

On Capacity for Single-Frequency Spatial Channels

Leif W. Hanlen¹
National ICT Australia
Canberra Australia
leif.hanlen@nicta.com.au

Alex J. Grant
University of South Australia
Adelaide Australia
alex.grant@unisa.edu.au

Rodney A. Kennedy
Australian National University
Canberra Australia
rodney.kennedy@anu.edu.au

Abstract —

We present information theoretic capacity results for information transfer in space, under the general assumption that the receiver applies a spatial filter to receive signals. Our work presents novel applications of linear, bounded, invariant operators to communication channels. We outline the procedure for calculating the information theoretic capacity for this restricted class of operator channels and apply this result to wireless information transfer.

I. INTRODUCTION

Recently, there has been great interest in the concept of wireless channels, where space is considered as a “continuous” parameter [1–7]. This interest is motivated by an understanding that the capacity of a spatially constrained channel is determined by the continuous nature of space, rather than the number of antenna elements used in a MIMO system [7]. Such results have been motivated by questions of the form “*Is there a limiting capacity, as the elements of a MIMO system become dense?*” It has been shown [2, 8] that standard MIMO vector channels [9] are inappropriate in the case where the number of “elements” are allowed to increase indefinitely.

The examination of signals which vary over continuous fields has been examined in detail in the continuous time setting as a “waveform channel” [10] and more generally as a “continuous channel” [11, sec.24]. These examples represent particular cases of bounded-linear operator channel in additive white Gaussian noise. We shall use [12] to provide preliminary steps toward answering the following question [1]: *Given a volume in space Ω_r , what is the information rate achievable within Ω_r , subject to power limit and wavefield constraints?*

This paper is arranged as follows: In Section II we examine the underlying properties of bounded-linear operators as communication channels, and provide a procedure for calculating information theoretic capacity results. We then apply this technique in two examples. In Section III we develop a capacity result for information transfer between two concentric bodies, abstracting an indoor environment, while in Section IV we investigate the capacity for a region of space. Section V draws conclusions.

II Transmitting between spaces

Consider transmitting over a channel

$$y(\mathbf{r}_r) = \Gamma x(\mathbf{r}_t) + s(\mathbf{r}_r) \quad (1)$$

where $y(\mathbf{r}_r) \in \mathcal{H}_r$ with $\mathbf{r}_r \in \mathbb{R}^m$ is the receive function, $x(\mathbf{r}_t) \in \mathcal{H}_t$ with $\mathbf{r}_t \in \mathbb{R}^n$ is the transmit function and $s(\mathbf{r}_r) \in \mathcal{H}_r$ is an additive noise process. The spaces \mathcal{H}_t and \mathcal{H}_r are assumed to be \aleph_0 dimensional Hilbert spaces, although we easily accommodate the case where one or both dimensions is finite. The functions $x(\mathbf{r}_t)$, $y(\mathbf{r}_r)$ and $s(\mathbf{r}_r)$ are random processes, while the operator Γ is fixed. The function $x(\mathbf{r}_t)$ is power limited:

$$\|x\|_{\mathcal{H}_t}^2 \leq P \quad (2)$$

We shall assume that the operator Γ defines an ergodic channel from \mathcal{H}_t to \mathcal{H}_r . Γ is a fixed bounded, linear operator taking functions in \mathcal{H}_t to functions in \mathcal{H}_r :

$$\Gamma : \{v(\mathbf{r}_t) \in \mathcal{H}_t \mapsto u(\mathbf{r}_r) \in \mathcal{H}_r : u(\mathbf{r}_r) = \Gamma v(\mathbf{r}_t)\} \quad (3)$$

Recall [13], a bounded operator $\Gamma : \mathcal{H}_t \mapsto \mathcal{H}_r$ may be represented by an infinite dimensional matrix², $\vec{\Gamma}$

$$\left[\vec{\Gamma} \right]_{ji} \triangleq \langle \Gamma \vartheta_i, \varphi_j \rangle_{\Omega_r} \quad (4)$$

where $\{\varphi_i\}_{i=1}^\infty$ and $\{\vartheta_j\}_{j=1}^\infty$ are complete, orthonormal sequences in \mathcal{H}_r and \mathcal{H}_t respectively. When $\{\varphi_i\}_{i=1}^\infty$ and $\{\vartheta_j\}_{j=1}^\infty$ are the left- and right- eigenfunctions (respectively) of the operator Γ , then $\vec{\Gamma}$ is diagonal.

