Effects of Basis-mismatch in Compressive Sampling of Continuous Sinusoidal Signals

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***Abstract*—The theory of compressive sampling (or compressed sensing) is very attractive in that it is possible to reconstruct some signals with a sub-Nyquist sampling rate provided that they are sparse in some basis domain. But if there exists a mismatch between the signal basis and the pre-defined reconstruction basis, the reconstruction performance is significantly degraded even if the signal is sparse enough. In this paper, the degradation due to this basis mismatch is investigated and a way to minimize the effects of basis mismatch in compressive sampling of continuous sinusoidal signals is discussed.**

I. Introduction

Compressive sampling (CS) [1]–[4] is an emerging data acquisition method which may be superior to the Nyquist-Shannon sampling theory in certain situations. It can reconstruct the original signal with a much lower number of samples than that of Nyquist in the case that the signal is sparse enough in some basis domain. The CS theory states that an s-sparse signal $\mathbf{x}$ of length N can be reconstructed with an overwhelming probability if measurements are taken in the order of $s \log_2 N$.

In CS, it is assumed that the signal is sparsely represented in a basis domain. To reconstruct the signal, the only task to do is to calculate the weights of basis functions with the reduced number of measurements. Generally, the weights are calculated through ℓ_1-minimization with a pre-defined set of reconstruction basis functions. This pre-defined set is called the CS-dictionary in this paper.

This leads to the question of what happens if the true basis of the signal is not included in the CS-dictionary. This kind of basis mismatch is very common in real physical environments. In wireless communications, for example, Doppler shifts caused by moving objects and frequency offsets between the transmitter and receiver oscillators [5] can become sources of basis mismatch.

In this paper, the effects of basis mismatch on the reconstruction performance of CS will be analyzed. Furthermore, a method to reduce the adverse effects of basis mismatch will be discussed. For this purpose, we will focus on the reconstruction of continuous-time sinusoidal signals.

II. Compressive Sampling

In this section, we briefly review the rich mathematical theory of CS in the literature [1], [2], [4], [6]–[9] in the context of its application to signal processing.

A. Data Acquisition in CS

Let the signal $f(t)$ be a continuous signal in time domain which can be expanded as follows

$$f(t) = \sum_{i=1}^{N} x_i \psi_i(t), \tag{1}$$

where $\psi_i(t)$ is the i^{th} orthonormal basis function, and N is the number of basis functions. This can be rewritten in matrix form as

$$f(t) = \mathbf{\Psi}(t)\,\mathbf{x}, \tag{2}$$

where $\mathbf{\Psi}$(t) is the vector of basis functions

$$\mathbf{\Psi}(t) = \big[\psi_1(t),\, \psi_2(t),\, \cdots,\, \psi_N(t)\big], \tag{3}$$

and $\mathbf{x}$ is the coefficient vector of size $N \times 1$

$$\mathbf{x} = [x_1,\, x_2,\, \cdots,\, x_N]^T. \tag{4}$$

The k^{th} sample of measurements of the signal $f(t)$, which is sampled by the delayed Dirac delta function $\varphi_k(t) = \delta(t - t_k)$, is expressed as

$$\begin{aligned} y_k &= \int_{-\infty}^{\infty} f(t)\,\varphi_k(t)\,dt \\ &= \sum_{i=1}^{N} x_i \int_{-\infty}^{\infty} \psi_i(t)\,\delta(t-t_k)\,dt, \end{aligned} \tag{5}$$

where $k = 1, 2, \ldots, M$ and M is the number of measurements. Equation (5) can be rewritten in the following matrix form

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \tag{6}$$

where $\mathbf{A}$ is the measurement matrix of size $M \times N$ whose element in row i and column j is defined as

$$A_{ij} = \int_{-\infty}^{\infty} \psi_j(t)\,\delta(t-t_i)\,dt. \tag{7}$$

The coefficient vector $\mathbf{x}$ in (6) is therefore acquired through the measurement matrix $\mathbf{A}$ with M samples represented in $\mathbf{y}$. Generally, M is much smaller than N in CS.

