

Deep Inference in Bi-intuitionistic Logic

Linda Postniece

Logic and Computation Group
College of Computer Science and Engineering
The Australian National University

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Introduction

- Int + dual-Int

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- \multimap dual to \rightarrow

$$\frac{A \vdash B, \Delta}{A \multimap B \vdash \Delta} \multimap_L \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow_R$$

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- Goal: show that deep inference can mimic residuation for BiInt
- Broader goal: proof search in display calculus

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 - Definitions
 - Shallow vs Deep Inference
- 3 **DBiInt**
 - Rules
 - Soundness
 - Completeness
 - Proof Search
- 4 **Conclusion**

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- $p \Rightarrow q, r \rightarrow ((p \multimap q) \wedge r)$ is not cut-free derivable in Rauszer's G1
- Derivation using cut:

$$\frac{\frac{\frac{p \Rightarrow p}{p \Rightarrow p} \text{Id} \quad \frac{q \Rightarrow q}{q \Rightarrow q} \text{Id}}{p \Rightarrow q, p \multimap q} \multimap_R \quad \frac{\frac{\frac{p \multimap q, r \Rightarrow p \multimap q}{p \multimap q, r \Rightarrow p \multimap q} \text{Id} \quad \frac{p \multimap q, r \Rightarrow r}{p \multimap q, r \Rightarrow r} \text{Id}}{p \multimap q, r \Rightarrow (p \multimap q) \wedge r} \wedge_R}{p \multimap q \Rightarrow r \rightarrow ((p \multimap q) \wedge r)} \rightarrow_R}{p \Rightarrow q, r \rightarrow ((p \multimap q) \wedge r)} \text{cut}$$

Syntax

- **Formula** where p is an atom:

$$A := p \mid \top \mid \perp \mid A \rightarrow A \mid A \multimap A \mid A \wedge A \mid A \vee A.$$

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- Nested **shallow sequent**: $X \Rightarrow Y$

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Example

$(p \triangleright q), t \triangleright r \multimap s$ is a deep sequent

$(p \triangleright q), t \Rightarrow r \multimap s$ is a shallow sequent

 $\langle \{t\}, \{r \multimap s\} \rangle$
 $\langle \{p\}, \{q\} \rangle$

Contexts and Polarities

- **Context:** $\Sigma[]$ a deep sequent with a single hole

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Contexts and Polarities

- **Context:** $\Sigma[\]$ a deep sequent with a single hole
- **Negative** context $\Sigma^-[\]$ means $X, [\] \triangleright Y$ is a substructure of Σ
- **Positive** context $\Sigma^+[\]$ means $X \triangleright [\]$, Y is a substructure of Σ
- Sequent translation to formula:

$$\begin{array}{llll}
 \tau^-(\emptyset) & = & \top & \tau^+(\emptyset) & = & \perp \\
 \tau^-(A) & = & A & \tau^+(A) & = & A \\
 \tau^-(X, Y) & = & \tau(X) \wedge \tau(Y) & \tau^+(X, Y) & = & \tau(X) \vee \tau(Y) \\
 \tau^-(X \triangleright Y) & = & \tau(X) \multimap \tau(Y) & \tau^+(X \triangleright Y) & = & \tau(X) \rightarrow \tau(Y)
 \end{array}$$

$$\tau(X \Rightarrow Y) = \tau(X \triangleright Y) = \tau^-(X) \rightarrow \tau^+(Y)$$

Filling a Context

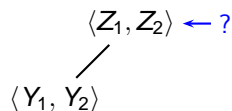
- $\Sigma[X]$: context $\Sigma[]$ filled with structure X

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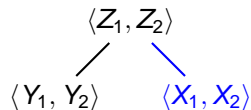
- $\Sigma[X]$: context $\Sigma[\]$ filled with structure X

Example

$$\Sigma^{-}[\] = Z_1, (Y_1 \triangleright Y_2), [\] \triangleright Z_2$$



$$\Sigma^{-}[X_1 \triangleright X_2] = Z_1, (Y_1 \triangleright Y_2), (X_1 \triangleright X_2) \triangleright Z_2$$



Top-level Formulae

- **Top-level:** $\{X\} = \{A \mid X = (A, Y) \text{ for some } A \text{ and } Y\}$

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Example

$$\{(X_1 \triangleright Y_1), A, B\} = \{A, B\}$$

Shallow vs Deep Inference

Shallow inference

- Apply rules to top-level sequent only

$$\frac{X \Rightarrow A, Y \quad X, B \Rightarrow Y}{X, A \rightarrow B \Rightarrow Y} \rightarrow_L$$