We shall assume that the spaces \mathcal{H}_t and \mathcal{H}_r have a common dimension t , such that the random processes $x(\mathbf{r}_t)$, $y(\mathbf{r}_r)$ and $s(\mathbf{r}_r)$ are ergodic over t . We may then consider our input parameter \mathbf{r}_t (respectively \mathbf{r}_r) as being composed

$$\mathbf{r}_t = \{t, \hat{\mathbf{r}}_t\}, \quad t \in \mathbf{T} \subseteq \mathbb{R} \quad (5)$$

$$\mathbf{r}_r = \{t, \hat{\mathbf{r}}_r\}, \quad t \in \mathbf{T} \subseteq \mathbb{R} \quad (6)$$

where t is a dummy variable over which we shall average to produce a capacity result³. We shall nominally assume that \mathbf{T} is an interval $\mathbf{T} = [0, \tau] \subset \mathbb{R}$ and allow the parameter τ to increase to infinity. We shall use the symbol t to remind ourselves that the dimension over which we average must have similar properties to that of “time” in the standard continuous channel [11] capacity derivation. We may write cf. [10]

$$C_{\mathbf{T}} = \frac{1}{\|\mathbf{T}\|} [\sup I(x; y)] \quad (7)$$

and

$$C = \lim_{\|\mathbf{T}\| \rightarrow \infty} C_{\mathbf{T}} \quad (8)$$

²We shall use the notation Γ to denote an operator, and $\vec{\Gamma}$ to denote its matrix representation.

³As part of the “average” we allow $\|\mathbf{T}\| \rightarrow \infty$ so that the dimension must be reasonably well behaved.

¹National ICT Australia is funded through the Australian Government’s *Backing Australia’s Ability* initiative, in part through the Australian Research Council

where $I(x; y)$ is the mutual information between input and output, cf. [10]. We shall also assume that the operator Γ is known at the transmitter and receiver ends of the communication link, so that the channel remains ergodic.

The continuous capacity results for $L^2(\mathbb{R})$ is extendable to more general Hilbert space frameworks. In order to apply the results of [10] we need to establish a valid noise model – so we may ensure the operator capacity results remain consistent with finite dimensional vector channels. We note that the operator – both noise and channel – acting over arbitrary Hilbert spaces, must be related back to a standard reference frame. The natural choice of reference frame is ℓ^2 – so that we may consider an equivalent additive-white-Gaussian operator channel, $\hat{\Gamma} : \ell^2 \mapsto \ell^2$. In sub-section A we provide details of the white noise equivalent for ℓ^2 functions. We outline a method for calculating the capacity of the operator channel in sub-section B, and provide the capacity in Lemma 1.

A Noise

Although we refer to $s(\mathbf{r})$ as “noise”, we note that $s(\mathbf{r})$ is not a white Gaussian noise process, since such processes are not strictly contained within an Hilbert Space. Fortunately, the work of [10] allows us to define white Gaussian noise as a random process *projected* into ℓ^2 . We shall assume that the noise corrupting y is *Gaussian* in ℓ^2 . This ensures that our capacity result may rely on the standard Gaussian noise results [10, 11].

Definition 1 (White Gaussian Noise in ℓ^2). *White Gaussian noise, in $\ell^2(\Omega_r)$ is given by a random process $z(\mathbf{r})$ for $\mathbf{r} \in \Omega_r \subseteq \mathbb{R}^N$, such that for any function $\psi(\mathbf{r}) \in \ell^2(\Omega_r)$ the complex scalar z*

$$z \triangleq \int_{\Omega_r} z(\mathbf{r}) \overline{\psi(\mathbf{r})} d\mathbf{r} = \langle z, \psi \rangle_{\ell^2} \quad (9)$$

is a zero mean, Gaussian random variable, with variance

$$E \{ |z_i|^2 \} = \frac{N_0}{2} \int_{\Omega_r} |\psi(\mathbf{r})|^2 d\mathbf{r} = \frac{N_0}{2} (\|\psi\|_{\ell^2})^2 \quad (10)$$

for constant N_0 independent of $\psi(\mathbf{r})$. cf. [10, eqn. 8.1.35]

If the functions $\psi_i(\mathbf{r})$ are orthonormal, then we may write $z = \{z_i\}_{i=1}^\infty$ as a vector of i.i.d. Gaussian random variables, with zero mean and variance $N_0/2$. Since all \aleph_0 dimensional Hilbert spaces are isomorphic to ℓ^2 [13, Thm 3.6-5], if we define $\{\alpha_i\}_{i=1}^\infty$ as a complete orthonormal sequence in \mathcal{H}_a and $\{\theta_j\}_{j=1}^\infty$ is a complete orthonormal sequence in ℓ^2 then the isomorphism P is given by the construction:

$$\vec{P}_{ij} = \langle \alpha_i, \theta_j \rangle_{\ell^2} \quad (11)$$

which allows us to define a Gaussian process in terms of its effect on the integral of a function – corresponding to the inner product in ℓ^2 – to give the noise in an arbitrary Hilbert space \mathcal{H} even though the noise may not be defined in terms of the inner product in \mathcal{H} .

