On the Capacity of Operator Channels

Leif W. Hanlen and Alex J. Grant and Rodney A. Kennedy

***Abstract*— We examine linear, bounded, fixed operators as channels between continuous input functions and continuous output functions. The time-frequency waveform channel and continuous space channel are both related to an abstract operator viewpoint of communication channels. The properties of the operator required in order to allow valid capacity estimation are examined, and the modelling of noise in a continuous setting is discussed. We outline the procedure for calculating the information theoretic capacity for this restricted class of operator channels and provide an example of operator channels in the form of a power constrained input signal.**

***Index Terms*— Operators, Capacity, Continuous Channels, Waveform Channels, Sobolev Spaces**

I. Introduction

Recently, there has been great interest in the wireless theoretic research community concerning the use of "continuous" channels as a means of describing wireless communication channels with spatial diversity [1]–[7]. This interest is motivated by an understanding that the capacity of a spatially constrained channel is determined by the continuous nature of space, rather than the number of antenna elements used in a MIMO system [8], [9].

This approach represents a novel application of the "waveform channel" of [10, ch.8], and the "continuous channel" of [11, sec.24]. In both the spatial case, and the time-frequency case, functions which are continuous in a parameter act as inputs for the channel. Through the action of the channel, these *continuous* inputs then produce *continuous* outputs which are corrupted by noise.

The waveform channel [10], for example, takes a power constrained function as its input, and has a function as its output. The output is then corrupted by additive Gaussian noise which has a given spectral density function. Typically, the Gaussian noise is assumed to be white, although this need not be the case, and the work of [10] is sufficiently general to accommodate coloured noise. The capacity results of [2], [4], [6], provide similar continuous channel results for a spatially diverse channel.

It is particularly interesting to note that the (time-frequency) channel described in [10], [11] and the (spatial) channel described in [3], [6] both correspond to particular forms of operators, acting as linear channels. Operators take functions as inputs, to produce functions as outputs. In this sense operators represent a natural abstraction of the spatial and time-frequency channels. Moreover, an operator viewpoint of channels leads us to examine the fundamental aspects of communication channels which are *independent of the particular channel instantiation.*

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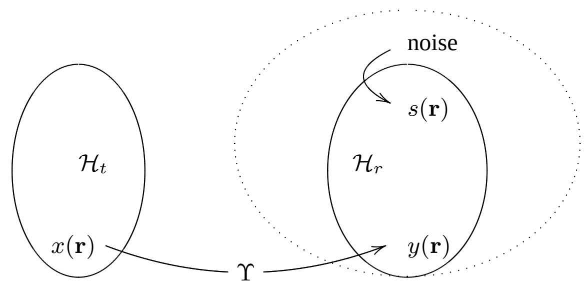

Fig. 1. Transmitting in Hilbert Spaces

We shall describe an additive Gaussian linear operator (AGLO) channel as *a channel Υ which takes functions x from an Hilbert space $\mathcal{H}_t$ and generates new functions y in an Hilbert space $\mathcal{H}_r$, such that y is corrupted by noise in the space $\mathcal{H}_r$*. In this way the operator "channel" is an abstraction of the well known linear channel [11] from communication theory. We shall restrict our investigation to those operators which produce bounded outputs, given bounded inputs[1].

In this paper we provide the framework necessary to initiate investigation into a general operator channel between two function spaces. In section II we outline the channel model for a bounded operator, and discuss the modelling of noise in III. Section V provides an example of the use the operator techniques and we draw conclusions in section VI.

II. Transmitting between spaces

Consider transmitting over a channel where the input $x(\mathbf{r})$ is a function taken from a Hilbert space $\mathcal{H}_t$, and the output $y(\mathbf{r})$ is a function taken from a Hilbert space $\mathcal{H}_r$, corrupted by noise $s(\mathbf{r}) \in \mathcal{H}_r$. This scenario is described in Fig. 1. We use the parameter of the continuous input and output functions as $\mathbf{r}$. In the work of [3], $\mathbf{r}$ corresponds to the 3-dimensional spatial vector $\{r, \theta, \phi\}$, while in [10], [11], the parameter $\mathbf{r}$ corresponds to time t.

