On the Dimensionality of Spatial Fields with Restricted Angle of Arrival

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Abstract—Wireless communication occurs through continuous fields, over regions of space. Although the communication channel may be modeled in terms of infinite dimensional vector spaces, it is paramount to develop finite dimensional approximations for this channel. We extend previous work which examined the fundamental finite dimensionality of fields over a finite region of space to incorporate restricted angles of signal arrival.

By considering a subspace of fields where direction of arrival of the field components, or source spatial distribution, is restricted we show the received field dimensionality is linearly related to the product of the radius of the region of interest and the angular restriction. This result provides a rigorous foundation for analysing the performance and capacity of MIMO systems in non isotropic environments. The proof is presented for the two dimensional case. It is apparent that this work can be extended to the case of three dimensions for a similar result.

I. INTRODUCTION AND MOTIVATION

Wireless communication is achieved through the interaction of continuous fields over regions of space. The additive white Gaussian noise vector channel is well established [1, 2] as a useful model for analysis of MIMO communications channels. Although the model serves as a useful mathematical abstraction it is also important to consider the fundamental physical processes occurring in the propagation of information through space [3, 4]. Typical vector model approaches reduce the properties of space to the statistics of random matrices [5–7]. Whilst such models may be useful, they often depend on secondary parameters such as correlation coefficients, and have come under recent scrutiny [8, 9].

Complete electromagnetic modelling is an alternative, however the complexities and sensitivity to specific physical embodiments of this approach are prohibitive. It is desirable to introduce an holistic approach to wireless channel modelling that naturally reflects the limiting characteristics of spatial information transfer whilst maintaining the elegance of the vector channel model. Towards this goal, in both acoustic and radio frequency wave propagation, the linear wave equation has been used as a model for prediction and analysis of the performance of physical systems. General solutions to the wave equation provide a basis that capture the wave propagation characteristics and allow a spatial field problem to be posed in linear algebra. The spherical harmonic modes provide the basis set that are most concentrated within a given finite spatial region and have been applied to problems in acoustics and field reproduction [10] and telecommunications capacity [11].

Previous work has addressed the richness or information content of a field observed in a finite region. In [12] the case of plane waves in two dimensions was considered, while [13] extended this to general source distributions outside of the observation region in three dimensions. Both works however, consider the case of an arbitrary source distribution with respect to the angular distribution.

Often in wireless communications the directions of arrival are constrained in direction or span only a partial sector of potential. This knowledge or restriction on the field should be incorporated into the model. It has been noted that the richness, dimensionality or degrees of freedom for a spatial field decrease as the angular diversity is reduced [11, 14].

While this relationship appears to be linear, it has not been formally stated or proven.

A reduction in angular diversity or sparse scattering environment is generally modelled as an increased antennae correlation [7, 9], and/or a line-of-sight component [15]. An alternate approach is to consider the properties of the continuous field. Restricted angular diversity decreases the degrees of freedom of a spatial field. This then directly relates to the theoretical capacity of the communication link. Proper formulation of this relationship can be used to reveal impact of angular diversity on the upper limit of capacity without reference to the antennae geometry or correlation models.

Our aim is to provide a rigorous foundation to the intuitive results that angular diversity restricts capacity, without relying on apriori model adjustments, such as correlation matrices. Further, we provide a spatial equivalent of the well known $2WT$ dimensionality of time-bandwidth constrained signals. The remainder of this paper is arranged as follows: Section II provides an overview of the mathematical tools we will require for the dimensionality result. Section III develops the main results of the paper, showing the relation between field dimensionality and angular diversity. In Section IV we provide simulation results, and then draw conclusions. Proofs are left to the appendix.

II. PROBLEM FORMULATION

A. Wave Model and Basis

Consider the linear wave equation for a narrow-band scalar field $\psi(x)$ [16, 17],

$$\nabla^2 \psi(x) + k^2 \psi(x) = 0,$$

(1)
Consider the relationship between the Herglotz kernel and the Herglotz Wave Function.

