

On the Complexity of Qualitative Spatial Reasoning: A Maximal Tractable Fragment of the Region Connection Calculus *

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Abstract

The computational properties of qualitative spatial reasoning have been investigated to some degree. However, the question for the boundary between polynomial and NP-hard reasoning problems has not been addressed yet. In this paper we explore this boundary in the “Region Connection Calculus” RCC-8. We extend Bennett’s encoding of RCC-8 in modal logic. Based on this encoding, we prove that reasoning is NP-complete in general and identify a maximal tractable subset of the relations in RCC-8 that contains all base relations. Further, we show that for this subset path-consistency is sufficient for deciding consistency.

1 Introduction

When describing a spatial configuration or when reasoning about such a configuration, often it is not possible or desirable to obtain precise, quantitative data. In these cases, qualitative reasoning about spatial configurations may be used.

One particular approach in this context has been developed by Randell, Cui, and Cohn [1992], the so-called *Region Connection Calculus* (RCC), which is based on binary topological relations. One variant of this calculus, RCC-8, uses eight mutually exhaustive and pairwise disjoint relations, called base relations, to describe the topological relationship between two regions (see also Egenhofer [1991]).

Some of the computational properties of this calculus have been analyzed by Grigni et al. [1995] and Nebel [1995]. However, no attempt has yet been made to determine the boundary between polynomial and NP-hard fragments of RCC-8, as it has been done for Allen’s [1983]

interval calculus [Nebel and Bürckert, 1995]. We address this problem and identify a maximal fragment of RCC-8 that is still tractable and contains all base relations.

As in the case of qualitative temporal reasoning, this proof relies on a computer generated case-analysis that cannot be reproduced in a research paper.¹ Further, we show that for this fragment path-consistency is sufficient for deciding consistency.²

2 Qualitative Spatial Reasoning with RCC

RCC is a topological approach to qualitative spatial representation and reasoning where *spatial regions* are subsets of topological space [Randell *et al.*, 1992]. Relationships between spatial regions are defined in terms of the relation $C(a, b)$ which is true iff the closure of region a is connected to the closure of region b , i.e. if they share a common point. Regions themselves do not have to be internally connected, i.e. a region may consist of different disconnected parts. The domain of *spatial variables* (denoted as X, Y, Z) is the whole topological space.

In this work we will focus on RCC-8, but most of our results can easily be applied to RCC-5, a subset of RCC-8 [Bennett, 1994]. RCC-8 uses a set of eight pairwise disjoint and mutually exhaustive relations, called *base relations*, denoted as DC, EC, PO, EQ, TPP, NTPP, TPP⁻¹, and NTPP⁻¹, with the meaning of *DisConnected*, *Externally Connected*, *Partial Overlap*, *EQual*, *Tangential Proper Part*, *Non-Tangential Proper Part*, and their converses. Examples for these relations are shown in Figure 1. In RCC-5 the boundary of a region is not taken into account, i.e. one does not distinguish between DC and EC and between TPP and NTPP. These relations are combined to the RCC-5 base relations DR for *DiscRete* and PP for *Proper Part*, respectively.

Sometimes it is not known which of the eight base relations holds between two regions, but it is possible

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¹The programs can be obtained from the authors.

²Full proofs can be found in our technical report [Renz and Nebel, 1997].

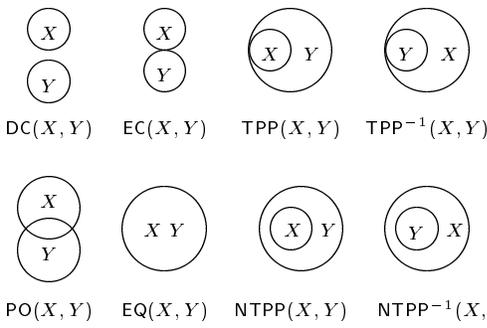


Figure 1: Two-dimensional examples for the eight base relations of RCC-8

to restrict to some of them. In order to represent this, unions of base relations can be used. Since base relations are pairwise disjoint, this results in 2^8 different relations, including the union of all base relations, which is called *universal relation*. In the following we will write sets of base relations to denote these unions. Using this notation, DR, e.g., is identical to $\{DC, EC\}$. *Spatial formulas* are written as XRY , where R is a spatial relation. Apart from union (\cup), other operations are defined, namely, converse (\smile), intersection (\cap), and composition (\circ) of relations. The formal definitions of these operations are:

$$\begin{aligned} \forall X, Y : X(R \cup S)Y &\leftrightarrow XRY \vee XSY, \\ \forall X, Y : X(R \cap S)Y &\leftrightarrow XRY \wedge XSY, \\ \forall X, Y : XR \smile Y &\leftrightarrow YRX, \\ \forall X, Y : X(R \circ S)Y &\leftrightarrow \exists Z : (XRZ \wedge ZSY). \end{aligned}$$

