Capacity Analysis of Correlated MIMO Channels

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Overview

- Background
- Analytic capacity for correlated channel
- “Large $n$” capacity
- Concept: “always linear” capacity growth
- Conclusions
Background

- Capacity of i.i.d MIMO channel well studied. (Telatar, 1999)
  - $t$ transmit, $r$ receive
  - “linear growth” with respect to minimum number of elements $m = \min(r, t)$.
  - reasonable match physical non-line-of-sight channel.

- Correlated MIMO channel more heuristic
  - LOS components
  - Chuah et al. (2002) noted that capacity growth still linear for correlated channels, through use of Stieljes transform, although LOS introduces equivalent power loss due to correlation.
Channel model

• General linear MIMO channel model:

\[ y = Hx + w \]

restriction: \( H \) random matrix, known at receiver, unknown at transmitter.

\[ C = \mathbb{E} \left\{ \log_2 \det \left( I_m + HH^* \cdot \frac{P}{t\eta} \right) \right\} \]

• For our model:
  
  – \( H \) is Gaussian random, with covariance \( \Sigma \).
  
  – \( H \) and \( \Sigma \) unknown at transmitter.

What is the analytic capacity of this channel?
Capacity of Correlated Gaussian MIMO Channel

\[ C = K_{\Sigma,m,n} \cdot \int_{\Lambda} \text{hypgeom} \left( -\frac{1}{2} \Sigma^{-1}, n\Lambda \right) \cdot J(\Lambda) \cdot \sum_{i=1}^{m} \log \left( 1 + \frac{P}{t} \lambda_i \right) \, d\Lambda \]

- Integral arises from definition of expectation.
- Problem: contains hypergeometric function
- Can be solved numerically, and \textit{under certain conditions} also analytically.
  1. \( \Sigma = I \): “famous” iid case, \( \text{hypgeom} \left( -\frac{1}{2} \Sigma^{-1}, n\Lambda \right) \) reduces to simpler form
  2. asymptotically for \( n \gg m \)
The next step... \( n \gg m \)

- Can assume \( \Sigma \) is diagonal,
  - \( \Sigma = \text{diag}\{\sigma_1, \cdots, \sigma_m\} \)
  - Also, \( \sigma_1 > \cdots > \sigma_k > \sigma_{k+1} = \cdots = \sigma_m \)
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- \( F_0(\cdot) \) separates into two (joint) distributions
  - Normal independent distribution for distinct eigenvalues and
  - Normal distribution, with conditional density for remaining equal eigenvalues
  - The distributions are jointly independent

- Separate capacity integral into two parts:
  \[
  C = C_d + C_{eq}
  \]
  - \( C_d \) for distinct eigenvalues \( \sigma_1 > \cdots > \sigma_{k-1} \)
  - \( C_{eq} \) for equal eigenvalues \( \sigma_k = \sigma_{k+1} = \cdots = \sigma_m \)
Split Capacity

- $C_d$ comprises joint distribution of independent random variables:

$$C_d = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{k} \int \exp \left( -\frac{z^2}{2} \right) \log \left( 1 + \frac{P}{t} n \lambda_i \left[ z \left( \frac{n}{2} \right)^{-1/2} + 1 \right] \right) \, dz$$

- $C_{eq}$ has similar distribution to standard i.i.d. MIMO case

$$C_{eq} = \int \log \left( 1 + \frac{P}{t} n \lambda \left[ z \left( \frac{n}{2} \right)^{-1/2} + 1 \right] \right)$$

$$\cdot \sum_{j=1}^{m-k} \frac{1}{2^j j! \sqrt{2\pi}} \left[ H_j \left( \frac{z}{\sqrt{2}} \right) \right]^2 e^{-z^2/2} \, dz$$

where $H_j(\cdot)$ is the $j$-th Hermite polynomial (Szegö, 1939).
Divide & Conquer

- $C_d$
  - requires special attention, use approximation of $n \to \infty$.

- $C_{eq}$
  - has same form as capacity for iid channel.

\[ C_{n \to \infty} = C_d + C_{eq} \]

\[ \sim \sum_{i=1}^{k} \log_2 \left( 1 + \frac{P}{t}n\sigma_i \right) + \int \log_2 \left( 1 + \frac{P}{t}n\sigma_m \right) \cdot f(\sigma_m) \]

This limit only converges for large $t$. 

Figure 1: $C_{eq}$ vs number of equal eigenvalues
**Large $n$ vs Large $t$**

- $n = t \to \infty$, limit converges
  - growth is linear with respect to $m = r$
    \[
    C_{t \to \infty} \sim \sum_{i=1}^{k} \log_2 \left( 1 + \frac{P}{\eta} \sigma_i \right) + (r - k) \log_2 \left( 1 + \frac{P}{\eta} \sigma_r \right)
    \]
  - $\sigma_i$ is the $i^{th}$ eigenvalue of $\Sigma$
  - Define equivalence class of correlation matrices $S_m \subset \mathbb{C}^{m \times m}$, with equal trace

- **Compare $\Sigma \in S'_m$ matrices**
  - number of distinct eigenvalues $k$ (or $\kappa = k/r$) and
  - ratio $\epsilon = \lambda_1/\lambda_m$
Capacity Surface $C_\infty(\Sigma)/r$

Capacity $C_\infty(\Sigma)/r$ for $\kappa$ and $\epsilon$
Capacity $C_\infty(\Sigma)$ for $\kappa$ and $r$, given $\epsilon = 15$
Conclusions

- Provided analytic capacity for correlated random MIMO channel.

- Provided asymptotic simplification for large $t$ and fixed $r$.
  - For full rank correlation matrices $\Sigma_m$, growth is linear
  - $\Sigma_m$ gives rate of growth

- i.i.d. channel has largest rate of growth $\alpha = \log_2(1 + P)$ for increasing $r$.
  - Equivalent $\beta$ dB power loss for LOS channel due to ratio $\epsilon$. 
References