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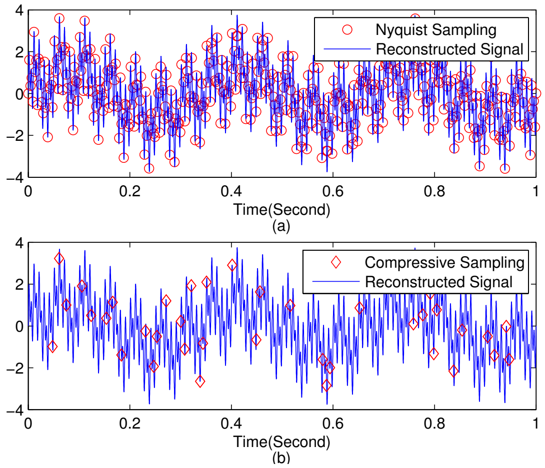

Fig. 1. Reconstruction of a signal which originally consists of 4 sinusoidals of 3Hz, 30Hz, 100Hz, and 190Hz; (a) Nyquist Sampling (b) Compressive Sampling.

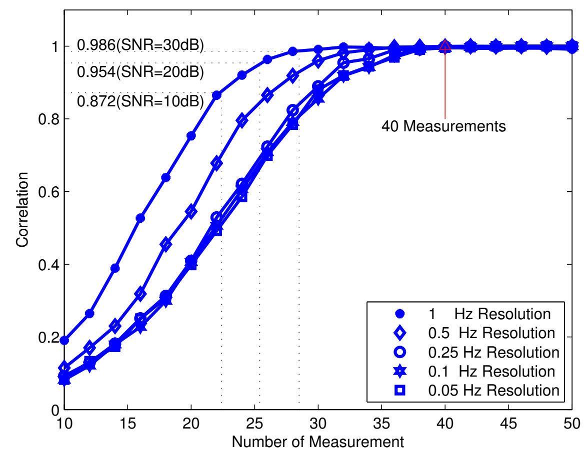

Fig. 2. Signal quality of reconstructed signal.

B. Signal Reconstruction

The reconstruction problem is ill-posed in general when $M < N$. There are many candidates of signal vector $\mathbf{x}$ for which $\mathbf{Ax} = \mathbf{y}$. However, reconstruction with a reduced number of samples is possible with a high probability using ℓ_1-minimization if the signal is sparse enough and the following conditions are met.

- Restricted isometry property (RIP) [2], [7]: For all s-sparse vectors $\mathbf{x}$, the matrix $\mathbf{A}$ obeys the RIP of order s if there exists a number δ_s not too close to one such that
$$(1-\delta_s) \leq \frac{\| \mathbf{Ax} \|_{\ell_2}^2}{\| \mathbf{x} \|_{\ell_2}^2} \leq (1+\delta_s) \tag{8}$$
which means that every s columns taken from $\mathbf{A}$ are nearly orthogonal.
- Minimum number of measurements [2]: M is bigger than $s \log_2 N$ if $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ are incoherent, or
$$M \geq C\,\mu^2\big(\mathbf{\Phi}(t), \mathbf{\Psi}(t)\big)\; s\; \log_2 N \tag{9}$$
in general, where μ is a measure of coherence between $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ and defined as
$$\mu\big(\mathbf{\Phi}(t), \mathbf{\Psi}(t)\big) = \max_{\substack{1\leq k\leq M\\ 1\leq j\leq N}} \left| \int_{-\infty}^{\infty} \varphi_k(t)\psi_j(t)\,dt \right|, \tag{10}$$
and C is a constant.

The reconstructed signal $f^\star(t)$ is given by
$$f^\star(t) = \mathbf{\Psi}(t)\,\mathbf{x}^\star, \tag{11}$$
where $\mathbf{x}^\star$ is the solution to the ℓ_1-minimization problem which can be cast to the linear program
$$\mathbf{x}^\star = \arg\min \|\mathbf{x}\|_{\ell_1} \quad \text{subject to } \mathbf{y} = \mathbf{Ax}. \tag{12}$$

If there is noise in the measurements, (12) is modified as
$$\mathbf{x}^\star = \arg\min \|\mathbf{x}\|_{\ell_1} \quad \text{subject to } \|\mathbf{Ax}-\mathbf{y}\|_{\ell_2} \leq \varepsilon, \tag{13}$$
where ε is a bound on noise power. The solution $\mathbf{x}^\star$ to (13) is known to obey [2]
$$\|\mathbf{x}^\star - \mathbf{x}\|_{\ell_2} \leq C_0 \frac{\| \mathbf{x} - \mathbf{x_s} \|_{\ell_1}}{\sqrt{s}} + C_1\,\varepsilon \quad \text{if } \delta_{2s} < \sqrt{2}-1, \tag{14}$$
where C_0 and C_1 are some constants.

