Abstract—In this paper low-density parity-check (LDPC) codes are considered for burst erasure channels. Multiple burst erasure correcting LDPC codes are constructed using superposition and the burst erasure correcting performances of the resulting codes are derived as a property of the stopping set size of the base matrices and the choice of permutation matrices for the superposition.

I. INTRODUCTION

A low-density parity-check (LDPC) code is a block code defined by a sparse parity-check matrix, \( H \), and decoded iteratively with message-passing decoding [1]. LDPC codes are well known to provide excellent decoding performances on memoryless channels (see e.g. [2]), and recent interest has focused on their performance on channels with burst errors or erasures [3], [4], [5], [6], [7]. Such channels encompass many real-world communications systems including fading environments, packet based communications, and magnetic storage devices where the burst errors caused by thermal asperity and media defects are the dominant error type.

Any system where the receiver is able to distinguish deep fades, for example by employing training sequences, can treat the fading period as an erasure burst. As well, packet losses in Internet transmissions can be modeled as erasure bursts, and forward error correction is becoming more attractive for these channels, particularly in real time or multicast applications where automatic repeat request schemes are less practical.

We will consider the classical burst erasure channel as defined in [8]. This channel has two states; the burst space, in which the channel output carries no information about the inputs, and the guard space, in which channel outputs are erased free or are randomly corrupted with some small background erasure probability \( p \) [8].

An erasure correcting code is optimal, and called maximum distance separable (MDS), if it achieves the Singleton bound. A rate \( k/n \) length \( n \) code is MDS if it can correct any \( n-k \) erasures. For the burst erasure channels we will say that a code is burstMDS if it can correct bursts of erasures with combined length of up to \( n-k \) bits. While non-binary codes which meet the Singleton bound exist over a range of lengths and rates (see e.g. [9]), our motivation for considering LDPC codes for these channels is twofold; firstly, we consider channels which also include random erasures, for which LDPC codes are known to perform very well, and secondly we wish to consider codes with low complexity implementation even for very long lengths, for which binary LDPC codes are an ideal choice.

LDPC codes have been previously designed to correct single erasure bursts. Yang and Ryan in [3] developed an efficient exhaustive search algorithm for finding the longest string of erasures guaranteed to be corrected, called the maximum resolvable erasure burst length, \( L_{\text{max}} \), for a given LDPC code. Song and Cruz in [6] presented a pseudo-random construction for LDPC codes with large good \( L_{\text{max}} \), and deterministic constructions were presented for single burst erasure correcting codes in [7].

In this paper we consider LDPC codes which can correct multiple erasure bursts, and present deterministic constructions for codes which achieve this. In Section II we describe the properties of the parity-check matrices of good multiple burst erasure correcting LDPC codes before designing LDPC codes which can correct multiple erasure bursts in the same codeword, in Section III, and concluding in Section IV.

II. BURST ERASURE CORRECTING LDPC CODES USING SUPERPOSITION

A binary LDPC code is represented by a sparse binary parity-check matrix, \( H \), and a bipartite graph, \( T \), called a Tanner graph. Each bit in the codeword corresponds to a column of \( H \) and a bit vertex of \( T \), and each parity-check equation satisfied by the codeword corresponds to a row of \( H \) and a check vertex of \( T \). The \((j, i)\)-th entry of \( H \) is ‘1’, and an edge joins the \( i \)-th bit vertex and \( j \)-th check vertex of \( T \), if the \( i \)-th codeword bit is included in the \( j \)-th parity-check equation.

The message-passing decoding of LDPC codes on erasure channels is particularly straightforward since a transmitted bit is either received correctly or completely erased. It is assumed that the receiver is able to detect an erasure and so deletions are not considered. If only one of the bits in any given code parity-check equation is erased, the erased bit can be determined by choosing the value which satisfies the parity-check equation. Conversely, if more than one bit in any given parity-check equation is erased, no correction can be made using that equation. Message passing iterative decoding of an LDPC code is a process of finding parity-check equations which check on only one erased bit and correcting it. The process is repeated until all the erasures are corrected or all the remaining uncorrected parity-check equations check on two or more erased bits. The latter will occur if the erased bits include a set of code bits, \( S \), which are a stopping set.

