Redeeming the Clinical Promise of Diffusion MRI in Support of the Neurosurgical Workflow

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Economic Cost of Brain Disorders in Europe 2010: € 798 billion ...

(N)MRI scanner
“everything must be made as simple as possible, but not one bit simpler”

attributed to Albert Einstein
operational model: brain = constrained water
operaPonal model: brain = constrained water

tissue microstructure imparts non-random barriers to water diffusion

operational model: brain = constrained water

Riemannian paradigm

extrinsic diffusion on Euclidean space $\approx$ intrinsic geometry of a Riemannian space
operational model: brain = constrained water

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Riemannian paradigm

extrinsic diffusion on Euclidean space $\approx$ intrinsic geometry of a Riemannian space
gauge figure = unit sphere = indicatrix = Riemannian metric = inner product
local gauge figure

Riemann geometry
local gauge figure
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local gauge figure

Riemann geometry
local gauge figure

Riemann geometry

length^2 = 6
local gauge figure
Riemann geometry

length$^2 = 9$
geodesic tractography
geodesic tractography

Riemannian length : Euclidean length

‘short’ geodesic  

5.0 : 6.0 < 1

‘long’ geodesic  

7.5 : 6.0 > 1
Diffusion Tensor Imaging
versus
local gauge figure
Diffusion Tensor Imaging

physics pipeline

physics in a nutshell:

nuclear spin quantization $\rightarrow$ Zeeman splitting $\rightarrow$ Boltzmann statistics $\rightarrow$ magnetization $\rightarrow$ Bloch-Torrey equation $\rightarrow$ DTI

mathematics in a nutshell:

DTI $\rightarrow$ local gauge figure $\rightarrow$ geodesic tractography
Diffusion Tensor Imaging
physics pipeline

physics in a nutshell:

**nuclear spin quantization** → Zeeman splitting → Boltzmann statistics → magnetization → Bloch-Torrey equation → DTI

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nuclear spin quantization → Zeeman splitting → Boltzmann statistics → magnetization → Bloch-Torrey equation → DTI

mathematics in a nutshell:

DTI → local gauge figure → geodesic tractography
nuclear spin quantization → Zeeman splitting

\[ E = -\mu \cdot B \]

\[ E_\downarrow = +\frac{1}{2} \gamma \hbar B_z \]

\[ E_\uparrow = -\frac{1}{2} \gamma \hbar B_z \]

\[ \Delta E = \gamma \hbar B_z = \hbar \omega_{\text{Larmor}} \]
Boltzmann statistics → magnetization
(typical clinical 3T MRI scanner)

\[
\frac{N_{\uparrow}}{N_{\downarrow}} = \exp \left[ \frac{\Delta E}{KT} \right] \approx 1 + \frac{\gamma \hbar B_z}{KT}
\]

\[
\frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}} \approx \frac{\gamma \hbar B_z}{2KT} \approx 10^{-5}
\]

\[
M_z = \frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}} M_{\text{max}} \approx (N_{\uparrow} + N_{\downarrow}) \frac{\gamma^2 \hbar^2 B_z}{4KT}
\]

‘low sensitivity modality’

‘big x small = measurable’
Bloch-Torrey equation → DTI

**Bloch-Torrey / Stejskal-Tanner / Basser-Mattiello-Le Bihan:**

Gaussian signal attenuation in q-space

\[
\frac{\partial M_\perp}{\partial t} = -i \gamma (M_\perp B_\parallel - M_\parallel B_\perp) - \frac{M_\perp}{T_2} + \nabla \cdot D \nabla M_\perp
\]

\[
S(x, q, \tau) = S_0(x) \exp(-\tau q \cdot D(x) q)
\]

\[
D(x) = -\frac{1}{\tau} \nabla_q^2 \ln \frac{S(x, q, \tau)}{S_0(x)}
\]
Bloch-Torrey equation → DTI

further reading
physics in a nutshell:

nuclear spin quantization $\rightarrow$ Zeeman splitting $\rightarrow$ Boltzmann statistics $\rightarrow$ magnetization $\rightarrow$ Bloch-Torrey equation $\rightarrow$ DTI

mathematics in a nutshell:

DTI $\rightarrow$ local gauge figure $\rightarrow$ geodesic tractography
geodesic tractography

mathematics pipeline

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mathematics in a nutshell:

DTI $\rightarrow$ local gauge figure $\rightarrow$ geodesic tractography
DTI \rightarrow \text{local gauge figure}