Our method of calculating the capacity is summarized as follows:

B Capacity Method

1. Provide a mapping $P : \ell^2 \mapsto \mathcal{H}_r$ to ensure noise in \mathcal{H}_r is related to (possibly correlated) Gaussian noise.
2. Write $y = \Gamma x + Pz$ where $z \in \ell^2$
3. Write $\hat{y} = P^{-1}y$ where $\hat{y} \in \ell^2$
4. Find capacity of “operator channel”

$$\hat{y} = \hat{\Gamma}x + z \quad \hat{\Gamma} = P^{-1}\Gamma : \ell^2 \mapsto \ell^2 \quad (12)$$

- (a) decompose the operator $\hat{\Gamma}$ into parallel, independent discrete AWGN channels
- (b) choose the input x to be zero mean, Gaussian random variables with variances chosen according to the water-filling [10] algorithm.

C Decomposition of Operator

In similar manner as for the eigen-decomposition of a matrix channel, we decompose the operator channel (1) into a set of parallel, independent additive white Gaussian noise channels. These channels are given by the eigenfunctions of the operator which may be found as solutions of:

Problem 1. *Given an operator $\hat{\Gamma} : \phi = \hat{\Gamma}\psi$ find the solutions $\lambda_i : \{\lambda_1 \geq \lambda_2 \cdots\}$ and function(s) ψ_i to:*

$$\lambda_i = \max_{\|\psi_i\|=1} \|\phi_i\|^2 = \max_{\|\psi_i\|=1} \|\hat{\Gamma}\psi_i\|^2 \quad (13)$$

such that $\langle \psi_j(\mathbf{r}_t), \psi_{i < j}(\mathbf{r}_t) \rangle = 0$.

The functions $\psi_j(\mathbf{r}_t)$ may be considered as the matched-filter responses for the channel. From Parseval’s theorem [13, pp.170] the value of $\|\phi\|^2$ is fixed, independent of the definition of the inner product, for all \aleph_0 dimension, complete separable Hilbert Spaces. Numerically we may perform (13) by finding the maximum eigenvalue λ_1 , and associated eigenfunction ψ_1 , and then successively restricting our search space to only functions which are orthogonal to ψ_1 , etc.

Lemma 1 (Capacity of Bounded Linear Invariant Operator). *Consider the operator channel (12), with eigenvalues λ_i , eigenfunctions $\varphi_i(\mathbf{r}, t)$ and input power constraint*

$$\|x\|^2 = P \|\mathbf{T}\| \quad (14)$$

The capacity of the channel (12) is given parametrically by

$$C_B = \lim_{\|\mathbf{T}\| \rightarrow \infty} \frac{1}{\|\mathbf{T}\|} \sum_{i \in \mathbb{Z}_B} \frac{1}{2} \log_2 (\lambda_i (\|\mathbf{T}\|) B) \quad (15)$$

$$P_B = \lim_{\|\mathbf{T}\| \rightarrow \infty} \frac{1}{\|\mathbf{T}\|} \sum_{i \in \mathbb{Z}_B} \left(B - \frac{1}{\lambda_i (\|\mathbf{T}\|)} \right) \quad (16)$$

and \mathbb{Z}_B is the set of integers associated with the eigenvalues λ_i , such that $B \geq \lambda_i$, and the parameter B is chosen to maximise (15). The capacity C is achieved by setting $x_i = \langle x, \varphi_i \rangle$ to be zero mean, independent Gaussian random variables with variance

$$E \{ |x_i|^2 \} = \max \left(0, B - \frac{1}{\lambda_i} \right) \quad (17)$$

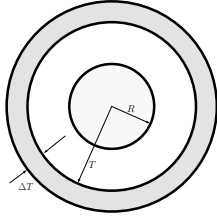


Figure 1: Transmit and receive regions with dimensions

III Example: Concentric spheres

Consider a spherical receiving volume Ω_r , of radius R which is enclosed in a spherical annulus Ω_t of radius T and thickness Δ_T as shown in Figure 1. We shall assume that R is small compared with the transmit radius ($R \ll T$) and the transmit annulus is thin $\Delta_T \ll T$. Figure 1 may be considered as an abstraction for a small wireless, sensing device inside a large room, we consider the “transmitter” to be the *entire volume of the annulus* Ω_t and the “receiver” to be the *entire volume of the sphere* Ω_r . We shall not be interested in “antenna elements” rather, the transmitter chooses continuous functions $f(\mathbf{r}_t)$ over the domain Ω_t . These functions produce corresponding continuous receive functions $g(\mathbf{r}_r)$ over the domain Ω_r , nominally via free-space wave propagation. We shall not be interested in polarity or other vector field effects in terms of capacity. Clearly, we may impose “antenna-like” properties on the transmit and receive fields in the same way that we might impose particular coding or modulation properties on a signal: we remove such constraints in an attempt to obtain insight into the fundamental restrictions on information theoretic capacity *due to free-space wave transmission*.