The compositions of the eight base relations are shown in Table 1. Every entry in the composition table specifies the relation obtained by composing the base relation of the corresponding row with the base relation of the corresponding column. Composition of two arbitrary RCC-8 relations can be obtained by computing the union of the composition of the base relations.

A *spatial configuration* can be described by a set Θ of spatial formulas. One important computational problem is deciding *consistency* of Θ , i.e. deciding whether it is possible to assign regions to the spatial variables in a way that all relations hold. We call this problem RSAT. When only relations of a specific set \mathcal{S} are used in Θ , the corresponding reasoning problem is denoted RSAT(\mathcal{S}). In the following $\widehat{\mathcal{S}}$ denotes the closure of \mathcal{S} under composition, intersection, and converse.

3 Encoding of RCC-8 in Modal Logic

In this work we use Bennett's [1995] encoding of RCC-8 in propositional modal logic.³ Bennett obtained this encoding by analyzing the relationship of regions to the universe \mathcal{U} . He restricted his analysis to closed regions

³We assume in the remainder that the reader is familiar with modal logic as presented, e.g., by Fitting [1993].

\circ	DC	EC	PO	TPP	NTPP	TPP ⁻¹	NTPP ⁻¹	EQ
DC	*	DR PO PP	DR PO PP	DR PO PP	DR PO PP	DR PO PP	DC	DC
EC	DR PO PP ⁻¹	DR, EQ PO, TPP TPP ⁻¹	DR PO PP	EC PO PP	PO PP	DR PO PP ⁻¹	DR DC	EC
PO	DR PO PP ⁻¹	DR PO PP ⁻¹	*	PO PP	PO PP	DR PO PP ⁻¹	DR PO PP ⁻¹	PO
TPP	DC	DR	DR PO PP	PP	NTPP	DR, EQ PO, TPP TPP ⁻¹	DR PO PP ⁻¹	TPP
NTPP	DC	DC	DR PO PP	NTPP	NTPP	DR PO PP	*	NTPP
TPP ⁻¹	DR PO PP ⁻¹	EC PO PP ⁻¹	PO PP ⁻¹	PO, EQ TPP TPP ⁻¹	PO PP	PP ⁻¹	NTPP ⁻¹	TPP ⁻¹
NTPP ⁻¹	DR PO PP ⁻¹	PO PP ⁻¹	PO PP ⁻¹	PO PP ⁻¹	PO, EQ PP PP ⁻¹	NTPP ⁻¹	NTPP ⁻¹	NTPP ⁻¹
EQ	DC	EC	PO	TPP	NTPP	TPP ⁻¹	NTPP ⁻¹	EQ

Table 1: Composition table for the eight base relations of RCC-8, where * specifies the universal relation

Relation	Model Constraints	Entailment Constraints
DC	$\neg(X \wedge Y)$	$\neg X, \neg Y$
EC	$\neg(\mathbf{I}X \wedge \mathbf{I}Y)$	$\neg(X \wedge Y), \neg X, \neg Y$
PO	—	$\neg(\mathbf{I}X \wedge \mathbf{I}Y), X \rightarrow Y,$ $Y \rightarrow X, \neg X, \neg Y$
TPP	$X \rightarrow Y$	$X \rightarrow \mathbf{I}Y, Y \rightarrow X, \neg X, \neg Y$
TPP ⁻¹	$Y \rightarrow X$	$Y \rightarrow \mathbf{I}X, X \rightarrow Y, \neg X, \neg Y$
NTPP	$X \rightarrow \mathbf{I}Y$	$Y \rightarrow X, \neg X, \neg Y$
NTPP ⁻¹	$Y \rightarrow \mathbf{I}X$	$X \rightarrow Y, \neg X, \neg Y$
EQ	$X \rightarrow Y, Y \rightarrow X$	$\neg X, \neg Y$