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$$\frac{X_2 \Rightarrow Y_2, Y_1}{(X_2 \triangleright Y_2) \Rightarrow Y_1} \triangleright_L \quad \frac{(X_1 \triangleright Y_1), X_2 \Rightarrow Y_2}{X_1, X_2 \Rightarrow Y_1, Y_2} S_L$$

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Deep inference

- Apply rules at any level

$$\frac{\Sigma[X, A \rightarrow B \triangleright A, Y] \quad \Sigma[X, A \rightarrow B, B \triangleright Y]}{\Sigma[X, A \rightarrow B \triangleright Y]} \rightarrow_L$$

Shallow vs Deep Inference

Shallow inference

- Apply rules to top-level sequent only

$$\frac{X \Rightarrow A, Y \quad X, B \Rightarrow Y}{X, A \rightarrow B \Rightarrow Y} \rightarrow_L$$

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$$\frac{X_2 \Rightarrow Y_2, Y_1}{(X_2 \triangleright Y_2) \Rightarrow Y_1} \triangleright_L \quad \frac{(X_1 \triangleright Y_1), X_2 \Rightarrow Y_2}{X_1, X_2 \Rightarrow Y_1, Y_2} s_L$$

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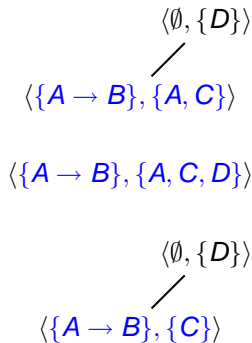
- Propagate some formulae between sequents at different levels

$$\frac{\Sigma^-[\{X\}, (X \triangleright Y)]}{\Sigma^-[X \triangleright Y]} \triangleright_{L1}$$

Example

Shallow inference

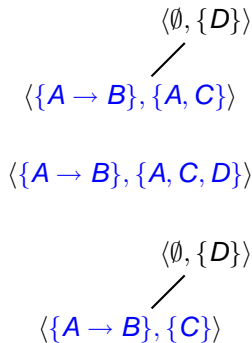
$$\frac{\frac{\frac{(A \rightarrow B \triangleright A, C) \Rightarrow D}{A \rightarrow B \Rightarrow A, C, D} s_L \quad \dots}{A \rightarrow B, A \rightarrow B \Rightarrow C, D} \rightarrow_L}{A \rightarrow B \Rightarrow C, D} c_L}{(A \rightarrow B \triangleright C) \Rightarrow D} \triangleright_L$$



Example

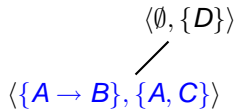
Shallow inference

$$\frac{\frac{(A \rightarrow B \triangleright A, C) \Rightarrow D}{A \rightarrow B \Rightarrow A, C, D} s_L \quad \dots}{\frac{A \rightarrow B, A \rightarrow B \Rightarrow C, D}{A \rightarrow B \Rightarrow C, D} c_L} \rightarrow_L \triangleright_L \frac{}{(A \rightarrow B \triangleright C) \Rightarrow D}$$



Deep inference

$$\frac{(A \rightarrow B \triangleright A, C) \triangleright D \quad \dots}{(A \rightarrow B \triangleright C) \triangleright D} \rightarrow_L$$



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 \end{array}
 \qquad
 \begin{array}{c}
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 \frac{A \rightarrow B \Rightarrow C, D}{A \rightarrow B \Rightarrow A, C, D} W_R \\
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 \frac{A \rightarrow B \Rightarrow A, C, D}{A \rightarrow B, A \rightarrow B \Rightarrow C, D} \dots \rightarrow_L \\
 \frac{A \rightarrow B, A \rightarrow B \Rightarrow C, D}{A \rightarrow B \Rightarrow C, D} C_L \\
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Rules

Logical rules:

$$\frac{\Sigma^- [A \multimap B, (A \triangleright B)]}{\Sigma^- [A \multimap B]} \multimap_L \qquad \frac{\Sigma^+ [A \rightarrow B, (A \triangleright B)]}{\Sigma^+ [A \rightarrow B]} \rightarrow_R$$

$$\frac{\Sigma[X, A \rightarrow B \triangleright A, Y] \quad \Sigma[X, A \rightarrow B, B \triangleright Y]}{\Sigma[X, A \rightarrow B \triangleright Y]} \rightarrow_L$$

$$\frac{\Sigma[X \triangleright Y, A \multimap B, A] \quad \Sigma[X, B \triangleright Y, A \multimap B]}{\Sigma[X \triangleright Y, A \multimap B]} \multimap_R$$