The field in Ω_t may be described in terms of a complete orthonormal set $\{\psi_i(\mathbf{r}_t)\}_{i=1}^{\infty}$ for $\mathbf{r}_t \in \Omega_t$, $\Omega_t = \mathbb{R}^3 \times \mathbf{T}$. Similarly, in Ω_r we may describe the receive field in terms of a (possibly different) complete orthonormal set $\{\phi_i(\mathbf{r}_r)\}_{i=1}^{\infty}$ for $\mathbf{r}_r \in \Omega_r$, $\Omega_r = \mathbb{R}^3 \times \mathbf{T}$. Then the channel may be described in terms of a bounded linear operator Υ ,

$$\Upsilon : L^2(\Omega_t) \rightarrow L^2(\Omega_r) \quad (18)$$

$$g(\mathbf{r}_r) = \Upsilon f(\mathbf{r}_t) + s(\mathbf{r}_r) \quad (19)$$

$$= \int_{\Omega_t} G(\mathbf{r}_r, \mathbf{r}_t) f(\mathbf{r}_t) d\mathbf{r}_t + s(\mathbf{r}_r) \quad (20)$$

where $g(\mathbf{r}_r)$ is the receive signal in Ω_r , $f(\mathbf{r}_t)$ is the transmit signal in Ω_t and $s(\mathbf{r}_r)$ is noise in Ω_r . The orthonormal functions $\phi_i(\mathbf{r}_r)$ are given by cf. [14]

$$\phi_i(\mathbf{r}_r) = \lim_{\tau \rightarrow \infty} \imath \frac{j_n(kr) Y_n^m(\theta, \vartheta)}{\mathbf{J}_n(k, R)^{1/2}} \cdot \frac{\exp(-\imath kt)}{\sqrt{\tau/2}} \quad (21)$$

$$\mathbf{r}_r = \{r, \theta, \vartheta, t\}, \mathbf{T} = [-\tau/2, \tau/2] \subset \mathbb{R} \quad (22)$$

where $\imath = \sqrt{-1}$, $j_n(z)$ is the n^{th} spherical Bessel function⁴ and $Y_n^m(\theta, \vartheta)$ is the m, n^{th} spherical Harmonic function,

⁴The spherical Bessel function $j_n(z)$ is related to the Bessel Function of the first kind through $j_n(z) = (\frac{\pi}{2z})^{1/2} J_{n+1/2}(z)$

which is orthonormal on the unit sphere. $\mathbf{J}_n(k, R)$ is a normalizing constant, cf. [1], to ensure orthonormality in Ω_r . Using [15, pp.658, 6.521.1]

$$\mathbf{J}_n(k, R) = \int_0^R r^2 [j_n(kr)]^2 dr = \frac{\pi R^2}{4k} [J_{n+1+1/2}(Rk)]^2$$

We may relate the indices i , m and n through the following enumeration:

$$i = n(n+1) + m \quad (23)$$

The point-source solution to the (source-free) wave equation at \mathbf{r}_r , with point-source at \mathbf{r}_t , $|\mathbf{r}_t| > |\mathbf{r}_r|$ defines $G(\mathbf{r}_r, \mathbf{r}_t)$, cf. [14, pp.9]

$$G(\mathbf{r}_r, \mathbf{r}_t) = \imath k \sum_{n,m} j_n(k|\mathbf{r}_r|) h_n^{(1)}(k|\mathbf{r}_t|) Y_n^m(\hat{\mathbf{r}}_r) \overline{Y_n^m(\hat{\mathbf{r}}_t)} \quad (24)$$

where $h_n^{(1)}(\cdot)$ is the spherical Hankel function⁵ of order n [15], and k is the wave-number, which we have assumed is a scalar: $k = 2\pi/\lambda = 2\pi f/c$. We use the symbol $\sum_{m,n} \equiv \sum_{n=0}^{\infty} \sum_{m=-n}^n$. Comparing (24) and (21) suggests the following orthonormal set for $\psi_i(\mathbf{r}_t)$:

$$\psi_i(\mathbf{r}_t) = (\imath)^{n-1} \frac{Y_n^m(\theta, \vartheta) h_n^{(2)}(kr)}{[\mathbf{H}_n(k, T + \Delta T) - \mathbf{H}_n(k, T)]^{1/2}} \quad (25)$$

where $\mathbf{H}_n(k, T)$ is a normalising factor,

$$\mathbf{H}_n(k, T) = \int_0^T r^2 |h_n^{(1)}(kr)|^2 dr \quad (26)$$

By construction, we see that $\{\psi_i(\mathbf{r}_t)\}_{i=0}^{\infty}$ and $\{\phi_i(\mathbf{r}_r)\}_{i=0}^{\infty}$ are the left- and right- eigenfunctions respectively of the Υ operator. We may write the eigenvalues γ_i as γ_n^m with the enumeration (23). The eigenvalues are given by:

$$|\gamma_n^m|^2 = \left| \int_{\Omega_r} \int_{\Omega_t} \phi_i(\mathbf{r}_r) G(\mathbf{r}_r, \mathbf{r}_t) \psi_i(\mathbf{r}_t) d\mathbf{r}_t d\mathbf{r}_r \right|^2 \quad (27)$$

$$= k^2 \mathbf{J}_n(k, R) [\mathbf{H}_n(k, T + \Delta T) - \mathbf{H}_n(k, T)] \quad (28)$$

where we note that there will be groups of equal magnitude eigenvalues, with the magnitudes diminishing for increasing n .