\textbf{DTI signal model:} \quad S(x, q, \tau) = S_0(x) e^{-\tau q^T D(x) q}

\textbf{Riemann metric:} \quad G(\xi, \xi) |_x = \xi^T G(x) \xi

\textbf{Lenglet et al. / O’Donnell et al.:} \quad G(x) \doteq D^{\text{inv}}(x)

\textbf{Fuster et al.:} \quad G(x) \doteq D^{\text{adj}}(x)
local gauge figure → tractography

\[ G(v, v) = \|v\|^2 \]  
Riemann metric: lengths & angles

\[ \nabla_{\dot{x}} \dot{x} = 0 \]  
Levi-Civita connection: parallel transport

\[ \ddot{x}^i + \sum_{jk} \Gamma^i_{jk}(x) \dot{x}^j \dot{x}^k = 0 \]  
Christoffel symbols: “pseudo-forces” (relative to local coordinate frames)
local gauge figure $\rightarrow$ tractography

$$G(v, v) = \|v\|^2$$

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local gauge figure → tractography
Euclidean geodesic

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local gauge figure → tractography

Euclidean geodesic
Riemannian geodesic:

\[ G(v, v) = \|v\|^2 \]
\[ \nabla_{\dot{x}} \dot{x} = 0 \]
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Riemann metric: lengths & angles

Levi-Civita connection: parallel transport

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local gauge figure \(\rightarrow\) tractography

Riemannian geodesic
Riemann-DTI paradigm:

- neural fiber bundles correspond to relatively short geodesics in a Riemannian ‘brain space’
- the Riemannian structure can be inferred from DTI
Riemann-DTI paradigm
pros & cons

geodesic completeness
= redundant connections

pro: pixels $\rightarrow$ geodesic congruences

ellipsoidal gauge figure
= poor angular resolution

con: destructive interference of orientation preferences
specific model (DTI):

\[ S(x, q, \tau) = S_0(x) \exp(-D(x, q, \tau)) \]

with

\[ D(x, q, \tau) = \tau q^T D(x) q \]

6 d.o.f.’s per point sample
6 d.o.f.’s of local gauge figure

generic model (HARDI):

\[ S(x, q, \tau) = \sum_{k=0}^{\infty} S^{i_1 \ldots i_k}(x, \tau) \phi_{i_1 \ldots i_k}(q) \]

or

\[ D(x, q, \tau) = \sum_{k=0}^{\infty} D^{i_1 \ldots i_k}(x, \tau) \psi_{i_1 \ldots i_k}(q) \]

∞ d.o.f.’s per point sample
beyond the Riemann-DTI paradigm
a paradigm shift

\[
\text{DTI} \subset \text{HARDI}
\]
beyond the Riemann-DTI paradigm

a paradigm shift

DTI \subset \text{HARDI}

\downarrow

Riemann geometry
beyond the Riemann-DTI paradigm

a paradigm shift

\[ \text{DTI} \subset \text{HARDI} \]

\[ \text{Riemann geometry} \subset \ldots \]
beyond the Riemann-DTI paradigm

a paradigm shift

DTI \subseteq \text{HARDI}

\text{Riemann geometry} \subseteq \text{Finsler geometry}
Finsler geometry
heuristics
Finsler geometry
heuristics

base manifold: $x \in \mathbb{R}^3 \rightarrow (x, w) \in \mathbb{R}^3 \times \mathbb{S}^2$ (sphere bundle)

base manifold: $x \in \mathbb{R}^3 \rightarrow (x, \xi) \in \mathbb{R}^3 \times \mathbb{T}\mathbb{R}^3 \setminus \{0\}$ (slit tangent bundle)
Finsler geometry

heuristics

gauge figure = unit sphere = indicatrix = Finsler metric ≠ inner product
2.3 Connections in Riemann-Finsler Geometry