C. An Example of CS

Fig. 1 is a simple reconstruction example of a signal which is composed of 4 sinusoidals, or 4-sparse. The frequencies of sinusoidals are 3Hz, 30Hz, 100Hz, and 190Hz and all sinusoidals have the same magnitude.

Fig. 1(a) displays the reconstruction by low-pass filtering the data samples taken at 440Hz, which includes about 15% margin of the Nyquist rate. Fig. 1(b) is the result of CS recovery with only 40 randomly-sensed samples, about 10% of the Nyquist rate. The CS-dictionary is composed of sinusoidals whose frequencies are from 1Hz to 200Hz.

Fig. 1 shows that the result of CS recovery is almost the same as that of Nyquist.

The reconstruction performance of an analog signal can be measured with the correlation coefficient ρ between the original signal and the reconstructed signal. The correlation coefficient ρ between signals X and Y is defined as
$$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2)-E(X)^2}\sqrt{E(Y^2)-E(Y)^2}}, \tag{15}$$
and SNR can be expressed using the correlation coefficient as
$$\text{SNR} = \frac{\rho^2}{1-\rho^2}. \tag{16}$$

Fig. 2 shows the quality of the reconstructed signal of the above CS example versus the number of measurements with 5 CS-dictionaries whose resolutions are 0.05Hz, 0.1Hz, 0.25Hz,

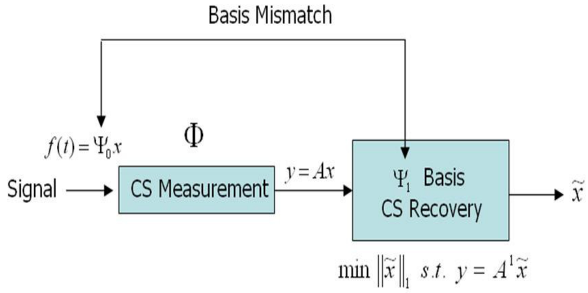

Fig. 3. Basis mismatch in CS.

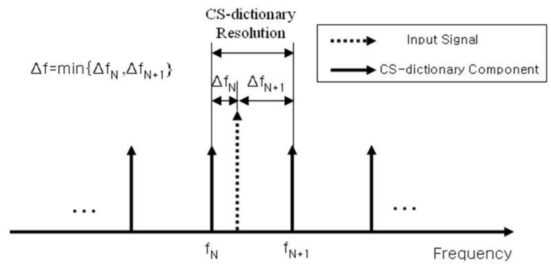

Fig. 4. Frequency offset.

0.5Hz, and 1Hz. From (9), the number of measurements should be at least about 30 ($\simeq 4\log_2 200$) to have a stable solution in the case of CS-dictionary of 1Hz resolution. It can be verified from Fig. 2 that an SNR over 30dB is achievable with 30 measurements. However, in this example, the number of measurements is set to 40 to maintain an SNR over 30dB for all 5 CS-dictionaries. This figure demonstrates two facts. One is that we need measurements much more than some threshold number to have a designated quality of the recovered signal. The other one is that the number of required measurements increases logarithmically if the number of CS-dictionary elements becomes larger.

III. Basis Mismatch in Compressive Sampling

Mismatch between the signal and reconstruction basis domains is very common, but the effects of such mismatch is not studied well in the literature. This section describes the effects of basis mismatch in CS.

A. Basis Mismatch

Let the signal $f(t)$ to be sampled be sparse in basis $\mathbf{\Psi_0}(t) = [\psi_0^1(t), \psi_0^2(t), \cdots, \psi_0^N(t)]$. The signal $f(t)$ can be reconstructed by the CS receiver under $\mathbf{\Psi_0}$ domain. But, if the recovery is conducted not under $\mathbf{\Psi_0}$ domain but under $\mathbf{\Psi_1}$ domain, $\mathbf{\Psi_1}(t) = [\psi_1^1(t), \psi_1^2(t), \cdots, \psi_1^L(t)]$, the quality of the recovered signal may be degraded.