A stopping set, \( S \), is a set of code bits with the property that every parity-check equation which checks on a bit in \( S \) checks on at least two bits in \( S \) [10]. If all of the bits in a stopping set are erased none of them can be corrected and so the stopping set distribution of an LDPC code determines the erasure patterns for which the message-passing decoding algorithm will fail [10]. The size of a stopping set is the number of bits it includes, and the minimum stopping set size, \( S_{\text{min}} \), determines the minimum number of erased bits which will cause a decoding failure.
On the burst erasure channel the location of the stopping set bits within the codeword, rather than just the size of the stopping sets, will be the important factor in determining the code's performance. A measure of an LDPC codes performance on a burst erasure channel

We design LDPC codes for burst erasure channels by using superposition (see e.g. [11]), starting with a $M \times N$ base matrix $H_{\text{base}}$, and replacing each non-zero entry in $H_{\text{base}}$ with a $v \times v$ permutation matrix, and each zero entry with the $v \times v$ all zeros matrix, to create an $m \times n$ LDPC code parity-check matrix $H$ where $m = Mv$ and $n = Nv$. The minimum stopping set size of the base matrix and choice of the superposition matrices determine the burst erasure correction capability of the final code.

We have previously used superposition to construct LDPC codes for packet loss channels in [12]. For the packet loss channel the erasure bursts can only occur within the packet boundaries and so it is sufficient to use arbitrary permutation matrices for the superposition to produce codes which can correct any $S_{\text{min}} - 1$ lost packets [12]. For burst erasure channels however, codes which can correct bursts occurring across the columns of two or more adjacent superposition matrices are required.

The case where every non-zero entry in $H_{\text{base}}$ is replaced with the same $v \times v$ permutation matrix corresponds to independently encoding $v$ codewords using $H_{\text{base}}$ and interleaving them so that bits from the same codeword are always $v$ bits apart. Interleaving, or interlacing, codewords in this way is commonly used in bursty channels and the resulting codes will be burstMDS if the base matrix is MDS (see e.g. [9]). However, the resulting LDPC codes are not very effective in the presence of guard band erasures since each copy of $H_{\text{base}}$ is a disjoint subgraph in the Tanner graph. Furthermore, binary MDS base matrices only exist for a limited range of rates and so binary burstMDS codes cannot be formed in this way for most code parameters of interest when two or more bursts are considered.

Here we use permutation, or circulant, matrices to achieve codes which are almost burstMDS. These codes are effective over a wider range of code rates and produce significantly improved performances in channels which include random erasures in the guard band. Firstly, we show that the LDPC codes formed by superposition of the same permutation matrix into MDS base matrices are burstMDS as we will build on this result in the following.

**Lemma 1:** If $H_{\text{base}}$ has minimum stopping set size $S_{\text{min}}$ and superposition with the same $v \times v$ permutation matrix replaces every non-zero entry in $H_{\text{base}}$ to form $H$, then any $S_{\text{min}} - 1$ bursts of length up to $v$ bits can be corrected provided that the guard band is erasure free.

**Proof:** Since the same permutation matrix is used in the superposition, $H$ contains $v$ copies of every stopping set in $H_{\text{base}}$ and so will also have a minimum stopping set size of $S_{\text{min}}$. However, since the same permutation matrix is used for the superposition each column in the stopping set is at least $v$ columns apart. Therefore no burst of length $v$ can erase two of the bits in a stopping set and so at least $S_{\text{min}}$ bursts are required to erase a full stopping set and the proof follows.

**LDPC codes which are MDS with message passing decoding have $S_{\text{min}} = N - K + 1$ and so, applying Lemma 1, the LDPC codes formed from interleaving MDS base matrices are burstMDS.**

The key to our constructions is that by using cyclic shifts of a base permutation matrix for the superposition, codes with connected Tanner graphs are produced which can also correct $S_{\text{min}} - 1$ bursts if they are of length slightly less than $v$.

**Lemma 2:** $H_{\text{base}}$ has minimum stopping set size $S_{\text{min}}$ and superposition with $v \times v$ permutation matrices is used to form $H$, such that every permutation matrix is either identical to, or a cyclic shift left by one column of, the closest permutation matrix on its left. If $r$ is the maximum number of cyclic shifts of any permutation matrix, $H$ can correct any $S_{\text{min}} - 1$ bursts of size $v - r$ on the burst erasure channel provided that the guard band is erasure free.

