We continue to investigate the idea of formal proof:

- Proof lines represent valid sequents.
- Deductive rules combine valid sequents to produce new valid sequents.
I want to approach the deductive rules from a different perspective, because I think it makes them easier to understand.
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We will think of the rules as ways of taking one or two valid sequents, and producing a different valid sequent.
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Recall that a sequent is a finite set (perhaps empty) of formulae called the premises, and a formula called the conclusion.
More notation (*don’t panic!*)

Consider the sequent

\[ P \rightarrow Q, \ P : P \& (P \rightarrow Q) \]
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\[ P \rightarrow Q, \ P : P \& (P \rightarrow Q) \]

We have metalinguistic variables \( A, B, \ldots \), which stand for formulae, so if we set \( A = P \rightarrow Q \), \( B = P \) and \( C = P \& (P \rightarrow Q) \) then we can write the sequent as

\[ A, B : C \]
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\[ P \rightarrow Q, \ P : P \& (P \rightarrow Q) \]

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\( C = P \& (P \rightarrow Q) \) then we can write the sequent as

\[ A, B : C \]

Now we introduce a new kind of metalinguistic variable which stands for a set of formulae. Capital Greek letters \( \Gamma, \Delta, \Phi, \ldots \) are used. Let \( \Gamma = \{ P \rightarrow Q, P \} \), and we can write our sequent as

\[ \Gamma : C \]
Proof lines are proven sequents

\[
\begin{align*}
\{1\} & \quad 1. \quad P \rightarrow Q \quad \text{premise} \\
\{2\} & \quad 2. \quad P \quad \text{premise} \\
\{1, 2\} & \quad 3. \quad P \& (P \rightarrow Q) \quad \&l 1, 2
\end{align*}
\]

This proves the sequent

\[ P \rightarrow Q, \quad P \vdash P \& (P \rightarrow Q) \]

The premises are the dependencies of the last line, the conclusion is the formula of the last line.
Proof lines are proven sequents

\[
\begin{align*}
\{1\} & \quad 1. \ P \rightarrow Q \quad \text{premise} \\
\{2\} & \quad 2. \ P \quad \text{premise} \\
\{1, 2\} & \quad 3. \ P \land (P \rightarrow Q) \quad \land I \ 1, 2
\end{align*}
\]

This proves the sequent

\[
P \rightarrow Q, \ P \vdash P \land (P \rightarrow Q)
\]

The premises are the dependencies of the last line, the conclusion is the formula of the last line.

Similarly, the first and second lines represent the (valid) sequents

\[
P \rightarrow Q \quad \vdash \quad P \rightarrow Q \\
P \quad \vdash \quad P
\]
Conjunction introduction as a sequent operation

We can think of the $\&$ introduction rule in that proof as combining the sequents of the first two lines.

$$
P \rightarrow Q \vdash P \rightarrow Q \quad P \vdash P
$$

$$
\frac{P \rightarrow Q \vdash P \rightarrow Q \quad P \vdash P}{P \rightarrow Q, \ P \vdash P \& (P \rightarrow Q)}
$$
Conjunction introduction as a sequent operation

We can think of the & introduction rule in that proof as combining the sequents of the first two lines.

\[
\begin{align*}
& P \rightarrow Q \vdash P \rightarrow Q & & P \vdash P \\
& P \rightarrow Q, P \vdash P & & P \vdash P \& (P \rightarrow Q)
\end{align*}
\]

We can think of the & introduction rule as a way of combining two valid sequents to give us a new valid sequent.

\[
\begin{align*}
& \frac{A \qquad B}{A \& B} & & \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \& B}
\end{align*}
\]
Conjunction introduction as a sequent operation

We can think of the & introduction rule in that proof as combining the sequents of the first two lines.

\[
\begin{align*}
P \to Q &\vdash P \to Q & P \vdash P \\
P \to Q, & P \vdash P \& (P \to Q)
\end{align*}
\]

We can think of the & introduction rule as a way of combining two valid sequents to give us a new valid sequent.

\[
\begin{align*}
A &\quad B \\
A \& B
\end{align*}
\]

\[
\begin{align*}
\Gamma &\vdash A \\
\Delta &\vdash B \\
\Gamma, \Delta &\vdash A \& B
\end{align*}
\]