The Herglotz representation of a wave field is described in Section II-A. A plane wave with direction of arrival \( \theta \) has coefficients \( \alpha_n = e^{i n \theta} \) and matching (2) and (5) gives us,

\[
\alpha_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{-in\theta} d\theta
\]

This relationship is an inverse Fourier transform. The restriction of \( g \in L^2(S^1) \) is analogous to \( \sum |\alpha_n|^2 < \infty \). It restricts fields to those that are reasonably behaved, in that

\[
\left\{ \psi : \lim_{R \to \infty} \frac{1}{R} \|\psi\|_R < \infty \right\}
\]

This slight restriction provides the significant advantage of placing the problem into a Hilbert space. With this space, any bound field, including a plane wave \( g \in L^1(S^1) \), can be represented to arbitrary precision over a finite volume [13].

### C. Restricted Direction of Arrival

**Definition 1 (Restricted direction field).** A restricted direction field is represented by a Herglotz Wave Function, \( g_A \in L^2(S^1) \),

\[
g_A(\theta) = \left\{ \sum_{n=-\infty}^{\infty} a_n e^{i n \theta} \pi/A \quad |\theta| < A \leq \pi \right. \text{ elsewhere}
\]

The space of Herglotz Wave Functions with restricted angle of arrival is a linear subspace of the full set of Herglotz Wave Functions. The Fourier expansion provides a complete countable basis for the space.

### III. Dimensionality

A useful approach to dimensionality is to consider the error in a reduced order approximation of the field [12, 13]. It has been shown that the modal basis (2) is optimal for the representation of a field over a disk of radius \( R \) where the source distribution is unconstrained. Thus,

\[
psi_N(x) = \sum_{n=-N}^{N} \alpha_n \beta_n
\]

provides an approximation of a globally bound field with exponentially decreasing error for \( N > (\pi e)/R/\lambda \) [13]. This provides a model of the field with \( 2N + 1 \) parameters \( \alpha_n \) for \( n = -N \ldots N \). Define dimensionality as the point beyond which an exponential improvement in the approximation error can be achieved regardless of the field, specifically:

**Definition 2 (Dimensionality).** Consider a Hilbert space of functions \( \{ \mathcal{X} : \| \cdot \| \} \). If for some value \( N_0 \) and choice of \( \{ \varphi_i \}_{i=0}^\infty \) and for any \( x \in \mathcal{X} \)

\[
\left\| x - \sum_{i=0}^{N_0} \langle x, \varphi_i \rangle \varphi_i \right\| \leq \epsilon < \infty
\]

and for any \( n > N_0 \),

\[
\left\| x - \sum_{i=0}^{n} \langle x, \varphi_i \rangle \varphi_i \right\| \leq \exp \{ -\alpha(n - N_0) \}
\]
for some fixed $\alpha > 0$, then we have say the space $X$ has dimension $N_o$.

Definition 2 is similar in application to the concept of “essential dimension” in operator approximations [22].

**Lemma 1.** The modal coefficients, as determined by (6), of a restricted direction field are an infinite band limited sequence.

**Lemma 2.** A restricted direction field is well approximated over a finite region, $\{ x : ||x|| \leq R \}$, by $2N + 1$ terms from an infinite band limited sequence where $N = [\pi e R / \lambda]$.

It is well known that a band limited sequence that is also confined is in reference to most of the energy being contained in a finite length. Strictly a band limited sequence cannot also be time limited.

**Definition 3** (Slepian approximation to truncated field). The $M$th order Slepian approximation to the $N$th order modal field is given by,

$$\hat{\Psi}_N(x) = \sum_{n=-N}^{N} \hat{\alpha}_n \beta_n(x) = \sum_{n=-N}^{N} \sum_{m=0}^{M-1} c_m v_{m+N}^n \beta_n(x)$$

**Lemma 3.** An arbitrary restricted direction field, can be well approximated by a Slepian approximation field, $\hat{\Psi}_N$, with $M \leq 2N + 1$ parameters provided where,

$$\frac{M}{2N + 1} \geq \frac{A}{\pi}$$

1Here confined is in reference to most of the energy being contained in a finite length. Strictly a band limited sequence cannot also be time limited.

