At last, it is time to learn about *proof*

- formal proof as a model of reasoning
- demonstrating validity
- metatheory
- natural deduction systems
- what a proof looks like and how it works
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Formal reasoning

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What do we mean by “justified”? 
Another validity test

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A proof takes us step by step from the premises to the conclusion. That is, from some formulae to a formula.
Proof is independent of semantics. We can therefore ask questions about deductive systems, such as

- **Soundness**: Are all the provable sequents actually valid?
- **Completeness**: Is there a proof for every valid sequent?
- **Decidability**: Is there some procedure which can find a proof for every valid sequent?
- **Computational Complexity**: How much work does it take to find a proof?
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We will look briefly at these questions next week.
The deductive systems we will learn are called “natural deduction” systems. They were invented in 1935 by Gerhard Gentzen, and the form we learn is due to E.J. Lemmon (Beginning Logic, 1965).
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They are called “natural” because they follow ordinary (mathematical) reasoning more closely than the more traditional Hilbert systems. (Remember Hilbert from the first lecture?)
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PL Deduction

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- **number**  line number, for referencing
- **formula** the formula which is established in this step
- **justification** the rule and previous lines which justifies this formula
- **dependencies** references to the premises and assumptions this line depends on
An example proof

\{1\} 1. \( P \rightarrow Q \)  premise
\{2\} 2. \( P \)  premise
\{1, 2\} 3. \( Q \)  MP 1, 2

This proves the sequent

\[ P \rightarrow Q, \ P \vdash Q \]
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We write \( \vdash \) instead of : when the sequent is proved.
An example proof

\[
\begin{align*}
\{1\} & \quad 1. \quad P \rightarrow Q \quad \text{premise} \\
\{2\} & \quad 2. \quad P \quad \text{premise} \\
\{1, 2\} & \quad 3. \quad Q \quad \text{MP 1, 2}
\end{align*}
\]

This proves the sequent

\[ P \rightarrow Q, \quad P \vdash Q \]

We write \( \vdash \) instead of : when the sequent is proved.

In fact, \textit{every} line of a proof represents a sequent whose premises are the dependencies, and whose conclusion is the lines formula.
Deductive rules

(See deductive rules handout)
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For each connective, we have an
Deductive rules

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For each connective, we have an

**introduction rule** which tells us under what circumstances a formula with that main connective is justified

**elimination rule** which tells us how we can deduce something from a formula with this main connective
Deductive rules

(See deductive rules handout)

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**introduction rule** which tells us under what circumstances a formula with that main connective is justified

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Some would claim that it is these rules, rather than the formal semantics, which give meaning to the connectives. That is, they say “meaning is use” rather than “statements are a picture.”
Exercises

Homework Assignment

Please do more than these, until you are confident you know what you are doing. But hand in those listed below.

- Page 53, 2.3.2 7
- Page 55, 2.4.1 2
- Page 63, 2.5.1 7
- Page 66, 2.6.1 4

Please note: I may not be able to mark the test before Tuesday’s lesson, in which case, I will show you the marked papers on Thursday.