**Proof:** If two adjacent permutation matrices, corresponding to the $j$-th row and $i$-th and $(i+1)$-th column of $H_{\text{base}}$, are identical then each pair of non-zero entries in each row across them are distance $v$ columns apart. Each cyclic shift left by one column in the $(i+1)$-th permutation matrix increases this distance by $v - 1$ for the row incident on the column shifted but reduces this distance by one for the remaining rows. Since any other permutation matrix corresponding to the $(i+2,j)$-th entry of $H_{\text{base}}$ must also be shifted by either the same number of columns or by one further column, the row incident on the shifted column from the $(i+1,j)$-th permutation will not be included in a column closer than $v - 1$ away in this permutation matrix. Thus any burst of size less than $v - 1$ can only erase one entry in any one parity-check equation. However, a single burst can still erase two of the columns in a stopping set if those columns are adjacent in $H_{\text{base}}$ but disjoint, and thus are incident in different circulants corresponding to different rows of $H_{\text{base}}$. In this case a shift left by $l$ places in the rightmost column of circulants, but not in the other, will bring the entries of the stopping set $l$ columns closer. The maximum possible difference in shifts, by $r$ places, will occur if the circulants in the first column are not shifted at all while the circulants in the second column are shifted by the maximum $r$ shifts.

Thus bursts of size $v - r$ or less cannot erase two columns of a stopping set. Therefore, restricting the burst length to $v - r$ ensures that at least $S_{\text{min}}$ bursts are required to erase a stopping set and the proof follows.

**Lemma 3:** Take any base matrix $H_{\text{base}}$ with minimum column weight $2$ containing a minimum stopping set $S$ with rows weight $2$. Form $H$ using superposition with $v \times v$ permutation matrices such that every entry of $S$ is replaced by the same permutation matrix $A$, except for the entries in the bottom most row involved in $S$, which we label as the $j$-th row of $H_{\text{base}}$. In this row the each entry in the $i$-th column

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1We use $N$, $M$, and $K = N - M$ for the dimensions of the base matrices and $n$, $m$, and $k = n - m$ for the dimensions of the resulting parity-check matrices.
is replaced by a cyclic shift left by \(i\) columns of \(A\). If \(r\) is the maximum difference in the number of cyclic shifts used between two entries in any row, then all but \(r\) consecutive bits corresponding to the columns of \(S\) in \(H\) are erased, and no other codeword bits are erased, all of the erased bits can be corrected.

Proof: Suppose that we wish to locate a stopping set \(S'\) within the columns in \(H\) corresponding to the stopping set \(S\) in \(H_{base}\). We start \(S'\) with the \((i)\)-th column of one of the permutation matrices from \(S\). As \(S\) is a minimum stopping set \(S'\) will include a column from every permutation of \(S\). Since all but the last row of \(S\) is replaced with \(A\), the \((i)\)-th column of each of the \(A\) matrices must be included in \(S'\). Every column of \(S\) has weight at least 2, so each column of \(S\) will include at least one copy of \(A\) and the \((i)\)-th column of every permutation matrix in \(S\) is included in \(S'\). The final permutation matrix in the \((j)\)-th row of \(S\), \(P_r\), was shifted left by \(r\) columns relative to the first, \(P_a\), and so the \((i-r)\)-th column in \(P_r\) will also need to be included in \(S'\) so that the check on the \(i\)-th bit in \(P_a\) checks on a second bit in all but the last row of \(S\) need to be included in \(S'\). Then the \((i-r)\)-th column of the other permutation matrix in the same column as \(P_r\), which must be \(A\), is also erased. As above, the \((i-r)\)-th column of every permutation matrix must now be included in \(S'\) to complete the stopping set on the checks of \(S'\). Again, the \((i-r) \mod v\)-th column in the last permutation matrix will subsequently need to be included in \(S'\) so that the check on the \((i-r)\)-th bit in the first permutation matrix in the \(j\)-th row checks on a second bit in \(S'\) and so on until every \(r\)-th column in all of the \(|S|v\) columns in \(H\) corresponding to \(S\) must be included to form a stopping set. Thus if any one of these columns is received the erased bits do not form a stopping set and the proof follows.

