The aim of this development is to show that the upward closed sets (upsets) of a preorder are a topology.

theory PreorderTopology = closure:

Let $\subseteq$ be a preorder (reflexive and transitive relation) on a set $X$. We call a subset $A \subseteq X$ a $\subseteq$ upset if anything bigger than something in $A$ is also in $A$. Symbolically, $A$ is an upset iff $\forall x, y. x \in A \subseteq y \rightarrow y \in A$. Then the set $T = \{A. A$ is a $\subseteq$ upset$\}$ is a topology over $X$.

locale preorder = var $X$ + var $R$ +
assumes on-carrier:
  $\forall x y. R x y \rightarrow x \in X \land y \in X$
and reflexive:
  $\forall x \in X. R x x$
and transitive:
  $\forall x \in X. \forall y \in X. \forall z \in X. R x y \land R y z \rightarrow R x z$
fixes upset
defines upset $A \equiv \forall x y. x \in A \land R x y \rightarrow y \in A$

Why is a finite intersection of upsets an upset? Let $F$ be a finite family of upsets, and $x$ be in the intersection, and $y$ be bigger than $X$. Then $x$ is in all the upsets in $F$, so $y$ is too, so $y$ is in the intersection.

lemma (in preorder) finite-intersection:
  assumes 1: $F \subseteq \{A. A \subseteq X \& \upset A\}$
  and 3: $F \neq \{\}$
  shows $\bigcap F \in \{A. A \subseteq X \& \upset A\}$
proof (simp only: mem-Collect-eq, intro conjI)
from 1 and 3 show $\bigcap F \subseteq X$ by auto
show upset ($\bigcap F$)
proof (unfold upset-def, intro allI impl, elim conjE)
  fix $x$ and $y$
  assume 4: $x \in \bigcap F$
  and 5: $R x y$
  from 4 have $\forall A \in F. x \in A$
    by auto
  have $\forall A \in F. y \in A$
    proof
fix \( A \)

assume 6: \( A \in F \)

with 1 have 7: upset \( A \) by auto

from 4 and 6 have \( x \in A \) by auto

with 5 and 7 show \( y \in A \)

by (unfold upset-def) blast

qed

thus \( y \in \bigcap F \)

by auto

qed

qed

Why is a union of upsets an upset? Anything \( x \) in the union is in one of the members \( A \) of the family, so anything \( y \) bigger than \( x \) is in \( A \) and hence in the union.

**lemma (in preorder) arbitrary-union:**

assumes 1: \( F \subseteq \{ A. A \subseteq X \ &\ upset A \} \)

shows \( \bigcup F \in \{ A. A \subseteq X \ &\ upset A \} \)

proof (simp only: mem-Collect-eq, intro conjI)

from 1 show \( \bigcup F \subseteq X \) by auto

show upset \( (\bigcup F) \)

proof (unfold upset-def, intro allI impl, elim conjE)

fix \( x \) and \( y \)

assume 2: \( x \in \bigcup F \)

and 3: \( R \ x \ y \)

from 1 and 2 obtain \( A \)

where 4: \( x \in A \)

and 5: \( A \in F \)

by auto

from 1 and 5 have upset \( A \)

by auto

with 3 and 4 have \( y \in A \)

by (unfold upset-def) blast

with 5 show \( y \in \bigcup F \)

by auto

qed

qed

**theorem (in preorder) upset-topology:**

assumes \( X \neq \{ \} \)

shows topological-space \( X \ \{ A. A \subseteq X \ &\ upset A \} \)

proof (intro topological-space.intro)

let \( ?T = \{ A. A \subseteq X \ &\ upset A \} \)

show \( X \neq \{ \} \).

show \( \forall A \in ?T. A \subseteq X \)

by auto

from on-carrier

show \( X \in ?T \)

qed
by (simp only: upset-def mem-Collect-eq) auto

show \{\} \in ?T
  by (unfold upset-def) auto

from finite-intersection

show \(\forall F. F \subseteq ?T \land \text{finite } F \land F \neq \{\} \rightarrow \bigcap F \in ?T\)
  by auto

from arbitrary-union

show \(\forall F. F \subseteq ?T \rightarrow \bigcup F \in ?T\)
  by auto

qed

end