We note that the noise process has been deliberately placed outside the space $\mathcal{H}_r$ in Fig. 1. This is to emphasize that the noise experienced by the receiver, within $\mathcal{H}_r$, cannot

[1] These are known as *bounded* operators [12], and the "bound" is given by the appropriate norm.

completely characterize the true noise function. We write the linear channel as:

$$y(\mathbf{r}) = \Upsilon x(\mathbf{r}) + s(\mathbf{r}) \tag{1}$$

where Υ is an operator taking functions in $\mathcal{H}_t$ to functions in $\mathcal{H}_r$:

$$\Upsilon : \{v(\mathbf{r}) \in \mathcal{H}_t \mapsto u(\mathbf{r}) \in \mathcal{H}_r : u(\mathbf{r}) = \Upsilon v(\mathbf{r})\} \tag{2}$$

We shall consider $\aleph_0$ dimensional separable Hilbert Spaces over multi-variable functions with complex fields. We assume familiarity with Hilbert Spaces, separability and completeness. The interested reader is referred to [12], or any standard Hilbert Space reference.

We wish to calculate the capacity of the channel (1), under the restriction that the communication channel operator Υ is fixed and bounded. This work follows the form of [10, ch.8], applying the waveform channel to more general Hilbert spaces. We shall be interested in bounded linear operators which admit a bounded self-adjoint representation. Recall [12], a bounded operator $\Upsilon : \mathcal{H}_t \mapsto \mathcal{H}_r$ satisfies

$$\|\Upsilon x\| \leq c\,\|x\| \tag{3}$$

for some constant c, independent of x, and may be represented by an infinite dimensional matrix[2], $\overrightarrow{\Upsilon}$

$$\left[\overrightarrow{\Upsilon}\right]_{ji} \triangleq \langle \Upsilon \vartheta_i, \varphi_j \rangle \tag{4}$$

where $\{\varphi_i\}_{i=1}^{\infty}$ and $\{\vartheta_j\}_{j=1}^{\infty}$ are complete, orthonormal sequences in $\mathcal{H}_t$ and $\mathcal{H}_r$ respectively. The norm $\|\cdot\|$, on the left hand side of (3) is over $\mathcal{H}_t$ while the norm on the right hand side is over $\mathcal{H}_r$.

We shall assume that the noise corrupting y is Gaussian in L^2. This ensures that our capacity result may rely on the standard Gaussian noise results [10], [11]. We shall assume that the input space $\mathcal{H}_t$ allows averaging on one dimension. This dimension replaces the "time" t in the standard continuous channel [11] capacity derivation. We may then consider our input parameter x as

$$\mathbf{r} = \{t, \hat{\mathbf{r}}\}, \quad t \in [0, T] \subset \mathbb{R} \tag{5}$$

where t is necessarily a dummy variable over which we shall average to produce a capacity result[3]. We may write cf. [10]

$$C_T = \frac{1}{\|T\|}\left[\sup I(x; y)\right] \tag{6}$$

and

$$C = \lim_{\|T\| \to \infty} C_T \tag{7}$$

where $I(x; y)$ is the mutual information between input and output, cf. [10]. Equations (6) and (7) do not imply that the channel (1) must have a temporal dimension, simply that there is one "excess" dimension, which we may ameliorate in our analysis. We shall also assume that the operator Υ is known at the transmitter and receiver ends of the communication link. This removes the need to account for non-ergodic channels.

Our method of calculating the capacity is summarized as follows:

A. Operator Capacity

1) Provide an isomorphism $\Gamma : L^2 \mapsto \mathcal{H}_r$ to ensure noise in $\mathcal{H}_r$ is related to (possibly correlated) Gaussian noise.
2) Write $y = \Upsilon x + \Gamma z$ where $z \in L^2$
3) Write $\hat{y} = \Gamma^{-1} y$ where $\hat{y} \in L^2$
4) Find capacity of "operator channel"

$$\hat{y} = \widehat{\Upsilon} x + z \tag{8}$$

where

$$\widehat{\Upsilon} = \Gamma^{-1}\Upsilon \tag{9}$$

   a) decompose the self-adjoint operator $H = \widehat{\Upsilon}\widehat{\Upsilon}^*$ into parallel, independent discrete AWGN channels
   b) choose the input x to be zero mean, Gaussian random variables with variances chosen according to the water-filling [10] algorithm.

