Capacity Analysis of Correlated MIMO Channels

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Abstract — **The capacity of correlated finite-dimensions MIMO channels, where the channel gains have a generalized Wishart distribution is found. Asymptotic expressions are given where one dimension is much larger than the other. For many transmitters, the asymptotic capacity can be divided into two components: one arising from the dominant eigenvalues of the correlation matrix, and the other from the remaining eigenvalues.**

I. Summary

The work of [1–3] has shown that under the assumption of an i.i.d. transfer matrix, the capacity of a MIMO channel grows in proportion to the minimum number of transmitters and receivers. Recently, [4] proposed the use of the *Stieltjes' transform* to estimate the capacity of a correlated channel, and suggested that the growth of the MIMO channel will remain linear under correlation, although the proportionality constant may change.

Consider a point-to-point communication link with t transmit antennas and r receive antennas. Define $m = \min\{r,t\}$ and $n = \max\{r,t\}$. At each symbol interval, $y \in \mathbb{C}^r$ depends on $x \in \mathbb{C}^t$,

$$y = Hx + w \tag{1}$$

Element y_j is the matched-filter output from antenna j, while x_i is the signal transmitted from antenna i, with the transmitter given a transmission power limit P. The matrix $H \in \mathbb{C}^{r\times t}$ has elements H_{ji}, which are the complex gains between transmit antenna i and receive antenna j. The vector $w \in \mathbb{C}^r$ contains i.i.d. circularly symmetric Gaussian noise samples $\mathsf{E}[ww^*] = \eta^2 I_r$. The H_{ji} are chosen from a complex Gaussian ensemble with zero mean and an $m \times m$ covariance matrix Σ. In the notation of [6], $H \sim N_{r,t}(0, \Sigma \otimes I_n)$. If $\Sigma = I_m$ we have the well known i.i.d. case [1].

Assume H is known at the receiver and that H, Σ are unknown at the transmitter. In this case, $\mathsf{E}[xx^*] = P \cdot I_t/t$ is optimal.

Theorem 1 (Correlated MIMO Capacity).
The capacity of the ergodic correlated MIMO channel (1) *with* $H \sim N_{r,t}(0, \Sigma \otimes I)$ *is given by*

$$C = \frac{n^{mn}\pi^{m(m-1)}}{2^{mn}\Gamma_m(n)\Gamma_m(m)\det(\Sigma)^n} \int_\Lambda {}_0F_0\left(-\frac{1}{2}\Sigma^{-1}, n\Lambda\right) \cdot \prod_{i=1}^m \lambda_i^{(n-m)} \prod_{i<j}^m (\lambda_i - \lambda_j)^2 \sum_{i=1}^m \log\left(I + \frac{P}{t}n\lambda_i\right) d\Lambda$$

where $\Gamma_m(a) = \pi^{m(m-1)/2}\prod_{i=1}^m \Gamma(a-i+1)$ *is the complex multivariate gamma function,* $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ *and* ${}_0F_0(\cdot)$ *is a hypergeometric function of two matrix arguments [6, p. 34].*

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Theorem 2 (Asymptotic Correlated MIMO Capacity).
Consider (1) *with* $H \sim N_{r,t}(0, \Sigma \otimes I)$ *such that* Σ *has eigenvalues* $\sigma_1 > \cdots > \sigma_k > \sigma_{k+1} = \cdots \sigma_m = \sigma > 0$. *[5, Corrollary 9.5.7]. Then the asymptotic capacity* C_∞*, as* $n \to \infty$ *with finite* m *is*

$$C_\infty \sim \sum_{i=1}^k \log\left(1 + \frac{P}{t}n\sigma_i\right) + \int_{-\infty}^{\infty} \log\left(1 + \frac{P}{t}n\sigma_m\left[z\left(\frac{n}{2}\right)^{-\frac{1}{2}} + 1\right]\right) \cdot \sum_{j=1}^{m-k} \frac{\left[H_j\left(\frac{z}{\sqrt{2}}\right)\right]^2}{2^j j!\sqrt{2\pi}} e^{-z^2/2}\, dz$$

where $H_j(\cdot)$ *is the* j*-th Hermite polynomial [7] and* $A \sim B$ *implies* $A/B \to 1$.

When $r = n \to \infty$ and $m = t$, capacity grows logarithmically without bound. For $r = m$ and $n = t \to \infty$, the right hand side above converges to a constant.

Corollary 1 (Large t).
Suppose $t \gg r$*, i.e.* $t = n \to \infty$ *and* $r = m$*, then*

$$\lim_{t=n\to\infty} C_\infty \approx \sum_{i=1}^k \log(1 + P\sigma_i) + (m-k)\log(1 + P\sigma_m)$$

where we have omitted constants of integration.

Corollary 1 presents two forms of growth for increasing r. The first term is given by the dominant eigenvalues of the covariance matrix and for $r \geq k$ is independent of r. The second term is linear in $r - k$ and corresponds to "linear" capacity growth of the i.i.d. channel, with proportionality constant $\alpha = \log(1 + P\sigma_m)$. Over the class of equivalent covariance matrices $S_m = \{\Sigma_m \in S_m : \operatorname{tr}(\Sigma_m) = m\}$ the i.i.d. channel has the largest rate of capacity growth $\alpha = \log(1 + P)$ for increasing r.

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