There is no "tangent" connection formulation for parallel transport on a Riemann-Finsler manifold. The Levi-Civita Cartan-Rashid and Hashiguchi connection may all be considered "natural" examples of the Levi-Civita connection in Riemann geometry. For instance, the (torsion free) Cartan-Rashid connection is defined by

\[ \frac{\partial}{\partial x^i} N_j^k(x, \xi) = \frac{\partial N_j^k}{\partial x^i} (x, \xi) + \mathcal{H}^l (x, \xi) \Gamma^j_{ki} \xi^l, \]

where \( N_j^k \) is the so-called \( \mathcal{H} \)-splitting symbol of Finsler geometry. If one replaces the Riemannian metric \( g(x, \xi) \) by the Riemann-Finsler metric \( F(x, \xi) \), Eq. (5), then it is called the Cartan-Rashid connection. The symbol \( \mathcal{H} \)-splitting is defined by

\[ \mathcal{H}^l (x, \xi) = \frac{\partial N_j^k}{\partial x^i} (x, \xi) \xi^i - \frac{\partial N_j^k}{\partial \xi^i} (x, \xi) x^i. \]

The coefficients \( N_j^k(x, \xi) \) define the so-called covector connection [35],

\[ N_j^k(x, \xi) = \frac{\partial N_j^k}{\partial x^i} (x, \xi) \xi^i, \]

in which the formal Covariant derivative of the vector field is introduced as

\[ \nabla_j (x, \xi) = \frac{\partial}{\partial x^j} (x, \xi) \xi^i - \frac{\partial N_j^k}{\partial x^i} (x, \xi) x^i, \]

Note that in the Riemannian limit, both Eq. (19) as well as Eq. (22) simplify to

\[ \frac{\partial N_j^k}{\partial x^i} (x, \xi) \xi^i = \frac{\partial \xi^i}{\partial x^j} (x, \xi), \]

These are the familiar Cartan-Galilean symmetries of the so-called Cartan's principal Cartan-Rashid connection in Riemannian geometry. A comparison reveals that

\[ \Gamma_j^k (x, \xi) = \frac{\partial \xi^i}{\partial x^j} (x, \xi) \xi^i - \frac{\partial \xi^i}{\partial x^j} (x, \xi) x^i, \]

in which relations have been based with the help of the Riemann-Finsler metric tensor

\[ N_j^k(x, \xi) = \frac{\partial N_j^k}{\partial x^i} (x, \xi) \xi^i + \mathcal{H}^l (x, \xi) \Gamma^j_{ki} \xi^l, \]

and in which the geodesic coefficients are defined as

\[ \nabla_j (x, \xi) = \frac{\partial N_j^k}{\partial x^i} (x, \xi) \xi^i + \mathcal{H}^l (x, \xi) \Gamma^j_{ki} \xi^l. \]

2.4 Horizontal-Vertical Splitting

The heuristic coupling of position and orientation is formalized in terms of the so-called horizontal and vertical basis vectors, recall Eq. (20),

\[ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_j^k (x, \xi) \frac{\partial}{\partial \xi^j}, \]

\[ \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - N_j^k (x, \xi) \frac{\partial}{\partial \xi^j}. \]

These constitute a basis for the horizontal and vertical tangent bundles over the slit tangent bundle:

\[ \mathcal{H}_j^k (x, \xi) TM = \text{span} \left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{V}_j^k (x, \xi) TM = \text{span} \left\{ \frac{\partial}{\partial x^i} \right\}. \]

Their direct sum yields the complete tangent bundle

\[ \text{TTM} \setminus \{0\} = \mathcal{H}_j^k \circ \mathcal{V}_j^k. \]

pointwise. By the same token one considers the horizontal and vertical basis covectors,

\[ dz^i \quad \text{and} \quad \delta z^i = \delta x^j \frac{\partial z^i}{\partial x^j}, \]

yielding the corresponding horizontal and vertical covariant bundles:

\[ \mathcal{H}_j^k (x, \xi) \circ \text{TTM} = \text{span} \left\{ dz^i \right\}, \quad \mathcal{V}_j^k (x, \xi) \circ \text{TTM} = \text{span} \left\{ \delta z^i \right\}. \]

such that

\[ T^\text{TTM} \setminus \{0\} = \mathcal{H}_j^k \circ \mathcal{V}_j^k. \]

pointwise.