This explains what we do when we calculate the dependencies of the &I line.
More rules seen this way

Contemplate the sequent forms of the rules, and see that if the sequent(s) above the line are valid, then the one below the line also must be valid.
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\[ \frac{A \& B}{A} \quad \& \quad E \text{ (left)} \]

\[ \frac{A \rightarrow B}{B} \quad \text{MP (→ E)} \]

\[ \frac{A \& B}{A} \quad \& \quad E \text{ (left)} \]

\[ \frac{A}{A \rightarrow B} \quad \text{MP (→ E)} \]

\[ \frac{A \rightarrow B}{B} \quad \text{MP (→ E)} \]

\[ \frac{A}{A} \quad \text{→ I (left)} \]

\[ \frac{A}{A \lor B} \quad \lor \text{ I (left)} \]

\[ \frac{A}{A \lor B} \quad \lor \text{ I (left)} \]
Implication Introduction

This rule, also called *conditional proof* (CP), is one which discharges an assumption. Hopefully this sequent view of the rules makes it easier to see why it works the way it does.
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\[
\begin{array}{c}
[A] \\
\vdash B \\
\hline
A \rightarrow B
\end{array}
\quad \text{CP (→ I)}
\]

\[
\begin{array}{c}
\Gamma, A \vdash B \\
\hline
\Gamma \vdash A \rightarrow B
\end{array}
\quad \text{CP (→ I)}
\]
Implication Introduction

This rule, also called *conditional proof* (CP), is one which discharges an assumption. Hopefully this sequent view of the rules makes it easier to see why it works the way it does.

\[
\begin{align*}
[A] \\
\vdots \\
B \\
\hline
A \rightarrow B
\end{align*}
\]
\[
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \text{ CP } (\rightarrow I)
\]

If some stuff and \( A \) guarantees that \( B \) must be true, then that stuff guarantees that “if \( A \) then \( B \)” is true.
The disjunction introduction rule also makes a bit more sense when we think of it in terms of sequents.
Disjunction Introduction

\[
\begin{array}{c|c|c}
\hline
A & \vdash & \lor \\
\hline
A \lor B & \vdash & C \\
\hline
C & \vdash & C \\
\hline
\end{array}
\]

The disjunction introduction rule also makes a bit more sense when we think of it in terms of sequents.

\[
\frac{\Gamma \vdash A \lor B \quad \Delta, A \vdash C \quad \Phi, B \vdash C}{\Gamma, \Delta, \Phi \vdash C} \lor E
\]

If the three sequents above the line are all valid, and all the formulae in \( \Gamma, \Delta \) and \( \Phi \) are true (in some given situation), then \( A \lor B \) is true there because of the first sequent, so either \( A \) or \( B \) is true there, so one of the other two sequents guarantees that \( C \) will be true in that situation.
Negation Introduction (RAA)

Contemplate the definition of validity, and the sequent on top of the RAA rule here.

\[
\begin{array}{c}
\text{[A]} \\
\vdots \\
B \& \sim B \quad \Gamma, A \vdash B & \sim B \\
\hline
\sim A \quad \Gamma \vdash \sim A \\
\end{array}
\]

RAA (\sim I)

In what situations will all the formulae in \(\Gamma, A\) be true? None! Therefore, \(\sim A\) is true whenever all the formulae in \(\Gamma\) are true.
Contemplate the definition of validity, and the sequent on top of the RAA rule here.

\[
\begin{array}{c}
[A] \\
\vdash \\
B \& \sim B \\
\sim A \quad \text{RAA (}\sim I) \\
\hline
\Gamma, A \vdash B \& \sim B \\
\Gamma \vdash \sim A \quad \text{RAA (}\sim I) \\
\end{array}
\]

In what situations will all the formulae in \( \Gamma, A \) be true? None! Therefore, \( \sim A \) is true whenever all the formulae in \( \Gamma \) are true.
Homework Assignment

Please do more than these, until you are confident you know what you are doing. But hand in those listed below.

- Page 93, 3.6.1 2
- Page 101, 3.9.1 7
- Page 272, 6.1.1 7
- Page 281, 6.2.1 7