Rules

Logical rules:

$$\frac{\Sigma^- [A \prec B, (A \triangleright B)]}{\Sigma^- [A \prec B]} \prec_L \quad \frac{\Sigma^+ [A \rightarrow B, (A \triangleright B)]}{\Sigma^+ [A \rightarrow B]} \rightarrow_R$$

$$\frac{\Sigma [X, A \rightarrow B \triangleright A, Y] \quad \Sigma [X, A \rightarrow B, B \triangleright Y]}{\Sigma [X, A \rightarrow B \triangleright Y]} \rightarrow_L$$

$$\frac{\Sigma [X \triangleright Y, A \prec B, A] \quad \Sigma [X, B \triangleright Y, A \prec B]}{\Sigma [X \triangleright Y, A \prec B]} \prec_R$$

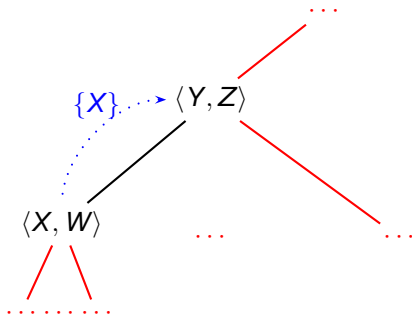
Propagation rules:

$$\frac{\Sigma^- [\{X\}, (X \triangleright Y)]}{\Sigma^- [X \triangleright Y]} \triangleright_{L1} \quad \frac{\Sigma^+ [(X \triangleright Y), \{Y\}]}{\Sigma^+ [X \triangleright Y]} \triangleright_{R1}$$

$$\frac{\Sigma [X \triangleright (W, (\{X\}, Y \triangleright Z))]}{\Sigma [X \triangleright (W, (Y \triangleright Z))]} \triangleright_{L2} \quad \frac{\Sigma [((X \triangleright Y, \{Z\}), W) \triangleright Z]}{\Sigma [((X \triangleright Y), W) \triangleright Z]} \triangleright_{R2}$$

Propagation Example

$$\frac{\Sigma[X \triangleright (W, (\{X\}, Y \triangleright Z))]}{\Sigma[X \triangleright (W, (Y \triangleright Z))]} \triangleright_{L2}$$



Uustalu's Example Revisited

$$\begin{array}{c}
 \frac{}{p \triangleright q, A, X, B, p \prec q, p} \text{id} \quad \frac{}{p, q \triangleright q, A, X, B, p \prec q} \text{id} \\
 \hline
 \frac{}{p \triangleright q, A, X, B, p \prec q} \prec_R \\
 \hline
 \frac{p \triangleright q, A, X, B, p \prec q}{p \triangleright q, A, (p, r \triangleright B, p \prec q)} \triangleright_{R1} \quad \dots \\
 \hline
 \frac{p \triangleright q, A, (p, r \triangleright (p \prec q) \wedge r)}{p \triangleright q, A, (r \triangleright (p \prec q) \wedge r)} \triangleright_{L2} \\
 \hline
 \frac{p \triangleright q, A, (r \triangleright (p \prec q) \wedge r)}{p \triangleright q, r \rightarrow ((p \prec q) \wedge r)} \rightarrow_R
 \end{array}$$

Where

$$\begin{aligned}
 A &= r \rightarrow ((p \prec q) \wedge r) \\
 B &= (p \prec q) \wedge r \\
 X &= p, r \triangleright B, p \prec q
 \end{aligned}$$

Soundness

Theorem (Soundness)

For any structures X and Y :

if $\vdash_{\text{DBilnt}} \Pi : X \triangleright Y$ then $\vdash_{\text{LBilnt}} \Pi' : X \Rightarrow Y$.

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Proof.

By induction on $|\Pi|$. Example case:

$$\frac{\Pi_1}{\frac{X \triangleright (Y_1 \triangleright Y_2), \{Y_2\}}{X \triangleright (Y_1 \triangleright Y_2)} \triangleright_{R1}} \rightsquigarrow \frac{\frac{\frac{\Pi'_1}{X \Rightarrow (Y_1 \triangleright Y_2), \{Y_2\}}{X, Y_1 \Rightarrow Y_2, \{Y_2\}} \text{SR}}{X, Y_1 \Rightarrow Y_2} \text{CR}}{X \Rightarrow (Y_1 \triangleright Y_2)} \triangleright_R$$

Where we obtain Π'_1 from Π_1 using IH and weakening. □

Admissibility of Residuation

Lemma (Admissibility of \triangleright_R)

If $\vdash_{\text{DBilnt}} \Pi : X, Y \triangleright Z$ then $\vdash_{\text{DBilnt}} \Pi' : X \triangleright (Y \triangleright Z)$.