The above vectors and covectors satisfy the following duality relations:

\[ dz^i \frac{\partial}{\partial z^i} = \delta x^j \frac{\partial}{\partial z^i} = 0, \quad \delta z^i \frac{\partial}{\partial z^i} = \delta z^j \frac{\partial}{\partial z^i} = 0. \]

Incorporating a natural scaling so as to ensure zero-symmetricity with respect to \( \xi \) (so that it indeed represents orientation rather than "velocity") we conclude that

\[ \text{TTM} \setminus \{0\} = \text{span} \left\{ \frac{\delta}{\delta x^i}, F(x, \xi) \frac{\partial}{\partial \xi^i} \right\}, \]

and similarly

\[ T^\text{TTM} \setminus \{0\} = \text{span} \left\{ \delta z^i, F(x, \xi) \frac{\partial}{\partial \xi^i} \right\}. \]

The so-called Finsler metric furnishes the slit tangent bundle with a natural Riemannian metric:

\[ g(x, \xi) = g_{ij}(x, \xi) dz^i \otimes dz^j + g_{ij}(x, \xi) \frac{\partial}{\partial \xi^i} \otimes \frac{\partial}{\partial \xi^j}. \]
Finsler geometry
axiomatics

Finsler metric:
\[ F^2(x, \xi) = g_{ij}(x, \xi)\xi^i \xi^j \quad \Leftrightarrow \quad g_{ij}(x, \xi) = \frac{1}{2} \partial_{\xi^i} \partial_{\xi^j} F^2(x, \xi) \]

Riemannian limit:
\[ F^2(x, \xi) = g_{ij}(x)\xi^i \xi^j \quad \Leftrightarrow \quad g_{ij}(x) = \frac{1}{2} \partial_{\xi^i} \partial_{\xi^j} F^2(x, \xi) \]

distance: \( d(x_1, x_2) = \inf \left\{ \int_\gamma F(\gamma(t), \dot{\gamma}(t)) \, dt \mid \gamma \in C^1([t_1, t_2], \mathbb{R}^3), \gamma(t_1) = x_1, \gamma(t_2) = x_2 \right\} \)

Cartan tensor:
\[ C_{ijk}(x, \xi) = \frac{1}{2} \partial_{\xi^k} g_{ij}(x, \xi) = \frac{1}{4} \partial_{\xi^i} \partial_{\xi^j} \partial_{\xi^k} F^2(x, \xi) \]

Riemannian limit: \( C_{ijk}(x, \xi) = 0 \) (Deicke’s Theorem)
Deicke’s Theorem:

Space is Riemannian iff the Cartan tensor vanishes.
Finsler-DTI paradigm
geodesic tractography

DTI ~ Riemannian geometry (inner product norm):

\[ F^*(x, q) = \sqrt{q^T D(x) q} \]
\[ F(x, y) = \sqrt{y^T D^{-1}(x) y} \]

HARDI ~ Finslerian geometry (generalized norm):

\[ F^*(x, \lambda q) = |\lambda| F^*(x, q) \]
\[ F(x, \lambda y) = |\lambda| F(x, y) \]
Finsler-DTI paradigm

geodesic tractography

Finsler metric: lengths

\[ G_v(v, v) = \|v\|^2_F \]

Chern-Rund (or other) connection: parallel transport

\[ \nabla_{\dot{x}} \ddot{x} = 0 \]

formal Christoffel symbols: “pseudo-forces”

\[ \ddot{x}^i + \sum_{jk} \Gamma^i_{jk}(x, \dot{x}) \dot{x}^j \dot{x}^k = 0 \]
Finsler-DTI paradigm

• neural fiber bundles correspond to relatively short geodesics in the 3-dimensional ‘horizontal part’ of a 5-dimensional ‘brain space’ furnished with a Finslerian structure
• this Finslerian structure can be inferred from diffusion MRI measurements
• the Finslerian dual metric can be interpreted as a ‘5-dimensional (3x3) DTI’ tensor
• Finsler geometry encompasses Riemannian geometry as a special case
• the Finsler metric admits $\infty$ d.o.f.’s per spatial point as opposed to 6 d.o.f.’s for the Riemannian limit
• the Finsler-DTI paradigm admits versatile dimensionality reduction in trade-off with acquisition time

Finsler-DTI paradigm

summary

Finsler function:

\[ F^*(x, \lambda q) = |\lambda|F^*(x, q) \]
\[ F(x, \lambda y) = |\lambda|F(x, y) \]
Finsler-DTI paradigm

**summary**

Finsler function:

\[
F^*(x, \lambda q) = |\lambda| F^*(x, q)
\]

\[
F(x, \lambda y) = |\lambda| F(x, y)
\]
applications & outlook
applications & outlook

Stephan Meesters et al., electronic poster 3476, ISMRM 2017, Hawaii
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preoperative

postoperative

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conclusion

- Finsler geometry is a generic and potentially powerful framework for diffusion MRI beyond classical DTI
- this framework allows us to exploit a rich body of knowledge gained over more than a century by great scientists