Here, we define $\mathbf{\Psi}_c(t) = [\psi_c^1(t), \psi_c^2(t), \cdots, \psi_c^N(t)]$ to be the closest basis vector to $\mathbf{\Psi}_0(t)$ within $\mathbf{\Psi}_1$, whose m^{th} element is

$$\psi_c^m(t) = \arg \min_{\psi(t)\in\Psi_1} \left\|\psi_0^m(t) - \psi(t)\right\|_{\ell_2}. \tag{17}$$

Basis error, $\mathbf{e}(t)$, between $\mathbf{\Psi}_0(t)$ and $\mathbf{\Psi}_c(t)$ is defined as

$$\mathbf{e}(t) = \mathbf{\Psi}_0(t) - \mathbf{\Psi}_c(t). \tag{18}$$

Then, if there exists a basis error, the compressively measured samples of (6) are given by

$$\mathbf{y} = \mathbf{A}^c\mathbf{x} + \mathbf{A}^e\mathbf{x}, \tag{19}$$

where $\mathbf{A}^c$ is the measurement matrix corresponding to $\mathbf{\Psi}_c$(t), whose elements are defined as

$$A_{ij}^c = \int_{-\infty}^{\infty} \psi_c^j(t)\,\delta(t-t_i)\,dt, \tag{20}$$

and $\mathbf{A}^e$ is the measurement matrix corresponding to $\mathbf{e}$(t), whose matrix elements are defined as

$$A_{ij}^e = \int_{-\infty}^{\infty} \big(\psi_0^j(t) - \psi_c^j(t)\big)\delta(t-t_i)\,dt. \tag{21}$$

Signal coefficient vector $\mathbf{x}$ can be found in the $\mathbf{\Psi_1}$ domain because $\mathbf{\Psi}_c(t)$ is a subset of $\mathbf{\Psi}_1$(t). And the term $\mathbf{A}^e\mathbf{x}$ behaves like noise in $\mathbf{\Psi}_1$domain. The recovery in the presence of basis mismatch is shown in Fig. 3 and is formulated as

$$\mathbf{x}^\star = \arg\min \|\widetilde{\mathbf{x}}\|_{\ell_1} \quad \text{subject to } \mathbf{y} = \mathbf{A}^1\widetilde{\mathbf{x}}, \tag{22}$$

where $\mathbf{x}^\star$ is the estimated coefficient vector in $\mathbf{\Psi_1}$ domain, and $\mathbf{A}^1$ is the measurement matrix of $\mathbf{\Psi}_1(t)$, whose matrix elements are defined as

$$A_{ij}^1 = \int_{-\infty}^{\infty} \psi_1^j(t)\,\delta(t-t_i)\,dt. \tag{23}$$

Equation (22) can be written in the form of (13)

$$\mathbf{x}^\star = \arg\min \|\widetilde{\mathbf{x}}\|_{\ell_1} \quad \text{subject to } \|\mathbf{A^1}\widetilde{\mathbf{x}} - \mathbf{y}\|_{\ell_2} \le \varepsilon, \tag{24}$$

where ε is an upper bound on the noise power, $\|\mathbf{A^e x}\|_{\ell_2}$. The physical meaning of (24) is that the error of the reconstructed signal is bounded by the rms of the basis mismatch.

B. A Reconstruction Example with Basis Mismatch

In this example, the recovered result is investigated in the presence of basis mismatch. For simplicity, the signal is the same as that used in the example of Fig. 1. However there are some offsets in frequencies. Due to the frequency offset $\triangle f$, the exact frequencies of the signal are not included in the CS-dictionary or the pre-defined basis functions.

The frequency offset and the CS-dictionary resolution are displayed in Fig. 4. And the offset-ratio is defined as

$$\text{Offset\_Ratio} = \frac{\triangle f}{\text{Resolution of CS-dictionary}} \times 100(\%) \tag{25}$$

The CS-dictionary of this example has frequencies from 1Hz to 200Hz with 1Hz resolution, and all the frequency offsets $\triangle f_k$ are set to 0.45Hz. The reconstruction is performed with 40 random measurements. These measurements are obtained during 1 second and can be expressed as

$$y_m = \sum_{k=1}^{4} \sin\big(2\pi(f_k + \triangle f_k)t_m\big), \quad t_m \in \{t\colon 0 \le t \le 1\}, \tag{26}$$