Given the receive functions, we may project the noise onto $\{\phi_i(\mathbf{r}_r)\}_{i=1}^{\infty}$. From the orthonormality of the receive basis functions, this projection is a zero-mean scalar random variable z_i with variance $E\{|z_i|^2\} = N0/2$.

A Equivalent parallel channel model

We note that the set $\{\psi_i\}_{i=1}^{\infty}$ is *not complete* over the functions in $L^2(\Omega_t)$, however, the set is complete over functions with

⁵The spherical Hankel function $h_n^{(1)}(z)$ is related to the Bessel Function of the third kind through $h_n(z) = (\frac{\pi}{2z})^{1/2} H_{n+1/2}^{(1)}(z)$

support in Ω_t , which have non-zero solutions to (24). As such, we may expand an arbitrary function $f(\mathbf{r}_t) \in L^2(\Omega_t)$ as:

$$f(\mathbf{r}_t) = f_\Delta(\mathbf{r}_t) + \sum_{i=1}^{\infty} f_i \psi_i(\mathbf{r}_t) = f_\Delta(\mathbf{r}_t) + \tilde{f}(\mathbf{r}_t) \quad (29)$$

such that

$$\Upsilon f = \Upsilon (\tilde{f} + f_\Delta) = \Upsilon \tilde{f} + 0 \quad (30)$$

$$= \sum_i \lambda_i \langle f(\mathbf{r}_t), \phi_i(\mathbf{r}_t) \rangle \quad (31)$$

B Capacity

We are now in a position to consider the free-space transfer between Ω_t and Ω_r in terms of an infinite set of parallel, independent, AWGN channels. Each channel i has a noise variance $N_0/2$ and gain magnitude $|\lambda_i|$ given by (28).

Theorem 1. Consider the channel (20), with transmitter shell of radius T and thickness ΔT , and receive sphere of radius R such that $T \gg R \gg \Delta T$. The noise variance is $N_0/2$ and transmitter is limited to power P .

The capacity of the channel is given by waterfilling – Lemma 1 – with

$$\lambda_i = k \sqrt{\mathbf{J}_n(k, R) [\mathbf{H}_n(k, T + \Delta T) - \mathbf{H}_n(k, T)]} \quad (32)$$

and using the enumeration (23).

Corollary 1.1 (Capacity bound for concentric shells). The capacity is bounded from above by

$$C \leq \overline{N}_c \log \left(1 + \frac{2P}{N_0} \gamma^2 \frac{1}{N_c} \right) \quad (33)$$

$$\leq \frac{e^2}{4} (kR)^2 \log \left(1 + \frac{2P}{N_0} \frac{3\pi^2 \Delta T}{8^2 k^4 R} \right) \quad (34)$$

where \overline{N}_c (54) and N_c (53) are upper and lower bounds (respectively) on the number of well connected modes.

We may show by symmetry that the same result holds if the role of the inner and outer spheres are reversed – i.e. the transmitter is placed inside the receiver. In Figure 2 we have plotted the upper bound from Theorem 1 with respect to the radius R of the inner sphere.

IV Example: Information Capacity for region of space

Our second example is intended as an initial step toward an information theoretic capacity for space. We shall consider a physical arrangement corresponding to a single-frequency f_c receiver, fixed in space. We shall again assume full transmitter knowledge of the channel. We wish to determine the information theoretic capacity of such an arrangement. We shall assume that the transmitter is power limited

$$\int_{\Omega_t} |f(\mathbf{x})|^2 d\mathbf{x} \leq P \cdot \|\Omega_t\| \quad (35)$$

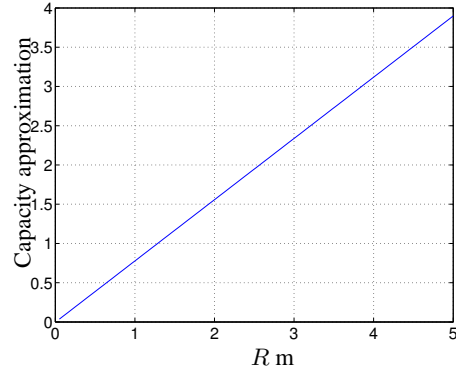


Figure 2: Capacity C bit/s/Hz for concentric spherical channel, at 3×10^8 Hz, and $P = 10$ dB.

