In this lecture we give precise *meaning* to formulae

- math stuff: sets, pairs, products, relations, functions
- PL connectives as truth-functions
- PL situations and truth-tables
- tautology, contradiction, contingency, consequence
- QL interpretations
Sets

- a set is several things considered together as one thing

- There are two ways of specifying a set:
  - List the items in it:
    - {Sydney, Toulouse, London}
    - {2, 3, 15, 328}
    - {}
  - Describe its members using a property:
    - The wheels on Greg's car
    - Students currently enrolled in Phil134
    - Goldfish currently enrolled in Phil134

- A set's members have no order:
  - {1, 2} = {2, 1}

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Set operations and relations

**Membership** we write \( x \in \{x, y, z\} \) to say that \( x \) is in the set.
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Set operations and relations

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- **Subset**: if every member of $A$ is also in $B$ we write $A \subseteq B$
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- **Intersection**: we write $A \cap B$ for the set of things that are in both $A$ and $B$
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**subtraction** $A - B$ is the elements from $A$ that are *not* in $B$
Exercise: Set operations

Let $A = \{1, 2, 3\}$ and $B = \{true, false\}$. Which of the following are correct?

1. $\{2\} \subseteq A$
2. $2 \in B$
3. $\{2, 3\} \in (A \cup B)$
4. $2 \in (A \cap B)$
5. $2 \in (A - \{1, 3\})$
Ordered Pairs

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  - (15, “turnip”)
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- the order matters: \((1, 2) \neq (2, 1)\)
- the same thing can be in a pair twice: \((1, 1)\) is an ordered pair
Products

The product operation takes an ordered pair of sets, and gives a set of ordered pairs.
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\[ A \times B = \{ \text{ordered pairs } (a, b) \text{ where } a \in A \text{ and } b \in B \} \]

eg. \{Sydney, Paris\} × \{2, 4, 11\} = \{(Sydney, 2), (Sydney, 4), (Sydney, 11), (Paris, 2), (Paris, 4), (Paris, 11)\}
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Relations and Functions

- A relation is a subset of a product.
- A function $A \rightarrow B$ is a relation $\subseteq A \times B$ which has exactly one pair $(a, b)$ for each $a \in A$. 

Do not confuse the function arrow with the "if ... then ..." arrow. If $f$ is a function $A \rightarrow B$ and $(a, b) \in f$ then we say $f(a) = b$. 

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Exercise: functions

Let \( A = \{1, 2, 3\} \) and \( B = \{true, false\} \).
Which of the following are functions \( A \to B \):

1. \( \{(1, \text{true}), (1, \text{false})\} \)
2. \( \{(1, 2), (2, 3), (3, 1)\} \)
3. \( \{(1, \text{true}), (2, \text{false}), (3, \text{true}), (1, \text{false})\} \)
4. \( \{(1, \text{true}), (2, \text{false}), (3, \text{true})\} \)
5. \( \{(1, \text{true}), (3, \text{true})\} \)
Now we are ready to give precise meaning to our formal languages.

- recall that a statement is true or false depending on the situation
- so, a PL situation must make each PL formula true or false
- the situation only needs to give truth values for the atomic formulae, because the truth of compound formulae are determined by the truth of their components
- therefore, a PL situation is a function
  \( \{P, Q, R, \ldots\} \rightarrow \{true, false\} \)
Propositional connectives as truth functions

- “I am rich and she is poor” is true when “I am rich” is true and “she is poor” is true, otherwise it is false.
- If $P = “I am rich”$ and $Q = “she is poor”, then $P \& Q$ is true in a situation if $P$ and $Q$ are both true there, false otherwise.
- So, the meaning of $\&$ is the function $\{T, F\} \times \{T, F\} \rightarrow \{T, F\}$,
  $\{((T, T), T), ((T, F), F), ((F, T), F), ((F, F), F)\}$
- We can write this much more conveniently as a truth-table.
**Truth tables**

*(careful: now $A$ and $B$ stand for PL formulae, not sets!)*

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$A$</td>
<td>$B$</td>
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<td>T</td>
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<tr>
<td>T</td>
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<tbody>
<tr>
<td><em>A</em></td>
<td><em>B</em></td>
<td><em>A</em> &amp; <em>B</em></td>
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- the value of the formula is written under its main connective
- if $A$ and $B$ were atoms, say $P$ and $Q$, the rows would be each possible situation
Truth tables as a validity test

Since a truth-table shows us the truth value for some formulae in every possible situation, we can use it to check validity of sequents.

For example, is $P \& Q$, $\sim P : Q$ a valid sequent?

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<th>: Q</th>
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More on this next time, but briefly, a situation for QL is a set $\mathcal{D}$ for the domain, a subset of the domain for each property letter, a relation $\subseteq \mathcal{D} \times \mathcal{D}$ for each binary relation letter..., and a member of $\mathcal{D}$ for each name letter. Atomic formulae are true when the named things are in the named property or relation.
Exercises

Homework Assignment

Tomassi, Chapter 4
4.1 evens
4.2.1 (for the formulae you worked in 4.1) 4.3: 2, 4, 6