Lemma 3 can be extended to the case where more than one row of the stopping set includes shifted circulants. The spacing between the columns of the resulting stopping set \(S'\) will depend on the difference in column shifts between the circulants of \(S\) on the same row. We will use the result for stopping sets of size three in the following and so present it here.

Lemma 4: Take any base matrix \(H_{base}\), containing a column weight-2 row weight-2 stopping set \(S\) with size 3. Suppose that both entries in the first row of \(S\), which we label entries \(P_1\) and \(P_2\), are replaced with the same permutation matrix \(A\) and the second entries in these two columns, entries \(P_3\) and \(P_4\), are a shifted version of \(A\) by \(b\) and \(c\) columns respectively. The two entries of the third column of \(S\), entries \(P_5\) and \(P_6\), are a shifted version of \(A\) by \(d\) and \(e\) columns respectively where the entries \(P_5\) and \(P_6\) are on the same row of \(S\). Then if all but \(r\) consecutive bits corresponding to the columns of \(S\) in \(H\) are erased, where \(r = |a - e + c + d - b|\) is greater than zero, and no other codeword bits are erased, all of the erased bits can be corrected.

Proof: We wish to find a stopping set \(S'\) within the columns of \(H\) corresponding to the columns of \(S\). To begin the \((i)\)-th column of \(P_1\) is included in \(S'\). Since \(P_2\) is the on the same row and contains the same permutation matrix as \(P_1\), the \((i)\)-th column of \(P_2\) is also in \(S'\). This requires the \((i-(c-a))\)-th column in \(P_5\) to be included in \(S'\) so that the corresponding check in \(P_3\) includes two stopping set bits. Then the \((i-(c-a)−(b-d))\)-th column in the \(P_1\) must now be included in \(S'\) so that the corresponding check in \(P_0\) includes two stopping set bits. If \((c-a)−(b-d) = 0\) we have a stopping set, otherwise the process begins again with the \((i-(c-a)−(b-d))\)-th column of \(P_1\) and the proof follows in the same manner as for Lemma 3.

In a column weight-2 code, a 6-cycle translates directly into a stopping set of size 3 and so the requirement on the allowed circulants for the quasi-cyclic codes in Lemma 4 is related to the restriction on the allowed circulants for quasi-cyclic codes to avoid 6-cycles [13].

To choose the base matrices we note firstly that all binary base matrices have \(S_{min} = 2\), since any stopping set requires two columns. In particular, we will use the matrices [12]

\[
H_{base} = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & 1 & 1 & \cdots & 1
\end{bmatrix}
\]

and

\[
H_{base} = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0
\end{bmatrix}
\]  

Lemma 5: The LDPC codes described by parity-check matrices in the form of (3) can correct any \(N-1\) bit erasures when decoded with message passing decoding.

Proof: Message-passing decoding on \(H_{base}\) will correct any \(N-1\) bit erasures if the minimum stopping set size of \(H_{base}\) is greater than \(N-1\), and so it is sufficient to show that the \(H_{base}\) matrices defined in (3) have a minimum stopping set size of \(N\). Start with any column of \(H_{base}\), which we label the \((i)\)-th column. Since \(H_{base}\) has weight 2 rows, to form a stopping set, \(S\), requires that each row of \(H_{base}\) incident on the \((i)\)-th column has the second column on which it is incident is also included in \(S\). Thus \(S\) must include the \((i-2)\)-th, \((i)\)-th and \((i+2)\)-th columns of \(H\). The same is true for the extra rows incident on the \((i-2)\)-th and \((i+2)\)-th columns which are not incident on the \((i)\)-th column and so on until the \((i-l)\)-th and \((i+l)\)-th columns of \(H_{base}\), for all \(l\) even, are included in \(S\). The two end columns however include adjacent checks and so are incident on one check in the set \(i \pm l\), \(l\) even and one in the set \(i \pm l\), \(l\) odd. The process of completing the stopping set is thus repeated to include the \((i-l)\)-th and \((i+l)\)-th columns, for all \(l\) odd. Therefore a stopping set can be formed only by including all of the columns in \(H_{base}\).