Of the above steps, 2 and 4a require most attention. We shall detail the noise isomorphism, and the decomposition of the channel below:

III. Noise

In order to develop a result for the capacity of the operator channel, we must define a noise model. The model for noise that we adopt must be consistent for different operator channels. This ensures that we do not develop a capacity result which is dependent upon the space in which it was developed. It is natural to relate any noise in an abstract Hilbert space to the well known independent Gaussian noise. Fortunately, the work of [10] allows us to define white Gaussian noise as a random process *projected* into L^2. We note that white Gaussian noise is not an L^2 function.

Definition 1 (White Gaussian Noise in L^2). *White Gaussian noise, in $L^2(\Omega)$ is given by a random process $z(\mathbf{r})$ for $\mathbf{r} \in \Omega \subseteq \mathbb{R}^N$, such that for any function $\psi(\mathbf{r}) \in L^2(\Omega)$ the complex scalar z*

$$z_i \triangleq \int_{\Omega} z(\mathbf{r})\overline{\psi_i(\mathbf{r})}\, d\mathbf{r} = \langle z, \psi \rangle_{L^2} \tag{10}$$

is a zero mean, Gaussian random variable, with variance

$$\mathcal{E}\left\{|z_i|^2\right\} = \frac{N_0}{2}\int_{\Omega} |\psi_i(\mathbf{r})|^2\, d\mathbf{r} = \frac{N_0}{2}\left(\|\psi\|_{L^2}\right)^2 \tag{11}$$

for constant N_0 independent of $\psi_i(\mathbf{r})$. cf. [10, eqn. 8.1.35]

In Definition 1 we have allowed the output parameter of the channel $\mathbf{r}$ to vary over an arbitrary number of dimensions *i.e.*, $\mathbb{R}^N$. We note that the standard waveform channel [10] has $\mathbf{r} \in \mathbb{R}$ and the continuous space channel [6] has $\mathbf{r} \in \mathbb{R}^3$.

If the functions $\psi_i(\mathbf{r})$ are orthonormal, then we may write $z = \{z_i\}_{i=1}^{\infty}$ as a vector of *i.i.d.* Gaussian random variables, with zero mean and variance $N_0/2$. It is important to recognise that by (10), the white noise "function" in L^2 has infinite power. In turn, this means we cannot use Bessel's inequality

[2] We shall use the notation Υ to denote an operator, and $\overrightarrow{\Upsilon}$ to denote its matrix representation.

[3] As part of the "average" we allow $T \to \infty$ so that the dimension must be reasonably well behaved.

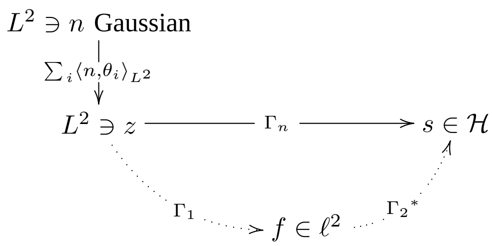

Fig. 2. Mapping from Gaussian Process (not L^2) to Gaussian noise $z \in L^2$ and to equivalent noise $s \in \mathcal{H}$

to determine the accuracy of the representation. Intuitively, we may take this as meaning that a white noise process has unbounded information, and consequently, any representation of the form (10) is necessarily incomplete. We shall shortly see that this does not present a problem within the arena of separable Hilbert spaces.

LEMMA 1 (ISOMORPHISM BETWEEN HILBERT SPACES). *Given a separable Hilbert space $\mathcal{H}_a$ with inner product $\langle \cdot, \cdot \rangle_a$ we may define an isomorphism Γ*

$$\Gamma : \mathcal{H}_a \mapsto \ell^2 \tag{12}$$

and the adjoint of Γ

$$\Gamma^* : \ell^2 \mapsto \mathcal{H}_a \tag{13}$$

if, and only if, the dimension of $\mathcal{H}_a$ is $\aleph_0$.

The proof of Lemma 1 is given in [12, Thm 3.6-5]. If we define $\{\alpha_i\}_{i=1}^{\infty}$ as a complete orthonormal sequence in $\mathcal{H}_a$ and $\{\theta_j\}_{j=1}^{\infty}$ is a complete orthonormal sequence in ℓ^2 then the isomorphism Γ is given by the construction:

$$\overrightarrow{\Gamma} = \begin{bmatrix} \langle \alpha_1, \theta_1 \rangle_{\ell^2} & \cdots & \langle \alpha_i, \theta_1 \rangle_{\ell^2} & \cdots \\ \vdots & & \vdots & \\ \langle \alpha_1, \theta_j \rangle_{\ell^2} & \cdots & \langle \alpha_i, \theta_j \rangle_{\ell^2} & \cdots \\ \vdots & & \vdots & \end{bmatrix} \tag{14}$$

LEMMA 2 (EQUIVALENT GAUSSIAN NOISE). *Let n be a Gaussian process and let $\mathcal{H}$ be a separable Hilbert space, of dimension $\aleph_0$. Let z be the projection of n onto L^2, then the equivalent noise s in $\mathcal{H}$ is given by*

$$s = \Gamma_n z \tag{15}$$

where Γ_n is the isomorphism $\Gamma_n : L^2 \mapsto \mathcal{H}$.

