A Canonical Model of the Region Connection Calculus

Jochen Renz

Institut für Informationssysteme
Technische Universität Wien
A-1040 Wien, Austria
renz@dbai.tuwien.ac.at

ABSTRACT. *Although the computational properties of the Region Connection Calculus* RCC-8 *are well studied, reasoning with* RCC-8 *entails several representational problems. This includes the problem of representing arbitrary spatial regions in a computational framework, leading to the problem of generating a realization of a consistent set of* RCC-8 *constraints. A further problem is that* RCC-8 *performs reasoning about topological space, which does not have a particular dimension. Most applications of spatial reasoning, however, deal with two- or three-dimensional space. Therefore, a consistent set of* RCC-8 *constraints might not be realizable within the desired dimension. In this paper we address these problems and develop a canonical model of* RCC-8 *which allows a simple representation of regions with respect to a set of* RCC-8 *constraints, and, further, enables us to generate realizations in any dimension* $d \geq 1$*. For three- and higher-dimensional space this can also be done for internally connected regions.*

KEYWORDS: *qualitative spatial representation, RCC-8, topological relations, spatial regions, modal logic.*

1. Introduction

The Region Connection Calculus (RCC) [RAN 92b] is a topological approach to qualitative spatial representation and reasoning [COH 97] where spatial regions are regular subsets of a topological space. Of particular interest for application purposes is RCC-8, a constraint language that uses eight mutually exhaustive and pairwise disjoint base relations definable in the RCC-theory. The computational properties of RCC-8 have been studied thoroughly [NEB 95, REN 99b, REN 99a] and efficient reasoning mechanisms were identified [REN 01].

Despite these advantages, there are still several problems with representing spatial regions within RCC-8. As the calculus is based on topology, spatial regions might be arbitrary subsets of a topological space which are not necessarily analytically describ-

Journal of Applied Non-Classical Logics.Volume 12 – n° 3-4/2002, pages 469 to 494.

able; therefore, it appears to be difficult to represent spatial regions in a computational framework.

Another representational drawback of using RCC-8 is that a topological space does not have a particular dimension, whereas most applications of qualitative spatial reasoning deal only with two- or three-dimensional space. It might, thus, be possible that a set of RCC-8 constraints is consistent but not realizable within a particular dimension. Lemon [LEM 96] gave an example of a set of RCC-8 constraints which is realizable in three dimensional space but not in two dimensional space if regions must be internally connected. Lemon used this result to argue that spatial logics like RCC are not an adequate formalism for representing space.

A further problem, which also depends on the ability to represent spatial regions, is finding a realization of a consistent and realizable set of RCC-8 constraints in a particular dimension, instead of just knowing whether the set is realizable or not.

In this paper, we will refer to these representational topics. In order to represent arbitrary spatial regions, it is necessary to have a *canonical model* of RCC-8, i.e., a structure that allows to model any consistent sentence of the calculus. Topological spaces are of course a canonical model, but, as described above, this does not seem to be very useful for representing regions. Therefore, we will present a new canonical model of RCC-8 that permits a simple representation of spatial regions by reducing them to their necessary topological features with respect to their spatial relations. Based on this model, we will prove that for any consistent set of RCC-8 constraints there are realizations in any dimension $d \geq 1$ when regions are not forced to be internally connected. This is still true even when regions are constrained to be sets of polytopes. Actually, internal connectedness of regions is not at all forced in the RCC-theory, so RCC can still be seen as an adequate representation formalism of space. We will also argue that forcing internal connectedness of all regions is too restrictive when dealing with spatial regions. Nevertheless, we will prove that in three- and higher dimensional space every consistent set of RCC-8 constraints can always be realized with internally connected regions. Using the new canonical model for representing spatial regions, it becomes possible to determine realizations of consistent sets of RCC-8 constraints. We will give algorithms for generating realizations of both internally connected and disconnected regions.

The remainder of the paper is structured as follows: In Section 2 we introduce RCC-8 and some basic topological notions. Section 3 sketches the modal encoding of RCC-8 and presents the new canonical model of RCC-8. In Section 4 we give a topological interpretation of this model which is used in Section 5 to prove the results about realizations in particular dimensions. Section 6 describes how models of sets of spatial relations can be determined and how realizations can be generated. In Section 7 we will discuss our results.

