Random Matrices and Wireless Communications

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Antonia M. Tulino and Sergio Verdu,
“Random Matrix Theory and Wireless Communications,”

Available from
Overview

- Introduction and Motivation
  - Eigenvalues of Random Matrices: Part 1
    - Marchenko – Pastur Law
  - Applications in Wireless Communications: Part 1
    - Capacity of MIMO Channel
  - Eigenvalues of Random Matrices: Part 2
    - Extended Marchenko – Pastur Law
  - Applications in Wireless Communications: Part 2
    - Linear MMSE Receivers
- Advanced Results and Extensions

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Random Matrices

- Originally studied in Statistics and Physics and more recently in Engineering
- Results now commonly used in diverse areas including attempts to prove Riemann Hypothesis, stochastic differential equations, condensed matter physics, statistical physics, chaotic systems, numerical linear algebra, multivariate statistics, signal processing, neural networks and communications and information theory
John Wishart (1898 – 1956)


Wishart died in a bathing accident in Acapulco, Mexico.
A quote from one of his postgraduate students …

“In those days he believed in his students keeping office hours. When he assigned me a desk in the Laboratory, he told me that he expected me to be sitting at the desk most of the day when not in class. He instructed me to do three hours computing a day on a table of the 1% level of z to 7 decimal places … Having anticipated a free and easy life as a graduate student, punctuated of course by periods of esoteric thinking when the spirit moved me, I didn't much like either the office hours or the computing, but I don't think they did me any harm.”
Eugene Wigner (1902 – 1995)


“After 60 years in the United States
I am still more Hungarian than American. ... much of American culture escapes me.”
Wigner’s Semicircle Law

Consider the symmetric, random, $n \times n$ matrix

\[ A = X + X' \]

where the elements of $X$ are i.i.d. random variables with zero mean and unit variance.

The eigenvalues of these random matrices were studied by Wigner in the 1950s connected to analysis of quantum mechanical systems.

What is the density of a randomly chosen eigenvalue of $A$ when $n$ is large?
Semicircle Law in Matlab

\begin{verbatim}
    n = 50;       % size of matrix
    s = 200;      % number of sample matrices
    e = [];       % eigenvalues

    for i = 1:s
        % n x n matrix with i.i.d. N(0,1) elements
        X = randn(n);
        % generate a symmetric Gaussian matrix
        A = (X + X') / sqrt(8*n);
        % collect eigenvalues
        e = [ e, eig(A) ];
    end;
\end{verbatim}
Semicircle Law in Matlab
Consider the linear memoryless vector Gaussian channel

\[ y = H x + n \]

where

- \( y \) is \( N \) – dimensional output vector
- \( x \) is \( K \) – dimensional input vector
- \( n \) is \( N \) – dimensional additive noise vector
- \( H \) is the \( N \times K \) random channel matrix
MIMO Channel

- Models CDMA systems where $K$ is the number of users, $N$ is the spreading gain, and the columns of the channel matrix $H$ are the spreading sequences.

- Models single user narrowband channels where $K$ and $N$ are the number of transmit and receive antennas respectively, and entries of $H$ model the propagation coefficients between each pair of transmit and receive antennas.
Model applies to many other systems of interest including multi-user, multi-antenna channels (possibly using CDMA) and multi-carrier CDMA.

Simplest model has $H$ with independent and identically distributed (i.i.d.) entries although many other models can be handled.
Capacity of MIMO Channel

The maximum (normalised) mutual information between the transmitted and received vectors given the channel is

$$\max_{f_x} \frac{1}{N} I(x; y|\mathbf{H}) = \frac{1}{N} \log \det (\mathbf{I} + \text{SNR} \ \mathbf{HH}^\dagger)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \log \left(1 + \text{SNR} \ \lambda_i(\mathbf{HH}^\dagger)\right)$$

We are interested in both the expected value of the above random quantity (ergodic capacity) as well as its distribution (outage capacity).
Linear MMSE Receivers

