Capacity Analysis of Correlated MIMO Channels

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Overview

- Background
- Analytic capacity for correlated channel
- “Large n” capacity
- Concept: “always linear” capacity growth
- Conclusions

Background

- Capacity of i.i.d MIMO channel well studied. (Telatar, 1999)
  - t transmit, r receive
  - "linear growth" with respect to minimum number of elements $m = \min(r, t)$.
  - reasonable match physical non-line-of-sight channel.

- Correlated MIMO channel more heuristic
  - LOS components
  - Chuah *et al.* (2002) noted that *capacity growth still linear* for correlated channels, through use of Stieljes transform, although LOS introduces *equivalent power loss* due to correlation.

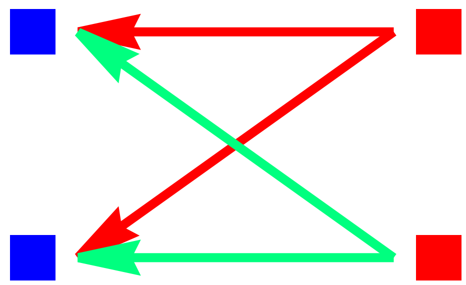
Channel model

- General linear MIMO channel model:

$$y = Hx + w$$

restriction: H random matrix, known at receiver, unknown at transmitter.

$$C = \mathsf{E}\left\{\log_2 \det\left(I_m + HH^* \cdot \frac{P}{t\eta}\right)\right\}$$



- For our model:
  - H is Gaussian random, with covariance Σ.
  - H and Σ unknown at transmitter.

What is the analytic capacity of this channel?

Capacity of Correlated Gaussian MIMO Channel

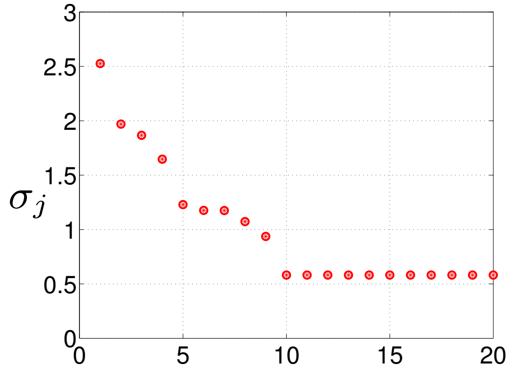
$$C = K_{\Sigma,m,n} \cdot \int_{\Lambda} {}_0F_0\left(-\frac{1}{2}\Sigma^{-1}, n\Lambda\right) \cdot J(\Lambda) \cdot \sum_{i=1}^{m} \log\left(1 + \frac{P}{t}\lambda_i\right) \, d\Lambda$$

- Integral arises from definition of expectation.
- Problem: contains hypergeometric function
- Can be solved numerically, and *under certain conditions* also analytically.

1. $\Sigma = I$: "famous" iid case, ${}_0F_0\left(-\frac{1}{2}\Sigma^{-1}, n\Lambda\right)$ reduces to simpler form
2. asymptotically for $n \gg m$

The next step... $n \gg m$

- Can assume Σ is diagonal,
  - $\Sigma = \text{diag}\{\sigma_1, \cdots, \sigma_m\}$
  - Also, $\sigma_1 > \cdots > \sigma_k > \sigma_{k+1} = \cdots = \sigma_m$



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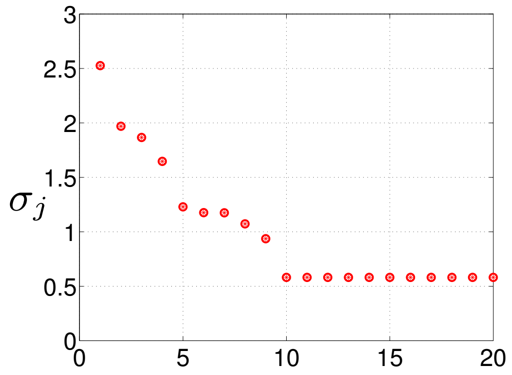

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- ${}_0F_0\left(\cdot\right)$ separates into two (joint) distributions
  - Normal independent distribution for distinct eigenvalues and
  - Normal distribution, with conditional density for remaining equal eigenvalues
  - **The distributions are jointly independent**