2. Qualitative Spatial Representation with the Region Connection Calculus

The Region Connection Calculus (RCC) developed by Randell, Cui, and Cohn [RAN 92b] is a topological approach to spatial representation and reasoning where *spatial regions* are non-empty regular subsets of some topological space $\mathcal{U}$. Spatial regions do not have to be *internally connected*, i.e., they might consist of (multiple) disconnected pieces. Since all spatial regions are regular subsets of the same topological space $\mathcal{U}$, all spatial regions have the same dimension, namely, the dimension of $\mathcal{U}$ (provided that $\mathcal{U}$ has a particular dimension).

RCC is based on a single primitive relation between spatial regions, the "connectedness" relation C. The intended topological interpretation of $\mathsf{C}(a, b)$, where a and b are spatial regions, is that a and b are connected if and only if their topological closures share a common point. With this interpretation it is not distinguished between open, semi-open, and closed regions which is different from previous approaches by Randell and Cohn [RAN 92a, RAN 89] and Clarke [CLA 85, CLA 81]. Using the connectedness relation C, a large number of different relations can be defined (cf. Gotts [GOT 94, GOT 96b]. Of particular interest are those relations that form a set of jointly exhaustive and pairwise disjoint relations, which are also denoted *base relations*. Base relations have the property that exactly one of them holds between any two spatial regions. If these relations are closed under composition they generate a relation algebra [LAD 94], thus, reasoning about these relations can be done using constraint satisfaction methods (cf. [MAC 77, MON 74, BEE 92]). Randell et al. [RAN 92b] suggested such a set of eight base relations, later denoted as RCC-8: DC (*DisConnected*), EC (*Externally Connected*), PO (*Partial Overlap*), EQ (*EQual*), TPP (*Tangential Proper Part*), NTPP (*Non-Tangential Proper Part*), and their converses TPP^{-1} and NTPP^{-1}. This set of relations is interesting for a number of reasons. It is the smallest set of base relations which allows topological distinctions rather than just mereological (being expressible by using the part-whole relationship) and which forms a relation algebra. Most other relations definable in the RCC theory are refinements of these relations. Furthermore, the semantics of these relations can be described by using propositional logics rather than first-order logics [BEN 94, BEN 96b], a property which allows us to prove decidability.

2.1. *The Region Connection Calculus RCC-8*

The Region Connection Calculus RCC-8 is the constraint language formed by the eight jointly exhaustive and pairwise disjoint base relations DC, EC, PO, EQ, TPP, NTPP, TPP^{-1}, and NTPP^{-1} and by all possible unions of the base relations. Unions of possible base relations are used to represent indefinite knowledge. Since the base relations are pairwise disjoint, this results in $2^8 = 256$ different RCC-8 relations altogether (including the empty relation and the universal relation). In some papers the set of base relations is denoted as RCC-8 while the set of all possible unions of base relations is denoted as 2^{RCC8}. We will, however, use RCC-8 to refer to the

RCC-8 *Relation*	*Topological Constraints*
DC(X, Y)	$X \cap Y = \emptyset$
EC(X, Y)	$i(X) \cap i(Y) = \emptyset$, $X \cap Y \neq \emptyset$
PO(X, Y)	$i(X) \cap i(Y) \neq \emptyset$, $X \not\subseteq Y$, $Y \not\subseteq X$
TPP(X, Y)	$X \subset Y$, $X \not\subseteq i(Y)$
TPP^{-1}(X, Y)	$Y \subset X$, $Y \not\subseteq i(X)$
NTPP(X, Y)	$X \subset i(Y)$
NTPP^{-1}(X, Y)	$Y \subset i(X)$
EQ(X, Y)	$X = Y$

Table 1. *Topological interpretation of the eight base relations of* RCC-8*. All spatial regions are regular closed, i.e.,* $X = c(i(X))$ *and* $Y = c(i(Y))$*.* $i(\cdot)$ *specifies the topological interior of a spatial region,* $c(\cdot)$ *the topological closure.*

set of all possible disjunctions of the base relations and $\mathcal{B}$ to refer to the set of base relations. Analogous to the general RCC-theory, spatial regions in RCC-8 are non-empty regular subsets of some topological space that do not have to be internally connected, and do not have a particular dimension. Without loss of generality (due to the intended interpretation of the C relation) we require spatial regions to be regular *closed* subsets of a topological space.