We are often interested in the performance of linear minimum mean square error (MMSE) receivers

\[ \frac{1}{K} \min_M \mathcal{E} \left[ ||x - M y||^2 \right] = \frac{1}{K} \text{tr} \left\{ \left( \mathbf{I} + \text{SNR} \mathbf{H}^\dagger \mathbf{H} \right)^{-1} \right\} \]

\[ = \frac{1}{K} \sum_{i=1}^{K} \frac{1}{1 + \text{SNR} \lambda_i(\mathbf{H}^\dagger \mathbf{H})} \]

Again we see that the performance is determined by the eigenvalues of the key matrix

\[ \mathbf{H}^\dagger \mathbf{H} \]
Random Matrices in Communications

- Capacity of multiple antenna channels
Random Matrices in Communications

- SIR of linear multiuser receivers for CDMA
Random Matrices in Communications

- Capacity of CDMA channels
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Empirical Distribution Functions

The empirical distribution function of the eigenvalues of an \( n \times n \) Hermitian matrix \( A \) is denoted \( F_A^n \) defined as

\[
F_A^n(x) = \frac{1}{n} \sum_{i=1}^{n} 1\{\lambda_i(A) \leq x\}
\]

where \( \lambda_1(A), \ldots, \lambda_n(A) \) are the eigenvalues of \( A \) and \( 1 \) is the indicator function.

So \( F_A^n(x) \) is the proportion of eigenvalues of \( A \) less than or equal to \( x \).
Marchenko – Pastur Law

Consider an $N \times K$ matrix $\mathbf{H}$ whose entries are independent zero-mean random variables (real or complex) with variance $1/N$. As $K, N \to \infty$ with $K/N \to \beta$, the empirical distribution function of the eigenvalues of $\mathbf{HH}^\dagger$ converges almost surely to a nonrandom limiting distribution with density

$$f_\beta(x) = (1 - \beta)^+ \delta(x) + \frac{\sqrt{(x - a)^+(b - x)^+}}{2 \pi x}$$

where $a = (1 - \sqrt{\beta})^2$ and $b = (1 + \sqrt{\beta})^2$. 
Marchenko – Pastur Law

\[ \beta = 0.2 \]

\[ \beta = 0.5 \]

\[ \beta = 1.0 \]
Marchenko – Pastur Law in MATLAB

- Generate a random N x K matrix $H$
- Calculate the eigenvalues of $HH^\dagger$
- Plot the empirical distribution function
- Observe what happens as N and K get large
Eigenvalue Convergence (K = 10)

Jumps indicate location of eigenvalues
Eigenvalue Convergence (K = 20)
Eigenvalue Convergence (K = 40)
Eigenvalue Convergence ($K = 160$)
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MIMO Channel

Consider the standard single-user MIMO channel with K transmit antennas and N receive antennas

\[ y = H x + n \]

where

- \( y \) is N – dimensional output vector
- \( x \) is K – dimensional input vector
- \( n \) is N – dimensional additive noise vector
- \( H \) is the N x K channel matrix
Capacity of MIMO Channel

With no knowledge of the channel at the transmitter, the maximum (normalised) mutual information is

\[
C(H) = \frac{1}{N} \log \det \left( I + \text{SNR} \; HH^\dagger \right)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \log \left( 1 + \text{SNR} \; \lambda_i( HH^\dagger ) \right)
\]

\[
= \int_0^\infty \log \left( 1 + \text{SNR} \; x \right) dF_{HH^\dagger}^N(x)
\]

where the integration is with respect to the empirical distribution function of the eigenvalues of the matrix \( HH^\dagger \).
Capacity of MIMO Channel

If $\mathbf{H}$ consists of i.i.d. elements with zero-mean and variance $1/N$ then as $N, K \to \infty$ with $K/N \to \beta$ then we have

$$C(\mathbf{H}) = \int_0^\infty \log (1 + \text{SNR } x) \ dF_{\mathbf{H}^\dagger \mathbf{H}}^N(x)$$

$$\to \int_0^\infty \log (1 + \text{SNR } x) \ f_\beta(x) \ dx$$

where

$$f_\beta(x) = (1 - \beta)^+ \delta(x) + \frac{\sqrt{(x - a)^+ (b - x)^+}}{2\pi x}$$

with $a = (1 - \sqrt{\beta})^2$ and $b = (1 + \sqrt{\beta})^2$. 
Capacity of MIMO Channel