Table 2: Encoding of the base relations in modal logic

that are connected if they share a point and overlap if they share an interior point. If, e.g. X and Y are disconnected, the complement of the intersection of X and Y is equal to the universe. Further, both regions must not be empty, i.e. the complements of both X and Y are not equal to the universe. In this way the eight base relations can be represented by constraints of the form ($m = \mathcal{U}$), called *model constraints*, and ($m \neq \mathcal{U}$), called *entailment constraints*, where m is a set-theoretic expression containing perhaps the topological interior operator i . Any model constraint must hold, whereas no entailment constraint must hold [Bennett, 1994].

The model and entailment constraints can be encoded in modal logic, where spatial variables correspond to propositional atoms and the interior operator i to a modal operator \mathbf{I} (see Table 2). The axioms for i must also hold for the modal operator \mathbf{I} , which results in the following axioms [Bennett, 1995]:

1. $\mathbf{I}X \rightarrow X$,
2. $\mathbf{I}IX \leftrightarrow \mathbf{I}X$,
3. $\mathbf{I}\top \leftrightarrow \top$ (for any tautology \top),
4. $\mathbf{I}(X \wedge Y) \leftrightarrow \mathbf{I}X \wedge \mathbf{I}Y$.

Axioms 1 and 2 correspond to the modal logics **T** and **4**, axioms 3 and 4 already hold for any modal logic **K**, so **I** is a modal **S4**-operator.

The four axioms specified by Bennett are not sufficient to exclude non-closed regions. In order to account for that, we add two formulas for each atom, which correspond to topological properties of closed regions. A closed region is the closure of an open region and the complement of a closed region is an open region:

$$X \leftrightarrow \neg \mathbf{I} \neg \mathbf{I} X, \quad \neg X \leftrightarrow \mathbf{I} \neg X.$$

In order to combine the different model and entailment constraints, Bennett [1995] uses another modal operator \square . $\square m$ is interpreted as $m = \mathcal{U}$ and $\neg \square m$ as $m \neq \mathcal{U}$. Any model constraint m can be written as $\square m$ and any entailment constraint as $\neg \square m$. If $\square X$ is true in a world w of a model \mathcal{M} , written as $(\mathcal{M}, w \models \square X)$, then X must be true in any world of \mathcal{M} . So \square is an **S5**-operator with the constraint that all worlds are mutually accessible. Therefore Bennett [1995] calls it a strong **S5**-operator. So the encoding of **RCC-8** is done in multi-modal logic with an **S4**-operator and a strong **S5**-operator.

Let Θ be a set of **RCC-8** formulas and $Reg(\Theta)$ be the set of spatial variables used in Θ , then $m(\Theta)$ specifies the modal encoding of Θ , where

$$m(\Theta) = \left(\bigwedge_{XRY \in \Theta} m_1(XRY) \right) \wedge \left(\bigwedge_{X \in Reg(\Theta)} m_2(X) \right).$$

$m_1(XRY)$ is a disjunction of the conjunctively connected model and entailment constraints for the base relations in R . m_2 results from the axioms of the **I**-operator and the additional properties of closed regions:

$$m_2(X) = \square(\mathbf{I}X \rightarrow X) \wedge \square(\mathbf{I}X \rightarrow \mathbf{I}X) \\ \wedge \square(\neg X \rightarrow \mathbf{I} \neg X) \wedge \square(X \rightarrow \neg \mathbf{I} \neg \mathbf{I} X).$$

$\square(\mathbf{I}X \rightarrow \mathbf{I}X)$, $\square(\mathbf{I} \neg X \rightarrow \neg X)$ and $\square(\neg \mathbf{I} \neg \mathbf{I} X \rightarrow X)$ are entailed by the other formulas and can be ignored. As follows from the work by Bennett [1995], Θ is consistent iff $m(\Theta)$ is satisfiable.

In order to refer to the single model and entailment constraints, we will introduce some abbreviations.

