns.

A. The MCM Problem

A binary matrix $G_{k \times m}$ can be represented as a bipartite graph $G(R, C, E)$, where $R = \{r_i, i \in [k]\}$ and $C = \{c_j, j \in [m]\}$ are two disjoint sets of vertices, called, the row and column vertex set, respectively. $E \subseteq R \times C$ denotes the set of edges.
lays the foundation that no two edges share a common vertex (i.e., if \((r_1, c_1), (r_2, c_2)\) and \((r_3, c_3), (r_4, c_4)\) are in \(E\), then \(i_1 \neq i_2 \) and \(j_1 \neq j_2\)). Such a set \(E'\) with maximum size is called the maximum cardinality matching (MCM) of \(G\).

**Definition 3 (Maximum Cardinality Matching of \(G\)).** Consider a binary matrix \(G\) represented as a graph \(G\). Let \(E'\) be an MCM of \(G\). Then, the maximum cardinality matching of \(G\) is defined as \(\text{mcm}(G) \triangleq |E'|\).

**Remark 2.** As mentioned above, for each two edges \((r_1, c_1)\) and \((r_2, c_2)\) in \(E'\), we must have \(i_1 \neq i_2\) and \(j_1 \neq j_2\). Consider the binary matrix \(G'\) associated with subgraph \(G(R, C, E')\). It can be easily observed that all the elements with value of one are positioned in distinct rows and columns. This implies that \(\text{mcm}(G)\) gives the maximum number of elements with value of one, which can be placed on the main diagonal after permuting the columns in a specific way. Thus,

\[
\text{mcm}(G) = \max_{v_{h_{i,j}}, j \neq 0} \text{rank } H.
\]

**V. THE MAXIMUM COLUMN DISTANCE (MCD) ALGORITHM**

In this section, the MCD algorithm is proposed to assign values to the nonzero elements of \(H_{k \times m}\) such that each submatrix \(H_{k \times l}, \forall L \in \mathcal{P}([m]), \forall i \in [k]\) achieves its possible maximum rank. In the following, Theorem 2 lays the foundation of the MCD algorithm.

**Definition 5 (Circuit Set).** \(L \subseteq [m]\) is considered to be a circuit set of \(H\) if

\[
\text{rank } H^{L \setminus \{j\}} = \text{rank } H^L = |L| - 1, \quad \forall j \in L,
\]

which means that for all \(j \in L\), the columns in \(H^{L \setminus \{j\}}\) are linearly independent, and the column \(h^j\) can be expressed as a linear combination of the other columns.

**Example 3.** Consider the following matrix \(H \in \mathbb{F}_3^{3 \times 4}\), where \(\mathbb{F}_3 = GF(3)\),

\[
H = \begin{bmatrix}
1 & 1 & 1 & 2 \\
0 & 2 & 1 & 2 \\
1 & 0 & 1 & 1
\end{bmatrix}.
\]

Then, \(L = \{1, 2, 4\}\) is a circuit set for \(H\) because we have

\[
\text{rank } H^{\{1,2,4\}} = 2,
\]

\[
\text{rank } H^{\{1,2\}} = \text{rank } H^{\{1,4\}} = \text{rank } H^{\{2,4\}} = 2.
\]

**Proposition 2.** Assume that \(L\) is a circuit set of \(H_{[k]}\). Then, in matrix \(H_{[k+1]}\) there exists a unique value for \(h_{k+1,j}, j \in L\) such that

\[
\text{rank } H^L_{[k+1]} = \text{rank } H^L_{[k]},
\]

which means that choosing this unique value does not increase the rank of \(H\) when we move from row \(k\) to row \(k+1\).

**Proof.** Let \(h^j\) be the \(j\)th column of \(H_{[k]}\). Since \(L\) is a circuit set of \(H_{[k]}\), then \(h^j\) can be expressed as a linear combination of the other columns as follows

\[
h^j = H^L_{[k]} f,
\]

where \(f = [f_1, \ldots, f_{|L|-1}]^T\) is a unique vector and \(f_i\) must be nonzero for all \(i \in [|L| - 1]\), since otherwise it contradicts (8).