Using these base matrices, the superposition results presented in this section will be applied to construct specific burst erasure correcting LDPC codes in Sections III. The codes formed from systematic base matrices are easily encoded regardless of which permutation matrices are used to form \(H\). For the non-systematic base matrices, however, the use of circulant permutations will ensure quasi-cyclic codes and thus ease of encoding using the method from [14].
III. CONSTRUCTIONS OF BURST ERASURE CORRECTING LDPC CODES

Firstly we design a single burst erasure correcting code using base matrices in the form of (3). Although these base matrices are not MDS, we show in the following that by judiciously selecting the circulant matrices for the superposition, codes with close to burstMDS performance can still be obtained.

Construction 1: Construct the $3 \times p^3$ base matrix $H_{\text{base}}$ from the concatenation of $p$ copies of the $3 \times 3$ base matrices from (3). Then construct a length $3pv$, rate $\approx (p-1)/p$, LDPC code $H$ using superposition. The last nonzero entry in the columns of $H_{\text{base}}$, corresponding to the $i$-th copy of (3), $i \in \{1, \ldots, p\}$, is replaced with the cyclic shift of the $v \times v$ identity matrix, $I$, by $i + 1$ columns left. Every other non-zero entry of $H_{\text{base}}$ is replaced by $I$ and every zero entry by $0$.

Lemma 6: The codes from Construction 1 have $L_{\text{max}} = 3v - p - 1$.

Proof: The base matrices from (3) have minimum stopping set size $S_{\text{min}} = 3$. Although concatenating multiple copies of these matrices reduces the minimum stopping set size of $H_{\text{base}}$ to 2, any adjacent set of 3 columns does not contain a stopping set of size less than 3. From Lemma 4 a burst across any one of these size 3 stopping sets can be corrected if $p + 1$ consecutive bits are received correctly and the proof follows.

For example, the rate-4/5 burst error correction codes

$$H = \begin{bmatrix}
0 & 1 & I & 0 & I & 0 & 0 & 0 & I & I \\
0 & 0 & I & I & 0 & I & 0 & 0 & 0 & I \\
I & 0 & I & 0 & I & 0 & I & 0 & I & 5 \\
I & 3 & 0 & I & 2 & 0 & I & 3 & 0 & I \\
I & 5 & 0 & I & 0 & I & 3 & 0 & I & 4 \\
I & 5 & 0 & I & 5 & 0 & I & 5 & 0 & I \\
\end{bmatrix}$$

have $L_{\text{max}} = 3v - 6$ when superposition with $v \times v$ circulants is used. The entry $I^2$ in (4) represents superposition with a circulant matrix that is the cyclic shift of the identity matrix by 1 columns left, and the entry 0 represents superposition with a $v \times v$ all zeros matrix.

Construction 1 can be generalized to all of the matrices in the form of (3) where the codes can correct bursts of length close to $vN$ bits if the sum of shifts in the entries allocated to the size $N$ stopping sets within adjacent columns, do not add to zero.

Construction 2: Construct the $N \times pN$ base matrix $H_{\text{base}}$ from the concatenation of $p$ copies of the $N \times N$ base matrices from (3). Then construct a length $Npv$, rate $\approx (p-1)/p$, LDPC code $H$ using superposition where the second non-zero entry of each column of $H_{\text{base}}$ is replaced by the permutation matrix $I^i$, with the shifts $i$ chosen so that the sum of the shifts in any adjacent set of $N$ columns is non-zero.

For example, the rate-1/2 burst error correction codes with

$$H = \begin{bmatrix}
0 & 0 & 0 & I & 0 & 0 & 0 & I & I \\
0 & 0 & I & I & 0 & I & 0 & 0 & I \\
I & 0 & I & 0 & I & 0 & 0 & I & 0 \\
I & 0 & I & 0 & I & 0 & 0 & I & 0 \\
I & 0 & I & 0 & I & 0 & 0 & I & 0 \\
I & 0 & I & 0 & I & 0 & 0 & I & 0 \\
\end{bmatrix}$$

have $L_{\text{max}} = 5v - 2$ when superposition with $v \times v$ circulants is used.

These constructions extend naturally to multiple burst erasure correcting codes.