Definition 3.1 *Abbreviations for the model constraints:*

$$\begin{array}{l|l} \alpha_{xy} \equiv \square(\neg(X \wedge Y)) & A_{xy} \equiv \square(\neg(\mathbf{I}X \wedge \mathbf{I}Y)) \\ \beta_{xy} \equiv \square(X \rightarrow Y) & B_{xy} \equiv \square(X \rightarrow \mathbf{I}Y) \\ \gamma_{xy} \equiv \square(Y \rightarrow X) & C_{xy} \equiv \square(Y \rightarrow \mathbf{I}X). \end{array}$$

As the entailment constraints are negations of the model constraints, they will be abbreviated as negations of the above abbreviations. When it is obvious which atoms are used, the abbreviations will be written without indices. The abbreviations can be regarded as “propositional atoms”. Then it is possible to write the modal

encoding $m_1(XRY)$ of every relation R of **RCC-8** as a “propositional formula” of abbreviations. We will call this formula the *abbreviated form* of R . In the remainder we will use the encoding of $m_1(XRY)$ such that the abbreviated form is in conjunctive normal form (CNF).

4 Computational Properties of **RCC-8**

In this section we prove that reasoning with **RCC-8** as well as **RCC-5** is NP-hard. A similar but weaker result has been proven by Grigni et al. [1995] (see Section 8).

In this paper NP-hardness proofs for different sets \mathcal{S} of **RCC-8** relations will be carried out. All of them use a reduction of a propositional satisfiability problem to $RSAT(\mathcal{S})$ by constructing a set of spatial formulas Θ for every instance \mathcal{I} of the propositional problem, such that Θ is consistent iff \mathcal{I} is a positive instance. These satisfiability problems include **3SAT**, **NOT-ALL-EQUAL-3SAT** where every clause has at least one true and one false literal, and **ONE-IN-THREE-3SAT** where exactly one literal in every clause must be true [Garey and Johnson, 1979].

The reductions have in common that every literal as well as every literal occurrence L is reduced to two spatial variables X_L and Y_L and a relation $R = R_t \cup R_f$, where $R_t \cap R_f = \emptyset$ and $X_L R_t Y_L$. L is true iff $X_L R_t Y_L$ holds and false iff $X_L R_f Y_L$ holds. Additional “polarity” constraints have to be introduced to assure that for the spatial variables $X_{\neg L}$ and $Y_{\neg L}$, corresponding to the negation of L , $X_{\neg L} R_t Y_{\neg L}$ holds iff $X_L R_f Y_L$ holds, and *vice versa*. Using these polarity constraints, spatial variables of negative literal occurrences are connected to the spatial variables of the corresponding positive literal, and likewise for positive literal occurrences and negative literals. Further, “clause” constraints have to be added to assure that the clause requirements of the specific propositional problem are satisfied in the reduction.

Theorem 4.1 *RSAT(**RCC-5**) is NP-hard.*

Proof Sketch. Transformation of **NOT-ALL-EQUAL-3SAT** to $RSAT(\mathbf{RCC-5})$ (see also Grigni et al. [1995]). $R_t = \{\mathbf{PP}\}$ and $R_f = \{\mathbf{PP}^{-1}\}$. Polarity constraints:

$$X_L \{\mathbf{PP}, \mathbf{PP}^{-1}\} X_{\neg L}, Y_L \{\mathbf{PP}, \mathbf{PP}^{-1}\} Y_{\neg L}, \\ X_L \{\mathbf{PO}\} Y_{\neg L}, Y_L \{\mathbf{PO}\} X_{\neg L}.$$

Clause constraints for every clause $c = \{i, j, k\}$:

$$X_i \{\mathbf{PP}, \mathbf{PP}^{-1}\} X_j, X_j \{\mathbf{PP}, \mathbf{PP}^{-1}\} X_k, X_k \{\mathbf{PP}, \mathbf{PP}^{-1}\} X_i, \\ X_i \{\mathbf{PO}\} Y_k, X_j \{\mathbf{PO}\} Y_i, X_k \{\mathbf{PO}\} Y_j. \quad \blacksquare$$

Since **RCC-5** is a subset of **RCC-8**, this result can be easily applied to **RCC-8**.