We shall *not* constrain the “transmit” field to satisfy the wave equation, so that any valid $L^2(\Omega_t)$ function is accepted. We assume that the field has zero power outside Ω_t . We note that for fields resulting from radiating sources, then (35) simply imposes the well known $1/D^2$ path loss constraint of the field, and Ω_t may be considered as a “large” sphere encompassing the “sources” and the receive region Ω_r . So that the effective power limit of Section III is $P \cdot \frac{4}{3} \pi T^3$.

We shall allow Ω_r and Ω_t to intersect – so that the source field may overlap the receive region: if the field does not intersect Ω_r then no information can be passed to the receiver! Some care must be taken here, since this does not necessarily imply that there are “sources” within Ω_r , simply that the energy from the transmitter is detectable within Ω_r .

We shall assume that the receiver does not use independent sources to measure the field in Ω_r . This is a more general constraint than assuming there are no sources within Ω_r – a source free Ω_r satisfies this assumption. We may consider the assumption in terms of the receiver applying a particular matched filter to the received field: the Helmholtz Scalar Wave Projection operator:

Definition 2 (Helmholtz Projection Operator P_H , cf. [1]). P_H projects functions with support in Ω_r , $u(\mathbf{r}_r) \in L^2(\Omega_r)$ onto functions with support in Ω_r which satisfy the Helmholtz Wave Equation

$$(\nabla^2 + kI) g(\mathbf{r}_r) = 0 \quad (36)$$

$$g(\mathbf{r}_r) = P_H f(\mathbf{r}_r) = \sum_{i=1}^{\infty} \langle f(\mathbf{r}_r), \phi_i(\mathbf{r}_r) \rangle_{\Omega_r} \phi_i(\mathbf{r}_r) \quad (37)$$

where $\{\phi_i(\mathbf{r}_r)\}_{i=1}^{\infty}$ are a complete orthonormal set, satisfying (36).

This projection operator generates a field $g(\mathbf{r}_r)$ for $\mathbf{r}_r \in \Omega_r$ which satisfies (36) which is the closest (in an RMS mean sense) to the actual field within Ω_r , [1, 3, 4]. Note that this filter is *information lossy* that is, there are fields within Ω_r which

cannot be uniquely determined by examining $g(\mathbf{r}_r)$. The operator P_H is the spatial equivalent of the (low-pass) linear filter representation of [10, ch. 8]. We will show shortly that this constraint does not cause significant problems. We shall also use the following Truncation projection operator cf. [1]:

Definition 3 (Truncation operator Π_{Ω_r}). *The Truncation operator forces a field to zero, outside a region Ω :*

$$\Pi_{\Omega} g(\mathbf{r}_r) = \begin{cases} g(\mathbf{r}_r) & \mathbf{r}_r \in \Omega \\ 0 & \text{else} \end{cases} \quad (38)$$

The channel is then given by

$$g(\mathbf{r}_r) = \Gamma f(\mathbf{r}_t) + s(\mathbf{r}_r) \quad (39)$$

$$= P_H \Pi_{\Omega_r} \Pi_{\Omega_t} f(\mathbf{r}_t) + s(\mathbf{r}_r) \quad (40)$$

For any function $f(\mathbf{r}_t)$, $\|g(\mathbf{r}_r)\|^2$ is maximised when $\Pi_{\Omega_r} \Pi_{\Omega_t} = \Pi_{\Omega_t}$, ie, when $\Omega_t \subseteq \Omega_r$. We may interpret this by noting that there is no benefit in expending energy to generate a field (or component of a field) which is outside the region of measurement Ω_r . We therefore wish to find eigenfunctions to the operator problem:

Problem 2 (Eigenfunctions for Ω_r). *Find ordered solution(s) λ_i and $\phi_i(\mathbf{r}_t)$ to:*

$$\lambda_i = \max \frac{\|P_H \Pi_{\Omega_r} \Pi_{\Omega_t} \psi_i(\mathbf{r}_t)\|^2}{\|\psi_i(\mathbf{r}_t)\|_{\Omega_t}^2} \quad (41)$$

$$= \max_{\|\psi_i(\mathbf{r}_t)\|_{\Omega_t}=1} \|P_H \Pi_{\Omega_r} \phi_i(\mathbf{r}_t)\|^2 \quad (42)$$

For an arbitrary receive body Ω_r , the solution to Problem 2 is intractable analytically. However, we may apply a bounding argument to Ω_r , such that any solution $\psi_i(\mathbf{r}_t)$ must have support within a sphere $S(\Omega_r)$ which encloses Ω_r . We may then apply a Gram-Schmidt orthonormalization on any (spherical-based) result, for approaches in the case of arbitrary bodies Ω_r see [1, 6]. For the following, we shall consider the case where Ω_r is a sphere of radius R .