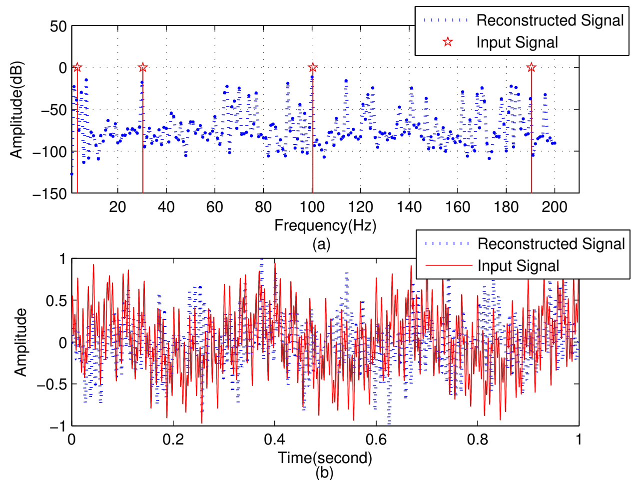

Fig. 5. Reconstruction with a CS-dictionary of 1Hz resolution in the presence of signal frequency offsets $\Delta f = 0.45Hz$; (a) Frequency domain (b) Time domain.

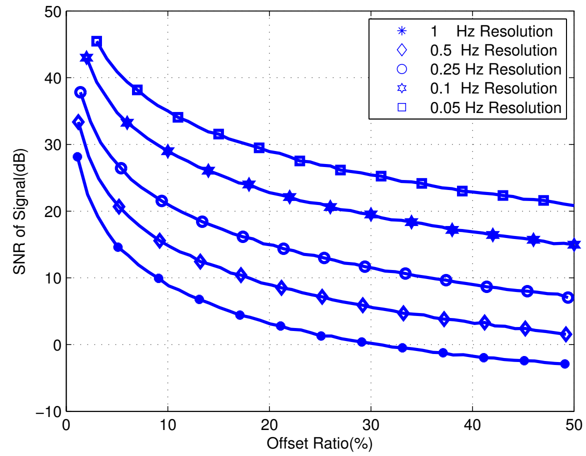

Fig. 6. SNR versus offset ratio.

where $m = 1, 2, \ldots, 40$. Fig. 5 is one snap-shot taken during the recovery of this example. Fig. 5(a) shows the coefficient of each basis as a result of applying (24). The spike noises are almost everywhere including near the correct frequency, and it is difficult to decide even the frequency elements of the signal because of the large magnitude of noise spikes. Fig. 5(b) is the time domain result which is reconstructed from the coefficients of Fig. 5(a). We can see that the reconstructed signal with the CS-dictionary of 1Hz resolution is quite different from the original signal in the presence of frequency offsets.

It can be inferred from this example that the reconstruction from CS may be impossible if the error from basis mismatch is not controlled.

IV. Minimizing Basis Mismatch Errors

To use CS as a data acquisition tool under basis mismatch, the noise from the basis mismatch should be much lower than the signal. To minimize the noise from the basis mismatch, one option is to build up the elements of CS-dictionary more finely. But the larger the number of CS-dictionary basis, the longer the time of reconstruction. Additionally, more measurements are needed if CS-dictionary becomes massive.

So the CS-dictionary should be made efficiently 1) to maintain a low noise level due to basis mismatch, 2) to be sized not to impose too much processing time during recovery, and 3) to be sized not to need much more measurements for reconstruction.

As already shown in Fig. 2, if there is no basis mismatch, an SNR above 30dB is achievable with 40 measurements by applying 5 given CS-dictionaries. So the approximation error from not having enough number of measurements can be ignored in this example.

A. An Enhanced Reconstruction Example with Basis Mismatch

Fig. 6 is the SNR of the measured data, the ratio of $E\{\|\mathbf{A}^c\mathbf{x}\|_{\ell_2}^2\}$ and $E\{\|\mathbf{A}^e\mathbf{x}\|_{\ell_2}^2\}$, versus the offset-ratio between the true basis of the signal and the basis of CS-dictionary. The SNR of the CS reconstructed signal also obeys this result according to (14) if the number of measurements is large enough for recovering the signal.

Fig. 6 shows that SNR is rapidly degraded if the frequency offset Δf increases. In the case of the example of Fig. 5, where the offset ratio is 45% (0.45Hz/1Hz), the achievable SNR may be about -2dB, at best, with the CS-dictionary of 1Hz resolution in Fig. 6. But, we can improve the SNR by adding more bases to the CS-dictionary. For example, SNR is improved from -2dB to 15dB if the resolution of CS-dictionary is changed from 1Hz to 0.1Hz even though the offset ratio is getting worse from 45% to 50%.