Proof: From Lemma 1 we may define Γ_n as the concatenation of $\Gamma_1 : L^2 \mapsto \ell^2$ and $\Gamma_2{}^* : \ell^2 \mapsto \mathcal{H}$. ■

The value of Lemma 2 is that we only need to define a Gaussian process in terms of its effect on the integral of a function – corresponding to the inner product in L^2 – to give the noise in an arbitrary Hilbert space $\mathcal{H}$ *even though the noise may not be defined in terms of the inner product in* $\mathcal{H}$.

We may use the definition (10) directly to give a vector z of noise "samples" in L^2, and then apply an isomorphism Γ_n to move from L^2 to the "new" space $\mathcal{H}$. This arrangement is shown in Figure 2 where we have emphasized the composition $\Gamma_n = \Gamma_1 \Gamma_2{}^*$. The effect of Γ_n may be interpreted as a colouring of the noise, as given by changing from one space to another. Mapping from L^2 to an arbitrary (equal dimension) space $\mathcal{H}$ corresponds to a change of coordinates in L^2. As such, the entropy of the "channel" may change, although the capacity will remain independent of the change in coordinates [11, sec.20].

IV. DECOMPOSITION OF OPERATOR

In similar manner as for the eigen-decomposition of a matrix channel, we decompose the operator channel (1) into a set of parallel, independent additive white Gaussian noise channels. Given the equivalent channel operator $\widehat{\Upsilon}$ we note that we do not use the eigen-channels of $\widehat{\Upsilon}$ directly, rather we choose independent channels *under the constraint that each channel will have maximum mutual information*. Maximising mutual information is equivalent to maximising the received signal power, which leads to solutions of the problem:

PROBLEM 1. *Given an operator*

$$\widehat{\Upsilon} : \phi = \widehat{\Upsilon} \psi \tag{16}$$

find the function(s) ψ which solve:

$$\nu = \max_{\|\psi\|=1} \|\phi\|^2 = \max_{\|\psi\|=1} \left\| \widehat{\Upsilon} \psi \right\|^2 \tag{17}$$

From Parseval's theorem [12, pp.170] the value of $\|\phi\|^2$ is fixed, independent of the definition of the inner product, for all $\aleph_0$ dimension, complete separable Hilbert Spaces. Equation (17) represents an eigenvalue decomposition of the self-adjoint operator $\widehat{\Upsilon} \widehat{\Upsilon}^*$.

LEMMA 3 (CAPACITY OF BOUNDED LINEAR INVARIANT OPERATOR). *Consider the operator channel* (8)*, with eigenvalues λ_i, eigenfunctions $\varphi_i(\mathbf{r}, t)$ and input power constraint*

$$\|x\|^2 = P \|T\| \tag{18}$$

The capacity of the channel (8) *is given parametrically by*

$$C_B = \lim_{\|T\| \to \infty} \frac{1}{\|T\|} \sum_{i \in \mathbb{Z}_B} \frac{1}{2} \log_2 \left(\lambda_i(\|T\|) B \right) \tag{19}$$

$$P_B = \lim_{\|T\| \to \infty} \frac{1}{\|T\|} \sum_{i \in \mathbb{Z}_B} \left(B - \frac{1}{\lambda_i(\|T\|)} \right) \tag{20}$$

and $\mathbb{Z}_B$ is the set of integers associated with the eigenvalues λ_i, such that $B \geq \lambda_i$, and the parameter B is chosen to maximise (19).