Corollary 4.2 *RSAT(**RCC-8**) is NP-hard.*

In order to identify the borderline between tractability and intractability, one has to examine all subsets of **RCC-8**. We limit ourselves to subsets containing all base

relations, because these subsets still allow to express definite knowledge, if it is available. Additionally, we require the universal relation to be in the subset, so that it is possible to express complete ignorance. This reduces the search space from 2^{256} subsets to 2^{247} subsets. We proved a property that has likewise been used in identifying the maximal tractable subset of Allen's calculus [Nebel and Bürckert, 1995] that can be used to further reduce the search space.

Theorem 4.3 $\text{RSAT}(\widehat{S})$ can be polynomially reduced to $\text{RSAT}(S)$

Corollary 4.4 Let S be a subset of RCC-8.

1. $\text{RSAT}(\widehat{S}) \in \text{P}$ iff $\text{RSAT}(S) \in \text{P}$.
2. $\text{RSAT}(S)$ is NP-hard iff $\text{RSAT}(\widehat{S})$ is NP-hard.

The first statement of Corollary 4.4 can be used to increase the number of elements of tractable subsets of RCC-8 considerably. With the second statement of Corollary 4.4, NP-hardness proofs of RSAT can be used to exclude certain relations from being in any tractable subset of RCC-8. The NP-hardness proof of Theorem 4.1, e.g., only uses the relations $\{\text{PO}\}$ and $\{\text{PP}, \text{PP}^{-1}\}$. So for any subset S with the two relations contained in \widehat{S} , $\text{RSAT}(S)$ is NP-hard. The following NP-hardness results can be used to exclude more relations.

Lemma 4.5 Let S be a subset of RCC-8 containing all base relations. If any of the relations $\{\text{TPP}, \text{NTPP}, \text{TPP}^{-1}, \text{NTPP}^{-1}\}$, $\{\text{TPP}, \text{TPP}^{-1}\}$, $\{\text{NTPP}, \text{NTPP}^{-1}\}$, $\{\text{NTPP}, \text{TPP}^{-1}\}$ or $\{\text{TPP}, \text{NTPP}^{-1}\}$ is contained in \widehat{S} , then $\text{RSAT}(S)$ is NP-hard.

Proof Sketch. When $R_f \cup R_t$ is replaced by $\{\text{TPP}, \text{NTPP}, \text{TPP}^{-1}, \text{NTPP}^{-1}\}$, $\{\text{TPP}, \text{TPP}^{-1}\}$ or $\{\text{NTPP}, \text{NTPP}^{-1}\}$, the transformation of Theorem 4.1 can be applied. For $\{\text{NTPP}, \text{TPP}^{-1}\}$ and $\{\text{TPP}, \text{NTPP}^{-1}\}$ ONE-IN-THREE-3SAT has to be used. ■

By computing the closure of all sets containing the eight base relations together with one additional relation, the following lemma can be obtained.

Lemma 4.6 $\text{RSAT}(S)$ is NP-hard for any subset S of RCC-8 containing all base relations together with one of the 72 relations of the following sets:

$$\begin{aligned} \mathcal{N}_1 &= \{R \mid \{\text{PO}\} \not\subseteq R \text{ and } (\{\text{TPP}, \text{TPP}^{-1}\} \subseteq R \text{ or} \\ &\quad \{\text{NTPP}, \text{NTPP}^{-1}\} \subseteq R)\}, \\ \mathcal{N}_2 &= \{R \mid \{\text{PO}\} \not\subseteq R \text{ and } (\{\text{TPP}, \text{NTPP}^{-1}\} \subseteq R \text{ or} \\ &\quad \{\text{TPP}^{-1}, \text{NTPP}\} \subseteq R)\}. \end{aligned}$$

5 Transformation of RSAT to SAT

For transforming RSAT to propositional satisfiability (SAT) we will transform every instance Θ of RSAT to a propositional formula in CNF that is satisfiable iff Θ is

consistent. We will start from $m(\Theta)$, the modal encoding of Θ , and show that whenever $m(\Theta)$ is satisfiable it has a Kripke model of a specific type. This model will then be used to transform $m(\Theta)$ to a propositional formula.