This integral has a simple closed-form

\[
\beta \log \left( 1 + \text{SNR} - \frac{1}{4} \mathcal{F}(\text{SNR}, \beta) \right) \\
+ \log \left( 1 + \text{SNR}_\beta - \frac{1}{4} \mathcal{F}(\text{SNR}, \beta) \right) \\
- \frac{\log e}{4\text{SNR}} \mathcal{F}(\text{SNR}, \beta)
\]

where

\[
\mathcal{F}(x, z) = \left( \sqrt{x(1 + \sqrt{z})^2 + 1} - \sqrt{x(1 - \sqrt{z})^2 + 1} \right)^2
\]
Illustration of Convergence

N = 3

N = 5

N = 10

N = 40

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Observations

- Final capacity expression depends only on SNR and $\beta$ and is relatively simple.
- Capacity limit is insensitive to the distribution of the elements of the channel matrix (provided elements are zero-mean and variance $1/N$).
- Convergence to limit is fast, especially for ergodic capacities.
- Ergodic behaviour is evident even when channel matrix does not vary ergodically over a codeword – in a large system, all channel realizations result in essentially the same capacity.
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Extended Marcenko – Pastur Law

Let $\mathbf{H}$ be an $N \times K$ matrix whose entries are i.i.d. random variables (real or complex) with zero-mean and variance $1/N$.

Let $\mathbf{T}$ be a $K \times K$ real diagonal random matrix whose empirical eigenvalue distribution converges almost surely to the distribution $F_T(\cdot)$.

Assume $\mathbf{H}$ and $\mathbf{T}$ are independent.
Stieltjes Transform

Let $X$ be a real-valued random variable with distribution $F_X(\cdot)$. Its Stieltjes transform is defined for complex arguments as

$$m(z) = \mathcal{E} \left[ \frac{1}{X - z} \right]$$

$$= \int_{-\infty}^{\infty} \frac{1}{\lambda - z} dF_X(\lambda)$$

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Extended Marcenko – Pastur Law

The empirical distribution function of the elements of $\mathbf{HTH}^\dagger$ converges, as $K, N \to \infty$ with $K/N \to \beta$, almost surely to a nonrandom limiting distribution whose Stieltjes transform satisfies

$$m(z) = \left[ -z + \beta \int_{-\infty}^{\infty} \frac{t}{1 + t m(z)} dF_T(t) \right]^{-1}$$
Extended Marcenko – Pastur Law

Consider the special case where $T$ is the identity matrix. We then have that

$$m(z) = \left[ -z + \beta \int_{-\infty}^{\infty} \frac{t}{1 + t m(z)} \delta(t - 1) \, dt \right]^{-1}$$

$$= \left[ -z + \beta \frac{1}{1 + m(z)} \right]^{-1}$$

This is a quadratic equation in $m(z)$ that can be solved to give the Stieltjes transform of the limiting distribution with density $f_\beta$. 
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Linear MMSE Receivers in CDMA

Standard real, baseband model for synchronous CDMA channel:

\[ y = S A x + n \]
\[ = a_1 x_1 s_1 + \sum_{k=2}^{K} a_k x_k s_k + n \]

- \( x_k \) is the data symbol of user \( k \)
- \( a_k \) is the channel gain for user \( k \)
- \( s_k \) is the signature sequence of user \( k \)
Linear MMSE Receivers in CDMA

Interest is in linear receivers that form estimates

\[ \hat{x}_1 = c^\dagger y = a_1 x_1 (c^\dagger s_1) + \sum_{k=2}^{K} a_k x_k (c^\dagger s_k) + (c^\dagger n) \]

Performance measure of interest is signal-to-interference ratio:

\[ \text{SIR} = \frac{a_1^2 (c^\dagger s_1)^2}{c^\dagger (S_1 A_1^2 S_1^\dagger + \sigma^2 I) c} \]

where \( S_1 = [s_2, \ldots, s_K] \) and \( A_1 = \text{diag}(a_2, \ldots, a_K) \).
Linear MMSE Receivers in CDMA