B. Computation Time

However, the time of reconstruction is also significantly increased if the elements of the CS-dictionary are increased. Fig. 7 shows the time needed for reconstructing the signal by applying various CS-dictionaries. The time of recovering becomes 1,000 times longer if the resolution is changed from 1Hz to 0.05Hz. But this may depend on the actual algorithm of the linear program [10] used for recovery. In this paper, the algorithm of [11] is used in the CS receiver. This time delay in computation is wholly due to the processing of a large measurement matrix.

C. Adjusting the CS-dictionary

Fig. 6 and Fig. 7 imply that the resolution of CS-dictionary should be determined considering the minimum required SNR and the processing time. Let the required SNR of the example of Fig. 5 be 15dB which should be satisfied even at the worst frequency offset ratio of 50%. The resolution of CS-dictionary

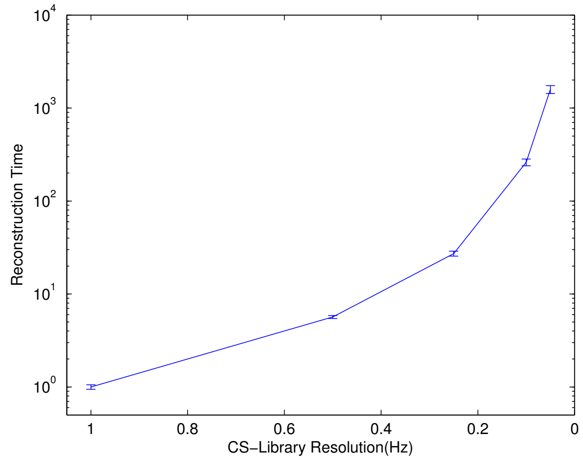

Fig. 7. Reconstruction time versus resolution (normalized by the reconstruction time when the CS-dictionary is 1Hz).

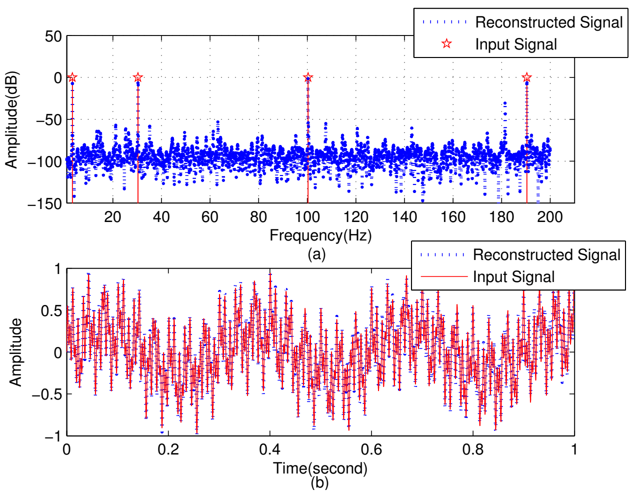

Fig. 8. Reconstruction with a CS-dictionary with 0.1Hz resolution when the normalized offset is 50%; (a) Frequency domain (b) Time domain.

should be at least 0.1Hz or smaller to meet the required SNR as shown in Fig. 6. Fig. 8 is the reconstruction result when the CS-dictionary with 0.1Hz resolution is applied to the example of Fig. 5. Even though there are some noise spikes as shown in Fig. 8(a), the noise level is much more negligible compared with the result of Fig. 5(a). The recovery of the original signal in time domain is also shown in Fig. 8(b), which is almost the same as the original signal except that the signal is somewhat noisy.

V. Conclusion

In this paper, we showed the effect of the CS basis mismatch in reconstructing continuous-time sinusoidal signals. Our result shows that uncontrolled basis mismatch may cause a severe noise which can make reconstruction of the original signal impossible. To reduce the noise from the basis mismatch, the resolution of CS-dictionary can be adjusted considering the required SNR, computation time, and the required number of measurements.

In our view, an adaptive selection among CS-dictionary candidates according to the required quality of recovered signal can be a good method. Even though we used low-frequency sinusoidal signals for simplicity, this approach can also be used in the signal processing of wireless communication.

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