- Separate capacity integral into two parts:

$$C = C_{\mathrm{d}} + C_{\mathrm{eq}}$$

  - C_{d} for distinct eigenvalues $\sigma_1 > \cdots > \sigma_{k-1}$
  - C_{eq} for equal eigenvalues $\sigma_k = \sigma_{k+1} = \cdots \sigma_m$

Split Capacity

- C_{d} comprises joint distribution of independent random variables:

$$C_{\text{d}} = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{k} \int \exp\left(-\frac{z^2}{2}\right) \log\left(1 + \frac{P}{t} n\lambda_i \left[z\left(\frac{n}{2}\right)^{-1/2} + 1\right]\right) dz$$

- C_{eq} has *similar* distribution to standard i.i.d. MIMO case

$$C_{\text{eq}} = \int \log\left(1 + \frac{P}{t} n\lambda \left[z\left(\frac{n}{2}\right)^{-1/2} + 1\right]\right) \cdot \sum_{j=1}^{m-k} \frac{1}{2^j j! \sqrt{2\pi}} \left[H_j\left(\frac{z}{\sqrt{2}}\right)\right]^2 e^{-z^2/2}\, dz$$

where $H_j(\cdot)$ is the j-th Hermite polynomial (Szegö, 1939).

Divide & Conquer

- C_{d}
  - requires special attention, use approximation of $n \to \infty$.
- C_{eq}
  - has same form as capacity for iid channel.

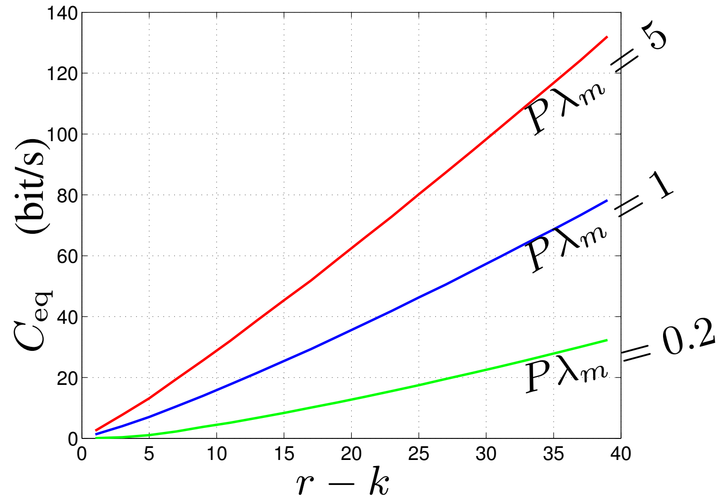


Figure 1: C_{eq} vs number of equal eigenvalues

$$C_{n\to\infty} = C_{\mathrm{d}} + C_{\mathrm{eq}}$$

$$\sim \sum_{i=1}^{k} \log_2\left(1 + \frac{P}{t} n\sigma_i\right) + \int \log_2\left(1 + \frac{P}{t} n\sigma_m\right) \cdot f(\sigma_m)$$

This limit only converges for large t.