The capacity C is achieved by setting $x_i = \langle x, \varphi_i \rangle$ to be zero mean, independent Gaussian random variables with variance

$$\mathcal{E}\left\{ |x_i|^2 \right\} = \max\left(0, B - \frac{1}{\lambda_i} \right) \tag{21}$$

Proof: The operator channel is bounded, so that $\widehat{\Gamma}$ may be written in matrix form $\overrightarrow{\widehat{\Gamma}}$. Since φ_i are eigenfunctions, $\overrightarrow{\widehat{\Gamma}}$ is diagonal and we use the parallel channel result of [10]. ■

V. Example L^2 to Sobolev

At this point we have outlined the methodology for calculating the capacity of a single user, linear operator channel. This analysis may be directly applied to that given in [10]. However, we now seek to show that such analysis is not restricted to the standard time-frequency continuous channel, nor the continuous space channel.

Consider Fig. 1, where we place a restriction on the input signals: that all input signals that must satisfy a more restrictive norm-constraint than the average power constraint. In particular, we shall define a constraint which encourages smooth input functions, and penalises input functions with higher order derivatives. The constraint is the norm associated with the m^{th} order Sobolev space. This example may be viewed in terms of an antenna constraint within a volumetric region, *i.e.,* that we cannot allow the antenna signal to change arbitrarily quickly across the region of space.

The Sobolev space of order 1, on Ω is denoted $\mathrm{H}^1(\Omega)$ for complex functions in $\mathbb{R}^N$:

$$\mathrm{H}^1(\Omega) = \left\{ u(\mathbf{r}) \in L^2(\Omega) : \frac{\partial u(\mathbf{r})}{\partial \mathbf{r}} \in L^2(\Omega) \right\} \tag{22}$$

and is clearly a sub-space of $L^2(\Omega)$, *i.e.,* $\mathrm{H}^1(\Omega) \subset L^2(\Omega)$. The space $\mathrm{H}^1(\Omega)$ is equipped with the inner product [13]:

$$\langle f, h \rangle_{\mathrm{H}^1} \triangleq \int_{\mathbf{r}\in\Omega} f(\mathbf{r})\overline{h(\mathbf{r})}\, d\mathbf{r} + \int_{\mathbf{r}\in\Omega} f^{(1)}(\mathbf{r})\overline{h^{(1)}(\mathbf{r})}\, d\mathbf{r} \tag{23}$$

where $f^{(i)}(\mathbf{r})$ denotes the i^{th} partial derivative of f with respect to $\mathbf{r}$. and since the Sobolev space H^1 is a Hilbert space, the norm $\|\cdot\|_{\mathrm{H}^1}$ is given by:

$$\|x\|_{\mathrm{H}^1} \triangleq \sqrt{\langle x, x \rangle_{\mathrm{H}^1}} \tag{24}$$

Consider the channel

$$y = x + z \tag{25}$$

with $x, y \in L^2$, and $z \in L^2$ is white Gaussian noise. The signal x is constrained to a power limit, subject to (24)

$$\|x\|^2_{\mathrm{H}^1} \le PT \tag{26}$$

with the norm given by (24). This places a more restrictive constraint on x than the usual average power constraint. We may consider the channel (25) as acting over all functions in L^2, with white noise applied, or we may consider the subspace of input functions which satisfy the norm constraint. In the second case, we may define an equivalent channel:

$$\hat{y} = \hat{x} + s \tag{27}$$

where now we consider $\hat{y}, \hat{x}, s \in \mathrm{H}^1$. The noise s is equivalent to the noise z, *projected into the new space* H^1. This provides a starting point on which to define the equivalent *Sobolev space noise* s using (14):

$$s = \Gamma_n z \tag{28}$$

Note, the Sobolev inner product (23) may be written in terms of the L^2 inner product:

$$\langle f, h \rangle_{\mathrm{H}^1} = \langle f, g \rangle_{L^2} + \left\langle f^{(1)}, g^{(1)} \right\rangle_{L^2} \tag{29}$$

Since $\mathrm{H}^1 \subset L^2$, any orthonormal sequence $\{\varphi_i\}_{i=1}^{\infty}$ which is complete in L^2 is also complete in H^1, although the sequence may not remain orthonormal. To generate an orthonormal sequence $\{\varphi'_i\}_{i=1}^{\infty}$ we perform a Gram-Schmidt orthonormalisation on the original sequence $\{\varphi_i\}_{i=1}^{\infty}$ using the new inner product (29).