From Section III we already know the eigenfunction solutions to (42) are given by (21). Using a similar argument as previously, we note that while the “transmit” function $f(\mathbf{r}_t) \in L^2(\Omega_t)$ need not satisfy the wave-equation, only those components of $f(\mathbf{r}_t)$ which project onto wave-equation solutions have non-zero eigenvalues λ_i . Consequently, a transmitter with full channel knowledge will choose only those functions $\hat{f}(\mathbf{r}_t) \in \mathcal{H}_t \subset L^2(\Omega_r)$ which have non-zero projections under P_H (corresponding to $\lambda_i > 0$). This places our work in close correspondence to [1, Problem 1]. We may write the transmit and receive functions in terms of (21):

$$x(\mathbf{r}_t) = \sum_i \phi_i(\mathbf{r}_t) x_i + x_\Delta(\mathbf{r}_t) \quad (43)$$

$$y(\mathbf{r}_r) = \sum_i \phi_i(\mathbf{r}_r) y_i + s_\Delta(\mathbf{r}_r) \quad (44)$$

$$= \sum_i \phi_i(\mathbf{r}_r) (\lambda_i x_i + z_i) + s_\Delta(\mathbf{r}_r) \quad (45)$$

where $s_\Delta(\mathbf{r}_r)$ is a noise component orthogonal to all $\phi_i(\mathbf{r}_r)$ and $x_\Delta(\mathbf{r}_t)$ is the component of the transmit function which is orthogonal to all $\phi_i(\mathbf{r}_t)$. We use the enumeration of (23) to map between i and m, n for the basis functions. Applying (21) to (42) we see that $\lambda_i = 1$.

Theorem 2. *Consider the channel (40) for a spherical receive region with radius R , and signal transmit frequency f , with wavenumber $k = 2\pi f/c$, noise variance $N_0/2$ and power limit $P \cdot 4\pi R^3/3$ the capacity of the channel is given by Lemma 1, with $\lambda_i = 1$.*

The capacity of this channel is bounded by noting that the eigenvalues are unity, and the number of parallel channels is bounded by (53) and (54):

Corollary 2.1. *The capacity C of the channel (40) is bounded by*

$$C \leq n \log \left(1 + \frac{P}{N_0/2} \frac{4}{3} \pi R^3 \frac{1}{n} \right) \leq \frac{P \frac{4}{3} \pi R^3}{N_0} \log e \quad (46)$$

V Conclusions

We have examined an abstraction of standard vector channel MIMO results, to incorporate bounded invariant linear operators acting over isomorphisms of ℓ^2 . We have provided capacity results for such operators.

The operator framework has been used to provide capacity results for otherwise intractable MIMO problems. We have shown two spatial examples, where we have calculated the fundamental limits to transmission of spatially detectable information between concentric shells, and found the limits to information capacity of a spherical region of space. Such results are extendible analytically to other simple geometries, and numerically to non-trivial geometries.

A Proofs

Proof of Lemma 1. The operator channel is bounded, so that $\hat{\Gamma}$ may be written in matrix form $\vec{\hat{\Gamma}}$. Any transmit function $f(\mathbf{r}_t) \in \mathcal{H}_t$ may be written as:

$$f(\mathbf{r}_t) = f_\Delta(\mathbf{r}_t) + \sum_i f_i \phi_i(\mathbf{r}_t)$$

where $f_\Delta(\mathbf{r}_t)$ is orthogonal to all ϕ_i . Since ϕ_i are eigenfunctions, $\vec{\hat{\Gamma}}$ is diagonal, so every receive function $g(\mathbf{r}_r) \in \mathcal{H}_r$ may be written as

$$g(\mathbf{r}_r) = g_\Delta(\mathbf{r}_r) + \sum_i \lambda_i f_i \phi_i(\mathbf{r}_r) + 0 \cdot f_\Delta(\mathbf{r}_r) + \sum_i z_i \phi_i(\mathbf{r}_r)$$

where $g_\Delta(\mathbf{r}_r)$ is an orthogonal noise component. The channels $g_i = \lambda_i f_i + z_i$ are independent (due to the orthogonality of ϕ_i and we use the parallel AWGN channel result of [10]. ■

Proof of Corollary 1.1. We note that while there are an infinite number of channels, only a finite subset N_c have non-negligible gains, ie. $|\lambda_i| > \epsilon, i < N_c$, and $|\lambda_i| < \epsilon, i > N_c$. Our aim is to estimate N_c . From [16]:

$$|J_{n+1+1/2}(Rk)| \leq \frac{(Rk)^{n+3/2}}{2^{n+3/2}\Gamma(n+5/2)} \quad (47)$$

and for $kT \gg 1$, and $T \gg \Delta T$,

$$\mathbf{H}_n(k, T + \Delta T) - \mathbf{H}_n(k, T) \approx \frac{\pi \Delta T}{4k^2} \quad (48)$$