Large n vs Large t

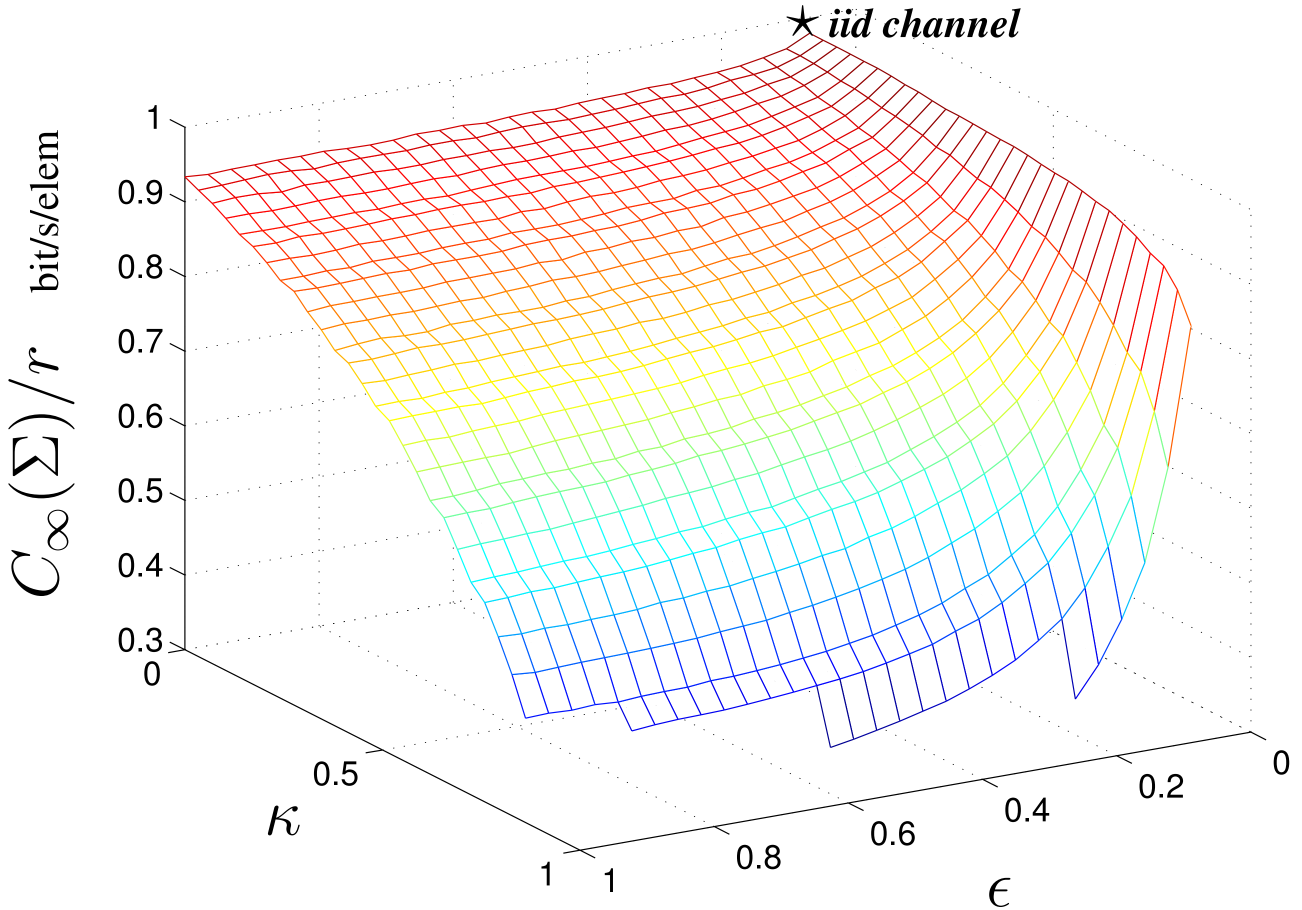
- $n = t \rightarrow \infty$, limit converges
  - growth is linear with respect to $m = r$

$$C_{t\rightarrow\infty} \sim \sum_{i=1}^{k} \log_2\left(1 + \frac{P}{\eta}\sigma_i\right) + (r-k)\log_2\left(1 + \frac{P}{\eta}\sigma_r\right)$$

  - σ_i is the i^{th} eigenvalue of Σ
  - Define equivalence class of correlation matrices $S_m \subset \mathbb{C}^{m\times m}$, with equal trace