Note that vector \(f\) can be achieved using reduced row echelon form (rref) as follows

\[
\text{rref } \begin{bmatrix} H^L_{[k]} & h^j \end{bmatrix} = \begin{bmatrix} I_{|L|-1} & f \\
0_{|L|-1} & 0 \end{bmatrix} \preceq \begin{bmatrix} I_{|L|-1} & f \end{bmatrix}.
\]

Let \(h_{k+1}\) be the \((k+1)\)-th row of \(H^L_{[k+1]}\). Then, we have

\[
\text{rref } H^L_{[k+1]} = \text{rref } \begin{bmatrix} H^L_{[k]} & h^j \\
h_{k+1} \end{bmatrix} \preceq \begin{bmatrix} I_{|L|-1} & f \\
h_{k+1} & h_{k+1} \end{bmatrix} \preceq \begin{bmatrix} I_{|L|-1} & f \\
0_{|L|-1} & h_{k+1} \end{bmatrix},
\]

where (11) is due to (10) and (12) is achieved by running \text{rref} over the last row such that

\[
h^j_{k+1} = h_{k+1,j} + H^L_{[k]} f.
\]
Thus, only the value $h_{k+1,j} = H_{\mathcal{L}(j)}^j \mathbf{f}$ will cause $h_{k+1,j}^* = 0$, which keeps the rank unchanged and leads to (9). In other words, if we select $h_{k+1,j} \neq H_{\mathcal{L}(j)}^j \mathbf{f}$, then the rank will increase by one.

**Definition 6 (Veto Value).** Since the unique value in the Proposition 2 is undesirable for the MCD algorithm, it is referred to as veto value and denoted by $h_{k+1,j}^*$.

**Example 4.** As seen in the Example 3, the set $\{1, 2\}$ is a circuit set for $H_{[3]}$. Now, we augment another row as a fourth row such that we have

$$H_{[4]}^{[1,2,4]} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & h_{4,4} \end{bmatrix}.$$  

It can be seen that $h^4 = H_{[4]}^{[1,2]} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So, the veto value of $h_{4,4}^{(1)}$ is $\begin{bmatrix} [2] & [2] \end{bmatrix} [1] = 1$. Thus, by choosing any value $h_{4,4} \in \{0, 1, 2\}\{1\} = (0, 2)$, we will have $H_{[4]}^{[1,2,4]} = H_{[3]}^{[1,2,4]} + 1$.

**Definition 7 (Veto Set).** Let $\mathcal{T}_j(L)$ denote the set of all $L' \subseteq L \setminus \{j\}$ such that each $L' \cup \{j\}$ is a circuit set of $H_{[k]}$. Then, we refer to the set $Z_j^L = \{h_{k+1,j} \in \mathbb{F}_q : \forall L' \in \mathcal{T}_j(L) \}$ as the veto set of column $j$ with column set $L$ at transmission $k + 1$.

**Theorem 2.** If rank $H_{[k]}^{L \cup \{j\}} < \min\{k, |L' + 1|\}$, then by selecting $h_{k+1,j} \in \mathbb{F}_q \setminus Z_j^L$, we will have

$$\text{rank } H_{[k+1]}^{L \cup \{j\}} = \text{rank } H_{[k]}^{L \cup \{j\}} + 1, \quad \forall L' \in \mathcal{P}(L),$$

(13)

which means that in order to increase the rank in (13) for all subsets in $\mathcal{P}(L)$, we just need to check the veto values of the subsets inside $\mathcal{T}_j(L)$ and pick a value from $\mathbb{F}_q \setminus Z_j^L$.  

**Proof.** First, if rank $H_{[k]}^{L \cup \{j\}} = \min\{k, |L'| + 1\}$, then the rank is maximum and cannot further increase. The proof is discussed through the following two cases:

- **(Case 1)** In this case, $|\mathcal{T}_j(L)| = 1$ which means that there is only one subset $L' \subseteq L \setminus \{j\}$ such that $L' \cup \{j\}$ forms a circuit set, and also the column $j$ is linearly dependent on the space spanned by the columns indexed by $L'$. Then, according to Proposition 2, by setting $h_{k+1,j} \neq H_{[k]}^{L \cup \{j\}}$, column $j$ will be linearly independent of the columns indexed by $L'$ and so will with all the columns indexed by $L$, which gives (13).