$m(\Theta)$ is satisfiable if it is true in a world w of a Kripke model $\mathcal{M} = \langle W, \{R_1 = W \times W, R_2 \subseteq W \times W\}, \pi \rangle$, where W is a set of worlds, R_1 the accessibility relation of the \Box -operator, R_2 the accessibility relation of the **I**-operator, and π a truth function that assigns a truth value to every atom in every world. The truth conditions for $\mathcal{M}, w \Vdash m(\Theta)$ can be specified as a combination of truth conditions of the single atoms according to the form of $m(\Theta)$. In this way $\mathcal{M}, w \Vdash \mathbf{I}\varphi$, e.g., can be written as $(\forall u : wR_2u. \mathcal{M}, u \Vdash \varphi)$ and $\mathcal{M}, w \Vdash \neg \mathbf{I}\varphi$ as $(\exists u : wR_2u. \mathcal{M}, u \Vdash \neg \varphi)$. We will call this form of writing $\mathcal{M} \Vdash m(\Theta)$ the *explicit form* of $m(\Theta)$.

Before transforming $m(\Theta)$ to a propositional formula, we have to show that there is a Kripke model of $m(\Theta)$ that is polynomial in the number of spatial variables n .

Definition 5.1 Let $u \in W$ be a world of the model \mathcal{M} .

- u is a world of level 0 if vR_2u only holds for $v = u$.
- u is a world of level $l + 1$ if vR_2u holds for a world v of level l and there is no world $v \neq u$ of level $> l$.

We assume that every occurrence of a sub-formula of $m(\Theta)$ of the form $\neg \Box \varphi$, where φ contains no \Box operators, introduces a new world of level 0. As these sub-formulas correspond to entailment constraints, the number of worlds of level 0 is polynomial in n .

For every spatial variable X and every world w there might be sub-formulas that force the existence of a world u with wR_2u where X is true or where $\neg X$ is true. Because there are n different spatial variables, $2n$ different worlds u with wR_2u are sufficient for each world w .

Definition 5.2 An RCC-8-frame $\mathcal{F} = \langle W, \{R_1, R_2\} \rangle$ has the following properties:

1. W contains only worlds of level 0, 1 and 2.
2. For every world w of level k ($k = 0, 1$) there are exactly $2n$ worlds u of level $k + 1$ with wR_2u .
3. For every world w of level k there is exactly one world u for every level $0 \leq l \leq k$ with uR_2w .

An RCC-8-model is based on an RCC-8-frame.

Lemma 5.3 $m(\Theta)$ is satisfiable iff $\mathcal{M}, w \Vdash m(\Theta)$ for an RCC-8-model \mathcal{M} with polynomially many worlds.

Now it is possible to transform the explicit form of $m(\Theta)$ to a propositional formula $p(m(\Theta))$ in CNF such that $p(m(\Theta))$ is satisfiable iff $m(\Theta)$ is satisfiable in a polynomial RCC-8-model \mathcal{M} . For this purpose, propositional atoms X_w are introduced which stand for the truth of atom X in world w of the RCC-8-model \mathcal{M} . Further, universally quantified truth conditions are transformed into conjunctions and existentially quantified

truth conditions are transformed into truth conditions on particular worlds, which can be determined using the structure of the RCC-8-frame and the modal formula.

Theorem 5.4 RSAT(RCC-8) can be polynomially reduced to SAT.

With Corollary 4.2 this leads to the following theorem.

Theorem 5.5 RSAT(RCC-8) is NP-complete.

6 Tractable Subsets of RCC-8

In order to identify a tractable subset of RCC-8, we analyze which relations can be expressed as propositional Horn formulas, as satisfiability of Horn formulas (HORN-SAT) is tractable.

Proposition 6.1 Applying the transformation p to the model and entailment constraints, to the axioms for \mathbf{I} , and to the properties of closed regions leads to Horn formulas.

Since the model constraints α and A are transformed to indefinite Horn formulas, the transformation of any disjunction of these constraints with any other constraint is also Horn. All relations with an abbreviated form using only abbreviations or disjunctions of abbreviations transformable to Horn formulas can be transformed to Horn formulas. In this way 64 different relations can be transformed to Horn formulas. We call the subset of RCC-8 containing these relations \mathcal{H}_8 .

Theorem 6.2 RSAT(\mathcal{H}_8) can be polynomially reduced to HORNSAT and therefore RSAT($\widehat{\mathcal{H}}_8$) \in P.