The Fourier functions are known to be complete and orthonormal [14] in $L^2(\Omega)$, and are their own derivatives (scaled). The form of (29) suggests using the Fourier functions as our choice of basis for L^2,

$$\varphi_k(\mathbf{r}) \triangleq \frac{1}{\sqrt{\|\Omega\|}} \exp\left\{ \frac{\iota 2\pi k \mathbf{r}}{\|\Omega\|} \right\} \tag{30}$$

c.f. [10, eqn. 8.1.18]. We may define a complete orthonormal sequence over H^1:

$$\varphi'_k(\mathbf{r}) = \varphi_k(\mathbf{r}) \cdot \frac{1}{\sqrt{1 + \left(\frac{2\pi k}{\|\Omega\|}\right)^2}} \tag{31}$$

From (14), substituting φ for θ and φ' for α, the isomorphism $\overrightarrow{\Gamma_n}$ is diagonal, with entries:

$$\left[\overrightarrow{\Gamma_n}\right]_{kk} = \sqrt{1 + \left(\frac{2\pi k}{\|\Omega\|}\right)^2} \tag{32}$$

and the noise variance for channel k is given by $(\Gamma_n)^2_{kk}$. *By definition we have chosen the orthonormal functions to diagonalize the channel.* This is described in Figure 2. We may therefore use the discrete, parallel channel model of [10], to calculate the capacity of the channel. We note that Γ_n is an unbounded operator, since the eigenvalues (diagonal entries) are unbounded – they increase indefinitely. However, we shall shortly consider Γ_n^{-1} which is clearly also diagonal, and bounded. Having an unbounded *noise* operator does not present significant problems for the capacity calculation. If the transmitter has full channel knowledge, it will simply avoid putting any signal power on those channels with excessively large noise power. However, the unbounded growth in noise power does imply that we are doomed to have negligible capacity if we blindly insist on transmitting equal power signals across all channels.

We write the capacity of the channel in terms of a parametric optimisation:

$$P = \int_{B \ge \mathcal{N}(\mu)} B - \mathcal{N}(\mu)\, d\mu \tag{33}$$

$$C = \int_{B \ge \mathcal{N}(\mu)} \frac{1}{2} \log_2 \left(\frac{B}{\mathcal{N}(\mu)} \right) d\mathbf{r} \tag{34}$$

where

$$\mathcal{N}(\mu) = \sqrt{1 + (2\pi\mu)^2} \tag{35}$$

From Fig. 3 we see that the noise function is convex and symmetric, and hence we can find μ_0 which solves $B = \mathcal{N}(\mu_0)$ for B. Solving for B in (33) and (34), we have:

$$C = \frac{1}{2} \int_{-\mu_0}^{\mu_0} \log_2 \left(\frac{\mu_0^2 + 1}{1 + (2\pi\mu)^2} \right) d\mu \tag{36}$$

$$\approx c P^{1/3} \log_2(1 + P) \tag{37}$$

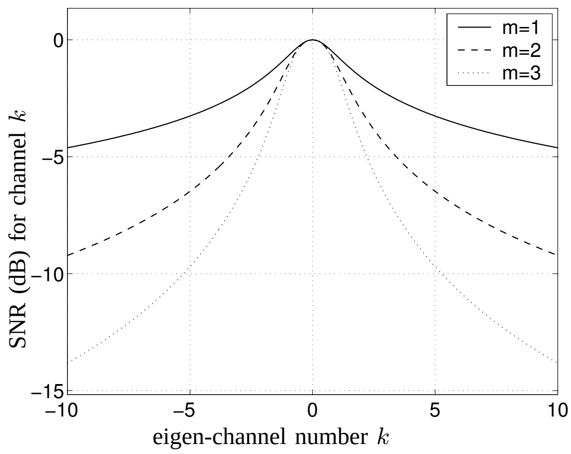

Fig. 3. Equivalent SNR for Sobolev H^m, shown with $m = 1$. We have also shown the equivalent SNR for higher order Sobolev spaces.