**Theorem 1** (Dimensionality of a Restricted Direction Field). In the same sense as defined in [12], a restricted direction of arrival field has a dimensionality of,

$$2N' + 1 \text{ where } N' = \left\lfloor \frac{\pi e R}{\lambda} \right\rfloor$$

**IV. Simulation**

The previous section set out a proof for the central dimensionality result of this paper. In this section we compare the bound obtained with numerical calculations of the true result. Here we lend heavily from the framework of [13] and assume that the reader is familiar with the correspondence between the energy concentration, eigenvalue and approximation problems.

In [29], Slepian derives a general method for finding the extent to which a set of orthogonal functions function, $\psi_n(x)$, and their Fourier transforms, $\hat{\psi}_n(u)$, can be simultaneously concentrated.

A field $\psi(x)$ with Fourier transform constrained to a region $\Delta$, will have a set of orthogonal functions maximally concentrated in the spatial region $\Omega = \{ x : ||x|| \leq R \}$, that are the solution of the following eigenfunction equation:

$$\lambda_n \hat{\Psi}(u) = \int_{\Delta} K_{\Omega}(u-v) \hat{\Psi}(v) dv, u \text{ in } \Delta$$

$$K_{\Omega}(u) = (2\pi)^{-2} \int_{\Omega} e^{i u \cdot v} dv$$

Where $K_{\Omega}u$ is the kernel specific to the spatial region $\Omega$. In this case, we are only interested in the eigenvalues $\lambda_n$ representing the maximal concentrations.

The wave equation constraint (1) and restricted direction field (Definition 1) leads to $\Delta = \{ u : u = ke^{i \theta}, |\theta| < A \}$. This allows us to cast the problem as a one dimensional equation,

$$\lambda_n g(\theta) = \int_{-A}^{A} K'_{\Omega}(\theta - \phi) g(\phi) k d\phi$$

with kernel for non-zero $\theta$,

$$K_{\Omega}(\theta) = \begin{cases} \frac{k R^2}{4\pi} & \theta = 0 \\ \frac{k R}{2\pi z} J_1(z R) & \text{otherwise} \end{cases}$$

$z = \sqrt{2 - 2 \cos \theta}$

This is a homogeneous Fredholm equation of the second kind, for which no standard analytical techniques can be applied. A large number of techniques exist to find the eigenvalues numerically [30]. Using the quadrature technique, we use a $Q$ point quadrature rule such that,

$$\int_{-A}^{A} g(\theta) \approx \sum_{i=1}^{Q} w_i g(\theta_i)$$

Then we can approximate (19) with the linear system of equations,

$$\lambda_n \Psi(\theta_i) = \sum_{j=1}^{Q} w_i K_{\Omega}(\theta_i - \phi_j) \Psi(\phi_j)$$
and thus numerical approximation of the to the first $Q$ eigenvalues is obtained from as the eigenvalues of the $Q \times Q$ matrix,

$$K_{i,j} = K_{\Omega}(\theta_i - \phi_j)w_j$$

(Fig. 1) Plot of numerical eigenvalues and bounds for a region $R = 2\lambda$ with successive restriction on the angle of arrival. The plots represent the concentration of energy of the component of the field within the region, and thus relate to a bound on the approximation of a field with reduced order. The dash-dot line is the error bound using the true $\alpha_i$ coefficients and the actual $\lambda_n$, Slepian sequence eigenvalues. The dotted line is based on the upper bound (33). The floor on the bound is due to the first term in (28) for which $N = 13$ was selected.

The first term can be made small by appropriate selection of $M$. Thus we can write

$$\hat{\beta}_m(x) \approx \sum_{n=-N}^{N} v_{n+N} \beta_n(x) \quad m < 2N' + 1$$

This result can be used to reduce the dimensionality of the problem space in problems such as direction of arrival estimation and channel capacity.

In conclusion, both the size of the region of interest and the extent of angular diversity are significant to the dimensionality of a spatial field.

**Proofs**

**Proof of Lemma 1.** From (6) we can express the modal coefficients of the field,

$$\alpha_n = \frac{1}{2\pi} \int_{-A}^{A} \sum_{m=-\infty}^{\infty} a_m e^{im\theta \pi / A} e^{-\lambda_m n} d\theta$$

$$= \frac{A}{\pi} \sum_{m=-\infty}^{\infty} a_m \text{sinc} (An - m\pi)$$

where $\text{sinc}(\theta) = \sin(\theta) / \theta$. The sinc functions are critically spaced effecting a low pass filter with cutoff $A$. Where the source direction is centred about an angle $\phi$ result is a filter band of $2A$ centred on $\phi$.