Theorem 6.3 $\widehat{\mathcal{H}}_8$ contains the following 148 relations:

$$\widehat{\mathcal{H}}_8 = \text{RCC-8} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3)$$

with \mathcal{N}_1 and \mathcal{N}_2 as defined in Lemma 4.6 and

$$\mathcal{N}_3 = \{R \mid \{\text{EQ}\} \subseteq R \text{ and } (\{\text{NTPP}\} \subseteq R, \{\text{TPP}\} \not\subseteq R) \text{ or } (\{\text{NTPP}^{-1}\} \subseteq R, \{\text{TPP}^{-1}\} \not\subseteq R)\}.$$

For proving that $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8, we have to show that no relation of \mathcal{N}_3 can be added to $\widehat{\mathcal{H}}_8$ without making RSAT intractable.

Lemma 6.4 The closure of every set containing $\widehat{\mathcal{H}}_8$ and one relation of \mathcal{N}_3 contains the relation $\{\text{EQ}, \text{NTPP}\}$.

Therefore it is sufficient to prove NP-hardness of RSAT($\widehat{\mathcal{H}}_8 \cup \{\text{EQ}, \text{NTPP}\}$) for showing that $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8.

Lemma 6.5 RSAT($\widehat{\mathcal{H}}_8 \cup \{\text{EQ}, \text{NTPP}\}$) is NP-hard.

Proof Sketch. Transformation of 3SAT to RSAT($\widehat{\mathcal{H}}_8 \cup \{\text{EQ}, \text{NTPP}\}$). $R_t = \{\text{NTPP}\}$ and $R_f = \{\text{EQ}\}$. Polarity constraints:

$$X_L\{\text{EC}, \text{NTPP}\}X_{-L}, Y_L\{\text{TPP}\}Y_{-L}, \\ X_L\{\text{TPP}, \text{NTPP}\}Y_{-L}, Y_L\{\text{EC}, \text{TPP}\}X_{-L},$$

Clause constraints for each clause $c = \{i, j, k\}$: $Y_i\{\text{NTPP}^{-1}\}X_j, Y_j\{\text{NTPP}^{-1}\}X_k, Y_k\{\text{NTPP}^{-1}\}X_i$. ■

Theorem 6.6 $\widehat{\mathcal{H}}_8$ is a maximal tractable subset of RCC-8.

It has to be noted that there might be other maximal tractable subsets of RCC-8 that contain all base relations.

As $\widehat{\mathcal{H}}_8$ is tractable, the intersection of RCC-5 and $\widehat{\mathcal{H}}_8$ is also tractable. We will call this subset $\widehat{\mathcal{H}}_5$.

Theorem 6.7 $\widehat{\mathcal{H}}_5$ is the only maximal tractable subset of RCC-5 containing all base relations.

7 Applicability of Path-Consistency

As shown in the previous section, RSAT($\widehat{\mathcal{H}}_8$) can be solved in polynomial time by first transforming a set of $\widehat{\mathcal{H}}_8$ formulas to a propositional Horn formula and then deciding it in time linear in the number of literals. This way of solving RSAT does not appear to be very efficient.

As RSAT is a Constraint Satisfaction Problem (CSP) [Mackworth, 1987], where variables are nodes and relations are arcs of the constraint graph, algorithms for deciding consistency of a CSP can also be used. A correct but in general not complete $O(n^3)$ algorithm for deciding inconsistency of a CSP is the path-consistency method [Mackworth, 1977] that makes a CSP path-consistent by successively removing relations from all edges using $\forall k : R_{ij} \leftarrow R_{ij} \cap (R_{ik} \circ R_{kj})$, where i, j, k are nodes and R_{ij} is the relation between i and j . If the empty relation occurs while performing this operation, the CSP is not path-consistent, otherwise it is.

In this section we will prove that path-consistency decides RSAT($\widehat{\mathcal{H}}_8$). This is done by showing that the path-consistency method finds an inconsistency whenever positive unit resolution (PUR) resolves the empty clause from the corresponding propositional formula. As PUR is refutation-complete for Horn formulas, it follows that the path-consistency method decides RSAT(\mathcal{H}_8). The only way to get the empty clause is resolving a positive and a negative unit clause of the same variable. Since the Horn formulas that are used contain only a few types of different clauses, there are only a few ways to resolve unit clauses using PUR.