Using (47), (48) and (28):

$$|\gamma_n^m|^2 \leq \frac{\pi R^2}{4k} \left(\frac{(Rk)^{n+3/2}}{2^{n+3/2}\Gamma(n+5/2)} \right)^2 \frac{\pi \Delta T}{4} \quad (49)$$

This gives a hard cut-off in eigenvalue magnitudes, although the result is difficult to interpret in terms of the original geometry. We may apply the following bound, as the operator $\|\Upsilon\|^2$ is compact, and has norm given by:

$$\|\Upsilon\|^2 = \sum_{n,m} |\gamma_n^m|^2 \approx \frac{\Omega_r \Omega_t}{(4\pi T)^2} \leq \frac{R^3 \Delta T}{3} \quad (50)$$

We note for $|Rk| \gg n$, and $T \gg \Delta T$, and using (48)

$$\mathbf{J}_n(k, R) \leq \frac{R}{2k^2} \quad (51)$$

$$\gamma_n^m \leq \sqrt{\frac{\pi R \Delta T}{8k^2}} \quad (52)$$

and both $\mathbf{J}_n(k, R)$ and $\mathbf{H}_n(k, R)$ diminish for large n (with fixed $k, R, \Delta T$). Combining (52) with (50), gives a (lower) bound on the number of well-connected modes:

$$N_c \geq \underline{N}_c = \frac{8}{3\pi} (kR)^2 \quad (53)$$

From [16] $\mathbf{J}_n(k, R)$ is exponentially decreasing for $n \geq (ekR)^2/4$, and $\mathbf{H}_n(k, T + \Delta T) - \mathbf{H}_n(k, T)$ is bounded by (48). This provides us with an upper bound on N

$$N_c \leq \overline{N}_c = \frac{e^2}{4} (kR)^2 \quad (54)$$

■

REFERENCES

- [1] R. A. Kennedy and T. D. Abhayapala, "Spatial concentration of wavefields: Towards spatial information content in arbitrary multipath scattering," in *4th Aust. Commun. Theory Workshop, AusCTW*, Melbourne, Australia, Feb. 4–5 2003, pp. 38–45.
- [2] L. W. Hanlen, "Channel modelling and capacity analysis for multi-input, multi-output, wireless communication systems," Ph.D. dissertation, The University of Newcastle, June 2003.
- [3] T. S. Pollock, T. D. Abhayapala, and R. A. Kennedy, "Fundamental limits of MIMO capacity for spatially constrained arrays," in *4th Aust. Commun. Theory Workshop, AusCTW*, Melbourne, Australia, Feb. 4–5 2003, pp. 7–12.
- [4] R. A. Kennedy, T. D. Abhayapala, and T. S. Pollock, "Modeling multipath scattering environments using generalized Herglotz wave functions," in *Proc. 4th Australian Communications Theory Workshop*, Melbourne, Australia, Feb. 4–5 2003, pp. 87–92.
- [5] J. W. Wallace and M. A. Jensen, "Intrinsic capacity of the MIMO wireless channel," in *Proc. IEEE Intl. Symp. Antennas and Propagation Society*, 2002, pp. 696–700.
- [6] L. W. Hanlen and M. Fu, "Wireless communications with spatial diversity: A volumetric approach," in *Proc. IEEE Intl. Conf. Commun., ICC*, Anchorage, Alaska, May 11–15 2003, pp. 3001–3005.
- [7] N. Chiurtu, B. Rimoldi, and E. Telatar, "Dense multiple antenna systems," in *IEEE Inform. Theory Workshop*, Cairns, Australia, Sept. 2–7 2001, pp. 108–109.
- [8] T. S. Pollock, "On limits of multi-antenna wireless communications in spatially selective channels," Ph.D. dissertation, The Australian National University, July 2003.
- [9] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *Euro. Trans. Telecomm.*, vol. 10, no. 6, pp. 585–595, Nov. 1999.
- [10] R. G. Gallager, *Information Theory and Reliable Communication*. New York, USA: John Wiley & Sons, 1968.
- [11] C. E. Shannon, "A mathematical theory of communication," *Bell System Tech. J.*, vol. 27, pp. 379–423, 623–656, July 1948.
- [12] L. W. Hanlen, A. J. Grant, and R. A. Kennedy, "On the capacity of operator channels," in *5th Aust. Commun. Theory Workshop, AusCTW*, Newcastle, Australia, Feb. 4–6 2004, pp. 23–27.
- [13] E. Kreysig, *Introductory functional analysis with applications*. New York, USA: John Wiley & Sons, 1978.
- [14] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd ed. Berlin, Germany: Springer-Verlag, 1998.
- [15] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. New York, USA: Academic Press, 2000.
- [16] R. A. Kennedy, T. D. Abhayapala, and H. M. Jones, "Bounds on the spatial richness of multipath," in *3rd Aust. Commun. Theory Workshop, AusCTW*, Canberra, Australia, Feb. 4–5 2002, pp. 76–80.