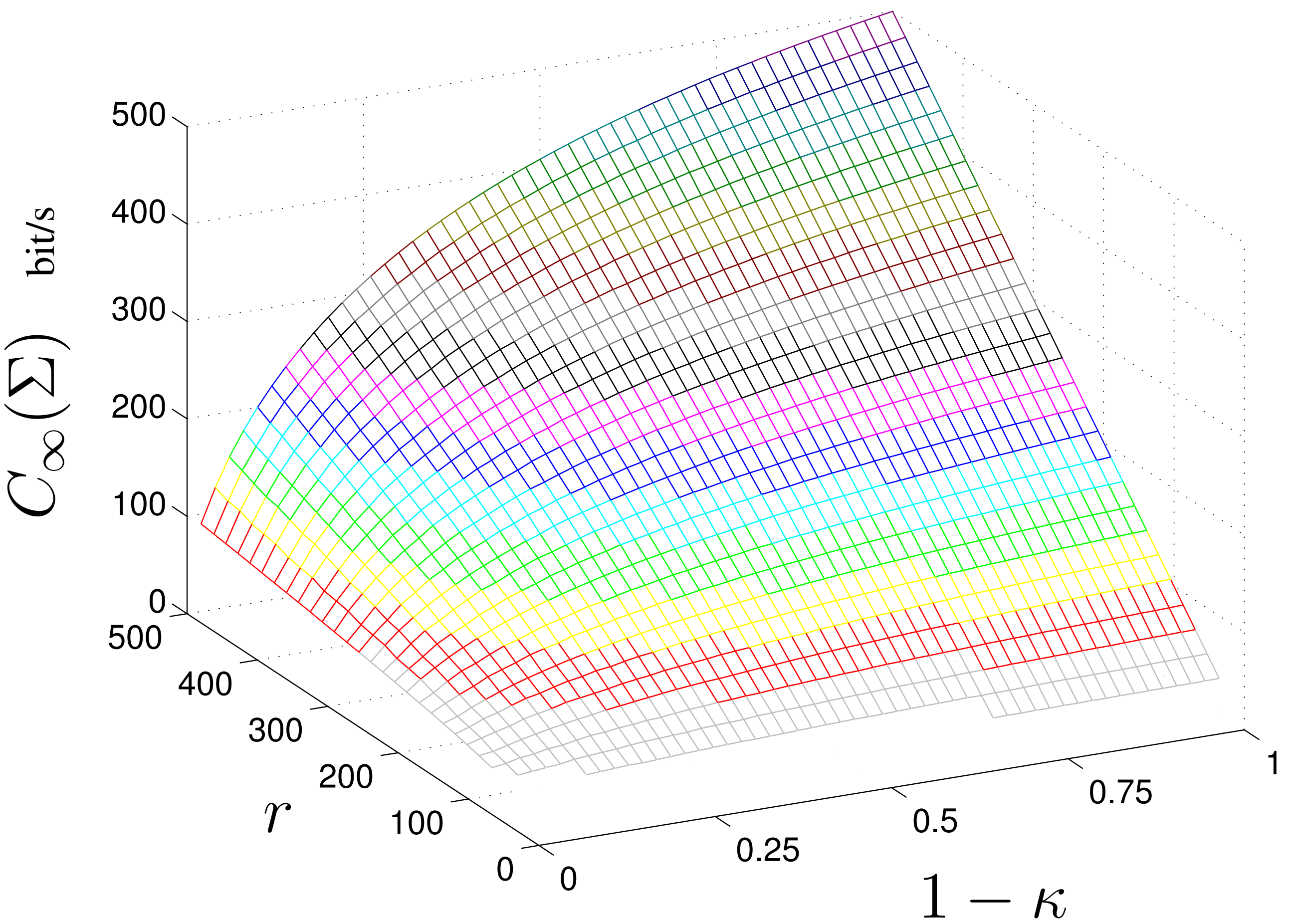
- Compare $\Sigma \in S_m$ matrices
  - number of distinct eigenvalues k (or $\kappa = k/r$) and
  - ratio $\epsilon = \lambda_1/\lambda_m$

Capacity Surface $C_\infty(\Sigma)/r$



Capacity $C_\infty(\Sigma)/r$ for κ and ϵ

Generalized Linear Growth



Capacity $C_\infty(\Sigma)$ for κ and r, given $\epsilon = 15$

Conclusions

- Provided analytic capacity for correlated random MIMO channel.
- Provided asymptotic simplification for large t and fixed r.
  - For full rank correlation matrices Σ_m, growth is linear
  - Σ_m gives **rate of growth**
- i.i.d. channel has largest rate of growth $\alpha = \log_2(1 + P)$ for increasing r.
  - Equivalent β dB power loss for LOS channel due to ratio ϵ.

References

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