Definition 7.1

- R_K denotes the set of relations of \mathcal{H}_8 with the conjunct K appearing in their abbreviated form.
- R_{K_1, K_2} denotes $R_{K_1} \cup R_{K_2}$.
- R_Γ denotes $R_\gamma \cup R_{\alpha\vee\gamma} \cup R_{A\vee\gamma} \cup R_C \cup R_{\alpha\vee C} \cup R_{A\vee C}$.
- An R_K -chain $R_K^*(X, Y)$ is a path from region X to region Y , where all relations between successive regions are from R_K .

Lemma 7.2 Let Θ be a set of \mathcal{H}_8 -formulas.

- A positive unit clause $\{X_w\}$ can only be resolved from $\{Y_w\}$ and a clause resulting from $XR_\Gamma Y \in \Theta$. When such a resolution is possible, $XR_{\alpha, A}Y$ cannot hold, so $XR_{\gamma, C}Y$ must hold.

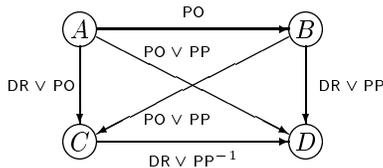
- A negative unit clause $\{\neg X_w\}$ can only be resolved from $\{Y_w\}$ and a clause resulting from $XR_{\alpha,AY} \in \Theta$.

Lemma 7.3 *If the positive unit clause $\{X_w\}$ can be resolved with PUR using an R_Γ -chain from X to Y , the path-consistency method results in $XR_{\gamma,CY}$.*

Using Lemma 7.3, it can be proven that the path-consistency method decides $\text{RSAT}(\mathcal{H}_8)$. Using the proof of Theorem 4.3, it is possible to express every relation of $\widehat{\mathcal{H}}_8$ as a Horn formula. Then the following theorem can be proven.

Theorem 7.4 *The path-consistency method decides $\text{RSAT}(\widehat{\mathcal{H}}_8)$.*

Another interesting question is whether the path-consistency method computes minimal possible relations on $\widehat{\mathcal{H}}_8$. As the following example demonstrates, this is not the case even for the set $\widehat{\mathcal{H}}_5$. $A\{\text{PP}\}D$ is impossible although the constraint graph is path-consistent:



8 Related Work

Nebel [1995] showed that $\text{RSAT}(\widehat{\mathcal{B}})$ can be decided in polynomial time, where \mathcal{B} is the set of the RCC-8 base relations. Since $\mathcal{B} \subseteq \mathcal{H}_8$, our result is more general. Further, $\widehat{\mathcal{B}}$ contains only 38 relations, whereas $\widehat{\mathcal{H}}_8$ contains 148 relations, i.e. about 58% of RCC-8.

Grigni et al. [1995] proved NP-hardness of problems similar to RSAT. For instance, they considered the problem of *relational consistency*, which means that there exists a path-consistent refinement of all relations to base relations, and showed that this problem is NP-hard. While our NP-hardness result on RSAT implies their result, the converse implication follows only using the above cited result by Nebel [1995].

In addition to this syntactic notion of consistency, Grigni et al. [1995] considered a semantic notion of consistency, namely, the *realizability* of spatial variables as internally connected planar regions. This notion is much more constraining than our notion of consistency. It is also computationally much harder.

9 Summary

We analyzed the computational properties of the qualitative spatial calculus RCC-8 and identified the boundary between polynomial and NP-hard fragments. Using a modification of Bennett's encoding of RCC-8 in a multimodal propositional logic, we transformed the RCC-8 consistency problem to a problem in propositional logic

and isolated the relations that are representable as Horn clauses. As it turns out, the fragment identified in this way is also a maximal fragment that contains all base relations and is still computationally tractable. Further, we showed that for this fragment path-consistency is sufficient for deciding consistency.

As in the case of qualitative temporal reasoning, our result allows to check whether the relations that are used in an application allow for a polynomial reasoning algorithm. Further, if the application requires an expressive power beyond the polynomial fragment, it can be used to speed up backtracking algorithms. Assuming that the relations are uniformly distributed, the average branching factor is reduced from 4.0 to 1.4375 using $\widehat{\mathcal{H}}_8$ instead of \mathcal{B} to split the relations (see also [Nebel, 1997]).

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