Construction 3: Given a $M \times N$ base matrix $H_{\text{base}}$, with minimum stopping set size $S_{\text{min}}$. Arrange the columns of $H_{\text{base}}$ such that the columns with final non-zero entry in the $M$-th row are first followed by the columns with final non-zero entry in the $(M-1)$-th row and so on. Form $H$ using superposition where the last non-zero entry in the $i$-th column, $i > 1$, of $H_{\text{base}}$ is replaced by $I^{i-1}$ and every other non-zero entry of $H_{\text{base}}$ is replaced by $I$.

Applying Lemma 2 the codes of Construction 3 can correct $S_{\text{min}} - 1$ length $v - N + 1$ erasure bursts. For example, using the length-7, rate-4/7 Hamming code parity-check matrix as the base matrix and applying superposition with $v = 100$ circulant matrices gives LDPC codes which can correct any two bursts of length up to 94 bits. However, codes constructed in this way only approach burstMDS performances for those rates for which binary MDS base matrices exist, i.e. the codes can be burstMDS for $n_b > 1$ bursts only if the code rate is $1/(n_b + 1)$.

To construct codes which are close to burstMDS for $n_b > 1$ and rate $\geq 1/2$ requires that more than $S_{\text{min}} - 1$ bursts can be corrected.

For codes constructed using superposition to be burstMDS for two erasure bursts requires that $H_{\text{base}}$ have only two rows. Unfortunately, the only binary base matrix with two rows and $S_{\text{min}} = 3$ is the rate-1/3 repetition code. Nevertheless, using the $2 \times N$ base matrices from (1) and applying Lemmas 2 and 3 we obtain codes which approach burstMDS performances.

Construction 4: Construct $H$ using superposition on the base matrix from (1). The $(2,i)$-th entry, $i > 2$, of $H_{\text{base}}$ is replaced by the permutation matrix $I^{i-2}$ and all other non-zero entries of $H_{\text{base}}$ are replaced by $I$.

Lemma 7: The codes from Construction 4 can correct any two bursts of size $v - N + 2$ on the burst erasure channel provided that the guard band is erasure free.

Proof: Applying Lemma 3 gives that any stopping set in the columns of $H$ corresponding to the weight-2 columns of $H_{\text{base}}$ must include at least every $(N-2)$-th column of two of the permutation matrices. With only two bursts of size $v - N + 2$ it is not possible to erase all of the columns in these stopping sets. Alternatively the weight-1 columns of $H_{\text{base}}$ are involved in stopping sets of size 3 in $H_{\text{base}}$, however, applying Lemma 2, gives that any two bursts of size $v - N + 2$ also cannot erase the corresponding stopping sets in $H$.

For example, rate-3/4, length 8v LDPC codes which can correct any two bursts of length up to $v - 4$ bits are given by:

$$H = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & I^1 & I^2 & I^3 \\
\end{bmatrix}$$

Non-binary BurstMDS codes for the same length and rate would be able to correct two bursts of length up to $v$ bits making the LDPC codes presented here very close to optimal on a burst erasure channel. Furthermore, the LDPC codes can correct these two bursts regardless of where they occur in the codeword, not just bursts which are aligned with the $v$-ary symbol locations in the codeword.

Where more than two bursts occur per codeword we also use base matrices with $M$ equal to the number of bursts and with $S_{\text{min}}$ less than $M + 1$. LDPC codes to correct three erasure bursts can be constructed similarly to Construction 2 with $p = 2$ but now to satisfy Lemma 4 the shifts must be chosen to ensure that the sum of the shifts for all size three stopping sets
in $H_{\text{base}}$ is greater than zero, rather than just those stopping sets with adjacent columns. However, this constraint alone does not guarantee a good burst error correction performance since three bursts can erase columns from as many as six different $H_{\text{base}}$ columns. For example, using

$$H = \begin{bmatrix} 0 & 0 & I & I & I & I \\ I & I & 0 & 0 & I^4 & I^5 \\ I & I^1 & I^2 & I^3 & 0 & 0 \end{bmatrix}$$

and superposition with $v \times v$ circulants, three bursts of size less than $v - 5$ cannot erase a stopping set in $H$ if the bursts are restricted to erase columns of $H$ within the boundaries of a circulant. Without this restriction, three short bursts can erase a stopping set of size 4. For example the stopping set corresponding to the $(2v+1)$-th, $4v$-th, $(4v+1)$-th, and $6v$-th columns of $H$ can be erased by three bursts of size 2 bits each. While these stopping sets will always exist whichever way we permute the columns of (7), we can avoid the situation that two of the columns are adjacent, and so erased by the same burst, by instead choosing

$$H = \begin{bmatrix} I & 0 & I & I & 0 & I \\ I & I & 0 & 0 & I & I^4 \\ 0 & I & I^1 & I^2 & I^3 & 0 \end{bmatrix}.$$  \tag{8}$$