The maximum SIR is achieved by setting

\[ c = a_1 \left( S_1 A_1^2 S_1^\dagger + \sigma^2 I \right)^{-1} s_1 \]

and the maximum SIR is

\[ \text{SIR} = a_1^2 s_1^\dagger \left( S_1 A_1^2 S_1^\dagger + \sigma^2 I \right)^{-1} s_1 \]

The performance depends on the signature sequences and powers of all users as well as the background noise. What happens when the signature sequences are random?
Linear MMSE Receivers in CDMA

We have

\[
SIR = a_1^2 \left[ s_1^\dagger (S_1 A_1^2 S_1^\dagger + \sigma^2 I)^{-1} s_1 \right]
\]

\[
\approx a_1^2 \left[ \frac{1}{N} \text{tr} \left\{ (S_1 A_1^2 S_1^\dagger + \sigma^2 I)^{-1} \right\} \right]
\]

\[
= a_1^2 \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i + \sigma^2} \right]
\]

\[
\rightarrow a_1^2 \left[ \int \frac{1}{\lambda + \sigma^2} dG(\lambda) \right]
\]

where \( G(\lambda) \) is the limiting empirical distribution function of the eigenvalues of \( S_1 A_1^2 S_1^\dagger \).
Linear MMSE Receivers in CDMA

So the SIR converges to

$$\text{SIR} \rightarrow a_1^2 \left[ \int \frac{1}{\lambda + \sigma^2} \, dG(\lambda) \right]$$

Recalling that the Stieltjes transform of $G(\lambda)$ is

$$m(z) = \int \frac{1}{\lambda - z} \, dG(\lambda)$$

we observe that the SIR is precisely $a_1^2 \, m(-\sigma^2)$ and we are led to the main result.
The Tse – Hanly Formula

Suppose that the elements of the signature sequences are chosen i.i.d. according to some arbitrary distribution with mean 0 and variance \( \frac{1}{N} \) and consider the limit \( N, K \to \infty \) with \( K/N \to \beta \). If the empirical distribution of the powers converges almost surely to a fixed distribution \( F(p) \), then the (random) SIR converges almost surely to \( a_1^2 \eta^* \), where \( \eta^* \) is the unique positive solution to the equation

\[
\eta = \left[ \sigma^2 + \beta \int_0^{\infty} \frac{p}{1 + p \eta} dF(p) \right]^{-1}
\]
Tse and Hanly

**Special Case: Equal Powers**

Suppose that $a_1^2 = a_2^2 = \cdots = a_K^2 = P$ and write $\text{SNR} = P/\sigma^2$. In this model, where all users are received at the same power, we have a simple expression for the limiting signal-to-interference ratio

$$\text{SIR} = \frac{1}{2} \sqrt{(1 - \beta)^2 \text{SNR}^2 + 2(1 + \beta)\text{SNR} + 1}$$

$$+ \frac{1}{2} (1 - \beta)\text{SNR} - \frac{1}{2}$$

This result can also be derived by evaluating the integral

$$\text{SIR} = \int_0^\infty \frac{1}{\lambda + \sigma^2} f_\beta(x) \, dx.$$
Special Case: Rayleigh Fading

Suppose that $a_1^2, a_2^2, \cdots, a_K^2$ are i.i.d. random variables with negative-exponential distribution of mean $P$ (Rayleigh fading). Let $\text{SNR} = P/\sigma^2$ denote the average SNR.

In this case, the limiting SIR is given by the fixed point of the iteration

$$
\eta_{i+1} = \text{SNR} \left[ 1 + \beta \text{SNR} \int_0^\infty \frac{x}{1 + x \eta_i} e^{-x} \, dx \right]^{-1}
$$
Multiuser Diversity

![Graph showing multiuser diversity](image)

- **Limiting SIR (dB)**
- **Rayleigh Fading**
- **Equal Powers**

System Load ($\beta$)

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Many models of interest lead to channel matrices which do not have i.i.d. elements. Examples include:

- Asynchronous CDMA
- CDMA with frequency-selective fading
- CDMA with multiple receive antennas
- Multi-carrier CDMA
- Cellular MIMO systems

More advanced techniques and results exist to deal with many of these models.