- **(Case 2)** In this case, $|\mathcal{T}_j(L)| > 1$. If we apply the rref to $H_{[k]}^{j}$, then its columns will be partitioned into two subsets as $L = L_1 \cup L_2$, where $L_1$ represents the set of linearly independent columns, and the columns indexed by $L_2$ are linearly dependent on the columns indexed by $L_1$. Now, it can be observed that column $j$ is linearly dependent on only one subset $L' \subseteq L_1$, which has been already discussed in Case 1. Now, we again run the rref on the columns indexed by $L_2$ to partition it into two subsets, where the first one will belong to Case 1, and for the second one, we partition it again. We repeat applying rref on the second set at each round until all subsets will finally belong to Case 1, which completes the proof of (13).

Therefore, first, we find the set $\mathcal{T}_j(L)$, then, choose the value $h_{k+1,j} \in \mathbb{F}_q \setminus Z_j^L$, outside of the veto set, and as a result, the rank increases by one as desired in (13).

**A. Description of the MCD Algorithm**

Given the binary matrix $G$, the MCD algorithm presented as Algorithm 2, generates a proper value for nonzero elements of $H$ such that every submatrix $H_{[k]}^{L} \forall L \in \mathcal{P}(\{m\}) \forall k \in [k]$ will reach its possible maximum rank. Here, we assume a sufficiently large field $\mathbb{F}_q$ is given. The MCD starts with the first row, and in each row it finds the proper value for each nonzero element one-by-one and then, moves to the next row and repeats this process until all the elements are assigned a proper value in each transmission.

In the $i$th row, the MCD selects one element $j$ randomly from $G_i$, whose value is yet to be determined, then sets $L = [m] \setminus G_i$, whose values are set. It then computes the veto set $Z_j^L$, and finally assigns a randomly chosen value from $\mathbb{F}_q \setminus Z_j^L$ to $h_{i,j}$ to increase the rank of all submatrices in (13). Now, we remove this element from $G_i$ (because the value is already found and fixed) and repeat this process for the remaining elements in $G_i$ (first the same row and then the next rows) until all the elements are assigned a value outside of their veto set.

<table>
<thead>
<tr>
<th>Algorithm 2: MCD Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> $G_{k \times m}$ &amp; $\mathbb{F}_q$</td>
</tr>
<tr>
<td><strong>Output:</strong> $H = MCD(G)$</td>
</tr>
<tr>
<td>initialization;</td>
</tr>
<tr>
<td>$i = 1$;</td>
</tr>
<tr>
<td>while $i \leq k$ do</td>
</tr>
<tr>
<td>while $G_i \neq \emptyset$ do</td>
</tr>
<tr>
<td>choose an element $j \in G_i$ at random;</td>
</tr>
<tr>
<td>$L = [m] \setminus G_i$;</td>
</tr>
<tr>
<td>find the veto set $Z_j^L$;</td>
</tr>
<tr>
<td>assign a value to $h_{i,j}$ from $\mathbb{F}_q \setminus Z_j^L$ at random;</td>
</tr>
<tr>
<td>$G_i = G_i \setminus {j}$;</td>
</tr>
<tr>
<td>end</td>
</tr>
<tr>
<td>$i \leftarrow i + 1$</td>
</tr>
<tr>
<td>end</td>
</tr>
</tbody>
</table>

**VI. CONCLUSION**

In this paper, a new coding scheme, referred to as update-based maximum column distance (UMCD) was proposed in which for each step of transmission, a linear coded message is designed with the aim of satisfying at least one receiver with the minimum size of side information. The coefficients of messages in each coded message are designed by the proposed maximum column distance (MCD) algorithm such that in each transmission, each column of the encoder matrix becomes linearly independent of the space spanned by other columns as much as possible. The problem is updated in each transmission using the polynomial-time Hopcroft-Karp algorithm, which is able to identify the satisfied receivers in each transmission. It was shown that the UMCD can outperform the ICC and recursive coding schemes. The vector version of our proposed scheme can also be achieved by having time-sharing between
the subproblems, where each is solved separately by the UMCD scheme.

The work in the last section of this paper can be extended in several directions. For example, given the sparse structure of the proposed UMCD encoder matrix (which is due to the fixing a number of elements to zero in each transmission), further investigation of how this sparsity would contribute to the lower computational complexity of the MCD algorithm can be interesting. Another exciting direction would be to determine a minimum required field size that still can guarantee the existence of a value outside of the veto set to assign to each nonzero element by the proposed MCD algorithm.

REFERENCES