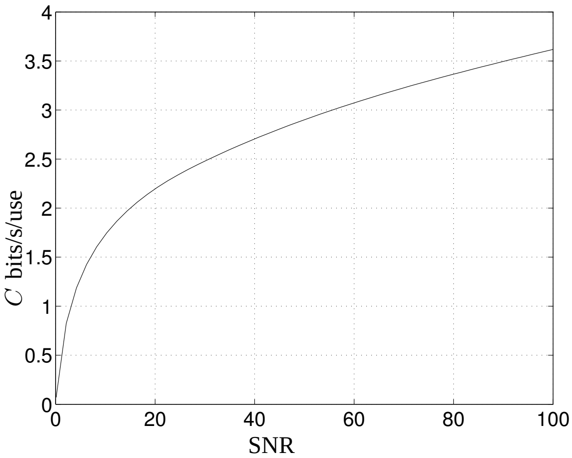

Fig. 4. Capacity growth rate for input constrained to Sorbolev norm, using water-filling (37).

for constant c. We have used [15, 2.731] for the simplification in (37). We have plotted the capacity of the channel using water-filling, for increasing signal to noise ratio in Fig. 4. We note, for the L^2 power constraint H^0, the capacity increases *linearly* with P.

$$C(W) = \lim_{W\to\infty} W \log_2\left(1 + \frac{P}{W}\right) = P \log_2 e \qquad (38)$$

The variable W is a "dummy" parameter, and may be associated with the equivalent "bandwidth" of the input signal. It can be seen that the capacity of the first-order Sobolev power constraint signal $x \in \mathrm{H}^1$ is significantly smaller than the L^2 case.

VI. Conclusion

We have shown how to calculate the capacity of a linear bounded invariant operator channel, and related the operator viewpoint to the well known waveform, and continuous space channels. We have shown where input constraints may be represented as a norm, we may use an effective change-of-coordinate to solve the channel capacity problem in an alternate space. We have shown how this "new" space has effectively coloured noise, determined by the relation between the norm constraint and the L^2 norm.

We have provided an example of a channel where a norm constraint leads us to consider the input-output channel in terms of a Sobolev space of order 1, and we have provided solutions for the capacity under this constraint. The details provided in this paper form a generalisation of the work of several seemingly disjoint aspects of wireless communication theory, and provide a starting point for the consideration of more general operators as communication channels.

References

[1] T. Pollock, "On limits of multi-antenna wireless communications in spatially selective channels," Ph.D. dissertation, The Australian National University, July 2003.

[2] T. S. Pollock, T. D. Abhayapala, and R. A. Kennedy, "Antenna saturation effects on MIMO capacity," in *IEEE Intl. Conf. on Commun., ICC'03*, Anchorage, Alaska, May. 11–15 2003, pp. 2301–2305.

[3] R. A. Kennedy and T. D. Abhayapala, "Spatial concentration of wave-fields: Towards spatial information content in arbitrary multipath scattering," in *Proc. 4th Australian Communications Theory Workshop*, Melbourne, Australia, Feb. 2003, pp. 38–45.

[4] J. W. Wallace and M. A. Jensen, "The capacity of MIMO wireless systems with mutual coupling," in *IEEE Veh. Technol. Conf. (VTC 2002-Fall)*, Vancouver, Canada, 2002, pp. 696–700.

[5] J. Wallace and M. Jensen, "Intrinsic capacity of the MIMO wireless channel," in *IEEE Intl. Symp., Antennas and Propogation Society*, 2002, pp. 696–700.

[6] M. Fu and L. W. Hanlen, "Capacity of MIMO channels: A volumetric approach," in *IEEE Intl. Conf. on Commun., ICC'03*, Anchorage, Alaska, May. 11–15 2003, pp. 2673–2677.

[7] N. Chiurtu, B. Rimoldi, and E. Telatar, "Dense multiple antenna systems," in *IEEE Information Theory Workshop*, Cairns, Australia, Sept. 2–7 2001, pp. 108–109.

[8] L. W. Hanlen, "Channel modelling and capacity analysis for multi-input, multi-output, wireless communication systems," Ph.D. dissertation, The University of Newcastle, June 2003.

[9] H. Jones, "On multipath spatial diversity in wireless multiuser communications," Ph.D. dissertation, The Australian National University, 2001.

[10] R. Gallager, *Information Theory and Reliable Communication*. New York, USA: John Wiley & Sons, 1968.

[11] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, pp. 379–423, 623–656, July 1948.

[12] E. Kreysig, *Introductory functional analysis with applications*. New York, USA: John Wiley & Sons, 1978.

[13] R. Adams, *Sobolev Spaces*, ser. Pure and applied mathematics. New York, USA: Academic Press, 1975, vol. 65.

[14] M. Birman and M. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, ser. Mathematics and its applications. Dordrecht, Holland: Kluwer Academic Publishers, 1986, translated from the Russian by S. Khrushchev and V. Peller.

[15] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. New York, USA: Academic Press, 2000.