**Proof of Lemma 2.** This is immediately apparent from Proposition 1 and (9) to take a finite truncation of the infinite bandlimited sequence. This holds as the space of restricted direction fields is a subspace of all fields.

**Proof of Lemma 3.** Given the orthogonality of $\beta_n$ on the region of interest $||x|| < R$, it can be seen that

$$||\psi - \hat{\psi}_N||_R = ||\psi - \psi_N||_R + ||\psi_N - \hat{\psi}_N||_R$$

The first term can be made small by appropriate selection of $N > \pi e R / \lambda$ [13]. The second term is the residual field error from the Slepian expansion of the $2N + 1$ terms. The Slepian series are also orthogonal for $n = -\infty \ldots \infty$ with the energy in this infinite extension is $\lambda^{-1}_m$ where $\lambda_m$ is the eigenvalue associated with the $m^{th}$ Slepian sequence. Thus we can write

$$\sum_{n=-\infty}^{\infty} |\alpha_n|^2 \geq \sum_{m=0}^{2N} |c_m|^2 \geq \frac{1}{\lambda_M} \sum_{m=M}^{2N} |c_m|^2$$

since $\lambda_m$ is strictly decreasing.

Given that $||\beta_n||_R$ is approximately bound by $R$, consider the second term of (28) and use the result (29),

$$||\psi_N - \hat{\psi}_N||_R \leq \sum_{m=-\infty}^{N} (\alpha_m - \hat{\alpha}_m)^2 ||\beta_n||_R$$

$$\leq R \sum_{m=M}^{2N} |c_m|^2$$

$$\leq R \lambda_M \sum_{n=-\infty}^{\infty} |\alpha_n|^2$$

**V. Conclusions and Discussion**

We have shown the dimensionality of a received field, in a finite region of space varies linearly with the angular diversity of the incident field. In general the dimensionality or degrees of freedom in a spatial field is proportional to the product $AR$ where $A$ represents the angular diversity of the sources, and $R$ represents the size (radius) of the region of interest.

We have provided bounds on the dimensionality based on Slepian sequences, and compared our bounds with numerical solutions of the wave equation, along with existing upper bounds.

This result is suggestive of the form of the solutions to the problem for constrained angle of incidence. For small angular diversity, the appropriate basis functions are well approximated by the low order Slepian sequences for an appropriate finite length, that is

$$\hat{\beta}_m(x) \approx \sum_{n=-N}^{N} v_{n+N} \beta_n(x) \quad m < 2N' + 1$$

Given that $||\beta_n||_R$ is approximately bound by $R$, consider the second term of (28) and use the result (29),

$$||\psi_N - \hat{\psi}_N||_R \leq \sum_{m=-\infty}^{N} (\alpha_m - \hat{\alpha}_m)^2 ||\beta_n||_R$$

$$\leq R \sum_{m=M}^{2N} |c_m|^2$$

$$\leq R \lambda_M \sum_{n=-\infty}^{\infty} |\alpha_n|^2$$
From the work of Slepian [23] we can take a bound for an approximation of $\lambda_m$,

$$\lambda_m \leq \frac{1}{2} \exp \left[ b \left( \frac{A(2N + 1)}{\pi} - m - 1/2 \right) \right]$$

(33)

$$b = \frac{\log(16N + 8) + \log(\sin(A)) + \gamma}{\pi^2}$$

(34)

where $\gamma = 0.577215\ldots$ is the Euler-Mascheroni constant. While this strictly bounds the approximation from [23] it was shown by numerical inspection to be a suitable upper for the actual $\lambda_m$ values.

Thus $\lambda_m < 0.5$ and decreases exponentially for $m \geq A(2N + 1)/\pi$. This combined with (32) completes the proof.

\section*{Acknowledgements}

National ICT Australia is funded through the Australian Government’s Backing Australia’s Ability initiative, in part through the Australian Research Council.

\section*{References}