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Proof.

By induction on $|\Pi|$. Example case:

$$\frac{\frac{\Pi_1}{(X_1 \triangleright X_2, \{Z\}), Y \triangleright Z} \triangleright_{R2}}{(X_1 \triangleright X_2), Y \triangleright Z} \triangleright_{R2} \quad \rightsquigarrow \quad \frac{\frac{\Pi'_1}{(X_1 \triangleright X_2, \{Z\}) \triangleright ((Y \triangleright Z), \{Z\})} \triangleright_{R2}}{(X_1 \triangleright X_2) \triangleright ((Y \triangleright Z), \{Z\})} \triangleright_{R1}}{(X_1 \triangleright X_2) \triangleright (Y \triangleright Z)}$$

Where we obtain Π'_1 from Π_1 using IH and weakening. □

Completeness

Theorem (Completeness)

For any structures X and Y :

if $\vdash_{\mathbf{LBiInt}} \Pi : X \Rightarrow Y$ then $\vdash_{\mathbf{DBiInt}} \Pi' : X \triangleright Y$.

Completeness

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Proof.

By induction on $|\Pi|$. Example case (logical rule):

$$\frac{\frac{\Pi_1}{X \Rightarrow A, Y} \quad \frac{\Pi_2}{X, B \Rightarrow Y}}{X, A \rightarrow B \Rightarrow Y} \rightarrow_L \rightsquigarrow \frac{\frac{\Pi'_1}{X, A \rightarrow B \triangleright A, Y} \quad \frac{\Pi'_2}{X, A \rightarrow B, B \triangleright Y}}{X, A \rightarrow B \triangleright Y} \rightarrow_L$$

Where we obtain Π'_1/Π'_2 from Π_1/Π_2 using IH and weakening. □

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For any structures X and Y :

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Proof.

By induction on $|\Pi|$. Example case (structural rule):

$$\frac{\Pi_1}{X, Y \Rightarrow Z} \triangleright_R \quad \rightsquigarrow \quad X \triangleright (Y \triangleright Z)$$

Where we obtain Π'_1 from Π_1 using IH and admissibility of \triangleright_R . □

Termination

- Non-termination caused by implicit contractions in \rightarrow_L and \leftarrow_R

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 - As in Int and dual-Int logic
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 - Extend traditional saturation methods to deep inference
 - Define blocking conditions in contexts

Blocking

Definition (Immediate super-structure)

$\widehat{\Sigma \square} = X \triangleright Y$ such that $X \triangleright Y$ is a sub-structure of Σ and $X = \square, X'$ for some structure X' or $Y = \square, Y'$ for some structure Y' .

Blocking

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$\widehat{\Sigma[]} = X \triangleright Y$ such that $X \triangleright Y$ is a sub-structure of Σ and $X = [], X'$ for some structure X' or $Y = [], Y'$ for some structure Y' .

Example

If $\Sigma[] = A, B \triangleright C, (D, (E \triangleright F) \triangleright [])$, then $\widehat{\Sigma[G]} = (D, (E \triangleright F) \triangleright G)$.

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Definition (Sequent superset)

$(X_1 \triangleright X_2) \supset (Y_1 \triangleright Y_2)$ iff $\{X_1\} \supset \{Y_1\}$ or $\{X_2\} \supset \{Y_2\}$.

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$(X_1 \triangleright X_2) \supset (Y_1 \triangleright Y_2)$ iff $\{X_1\} \supset \{Y_1\}$ or $\{X_2\} \supset \{Y_2\}$.

New \rightarrow_L rule:

$$\frac{\Sigma[X, A \rightarrow B \triangleright A, Y] \quad \Sigma[X, A \rightarrow B, B \triangleright Y]}{\Sigma[X, A \rightarrow B \triangleright Y]} \rightarrow_L$$

Where $\overbrace{\Sigma[X, A \rightarrow B \triangleright A, Y]} \supset \overbrace{\Sigma[X, A \rightarrow B \triangleright Y]}$

and $\overbrace{\Sigma[X, A \rightarrow B, B \triangleright Y]} \supset \overbrace{\Sigma[X, A \rightarrow B \triangleright Y]}$

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 - Characterisation of logics/axioms where residuation can be mimicked by deep inference