In this case we do have the problem that three adjacent bursts of size $2v/3$ or greater across the third and fourth columns of (8) will erase the stopping set in $H$ from the two adjacent copies of the same column in $H_{\text{base}}$. Nevertheless, a single received bit between any two of the bursts will allow the correction of all of the erased bits. Any other set of three bursts with length up to $v - 6$ bits can always be corrected using the base matrix from (8) and superposition with $v \times v$ circulants. Since the event of three consecutive bursts has a low probability of occurring the average decoder performance is not significantly affected by this case as we will see below.

The same principles can be extended to other base matrices with $S_{\text{min}} = 3$.

**Construction 5:** Construct $H_{\text{base}}$ from the concatenation of 2 copies of an LDPC base matrix with $S_{\text{min}} \geq 3$. Order the columns of $H_{\text{base}}$ to avoid small stopping sets in adjacent columns. Form $H$ using superposition where the final entry of each column of $H_{\text{base}}$ is replaced by the permutation matrix $I^l$, with the shifts $l$ chosen so that the sum of the shifts in any stopping set is non-zero.

For example,

$$H = \begin{bmatrix} I & I & I & I & 0 & 0 & I & I \\ I & I & 0 & 0 & I & I & I^6 & I^7 \\ I & I^1 & I^2 & I^3 & I^4 & I^5 & 0 & 0 \end{bmatrix}$$

\tag{9}$$
gives rate-5/8 3-burst correcting LDPC codes.

Figs. 1 and 2 show the erasure correction performance of binary codes from Constructions 4 and 5 on burst erasure channels with a fixed number of erasure bursts, both with and without erasures in the guard band. Also shown is the performance of interleaved codes and pseudo-randomly constructed LDPC codes with the same rate and length. The pseudo-random LDPC codes are constructed with weight-3 columns and small cycles avoided using the method from [15].

Fig. 1 shows the performance of length 800, rate-3/4 codes on the burst erasure channel with two randomly located erasure bursts per codeword. The burst erasure code from Construction 4 uses the base matrix shown in (6) and superposition with $v = 100$ circulant permutation matrices. Since there is no length-8, rate-3/4 binary MDS code, 100 codewords from the length-4, single erasure correcting MDS code are interleaved and two sets are concatenated to form length 800 interleaved codewords.

Fig. 2 shows the erasure correction performance of length-600, rate-1/2, binary codes on the burst erasure channel with three randomly located erasure bursts per codeword. The burst erasure correction code from Construction 5 uses the base matrix shown in (8) and superposition with $v = 100$ circulant permutation matrices. For the interleaved codes, 100 codewords from the rate half repetition code are interleaved and three sets are concatenated to form length 600 codewords.

While random LDPC codes do well in channels dominated by random erasures, when the channel is dominated by burst erasures the structured LDPC codes presented here produce by far the best results. By designing structured LDPC codes we
for the random codes, the structure of the channel has been taken into account in their design. More significantly they can also perform much better than traditional binary burst error correction schemes. Although the interleaved codes are burstMDS for one burst, since single erasure correcting binary MDS base matrices exist, they do not achieve the performance improvements gained by the structured LDPC codes which can correct multiple bursts occurring close proximity.

IV. CONCLUSION

In this paper we have designed LDPC codes for channels with erasure bursts. We have shown that, in burst erasure channels, structured LDPC codes constructed via superposition on carefully chosen base matrices can provide significant performance improvements over both pseudo-random LDPC codes and traditional code interleaving.

For memoryless channels it is well established that long pseudo-random LDPC codes provide capacity approaching performances. However, for channels with memory in the form of burst erasures we have shown that structured LDPC codes are certainly the best choice. Moreover, the ease of implementation of message-passing decoding on erasure channels, and the ease of encoding provided by the proposed codes, suggests that structured LDPC codes represent a promising candidate for applications which suffer from burst erasures and face low complexity or high throughput constraints.

REFERENCES