The RCC-8 relations can be given a straightforward topological interpretation in terms of point-set topology (see Table 1), which is almost the same as for the topological relations given by Egenhofer [EGE 91] (though Egenhofer places stronger constraints on the domain of regions, e.g., regions must be one-piece and are not allowed to have holes). Examples for the RCC-8 base relations are given in Figure 1.

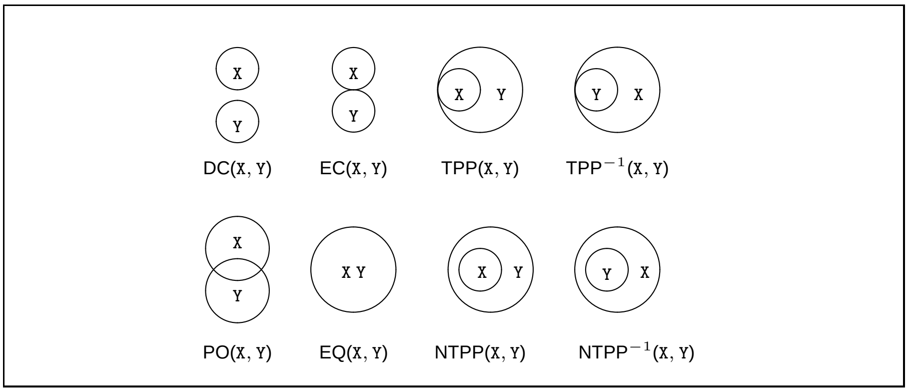

Figure 1. *Two-dimensional examples for the eight base relations of* RCC-8*.*

A spatial configuration can be described by specifying a finite set Θ of RCC-8 constraints, written as xRy or $R(x, y)$, where R is an RCC-8 relation and x, y are *spatial variables* over the infinite domain of all possible spatial regions. An important reasoning problem is deciding *consistency* of Θ, i.e., deciding whether there is an assignment of non-empty, regular closed regions of some topological space to variables of Θ in a way that all constraints are satisfied. Computational properties of reasoning with RCC-8 were studied in [NEB 95, REN 99b, REN 99a].

In this paper we deal with representational properties of RCC-8 for which it is necessary to go further into topology. In the next subsection we define some common topological terms and concepts which are used in the remainder of the paper.

REMARK 1. — Throughout this work we will use the following convention for referring to spatial regions, spatial variables, and propositional atoms corresponding to spatial regions or spatial variables:

– Spatial variables are written as x, y, z.

– Spatial regions are written as $\mathtt{X}, \mathtt{Y}, \mathtt{Z}$.

– Propositional atoms corresponding to spatial regions or spatial variables are written as $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$.

If the same letter is used in different fonts in the same context, it represents the same region. For instance, $\mathtt{X}$ is a possible instance of x, $\mathtt{Y}$ a possible instance of y, and X is the propositional atom corresponding to x or to $\mathtt{X}$.

2.2. *Topological Background*

In this subsection we introduce and define the topological concepts that are used in this paper. This includes the notion of a topological space, different kinds of regions such as open, closed, regular open, and regular closed regions, the notion of interior, exterior and boundary of a region, as well as neighborhoods, neighborhood systems, and points with different properties. These concepts are very basic and can be found in this or in a similar form in any book on general topology or point-set topology (e.g., [MUN 74, BAU 91]). We start with the formal definition of a topology and a topological space:

DEFINITION 2 (TOPOLOGY, TOPOLOGICAL SPACE). — *Let $\mathcal{U}$ be a non-empty set, the* universe. *A* topology *on $\mathcal{U}$ is a family T of subsets of $\mathcal{U}$ that satisfies the following axioms:*

1) $\mathcal{U}$ and $\emptyset$ belong to T,

2) the union of any number of sets in T belongs to T,

3) the intersection of any two sets of T belongs to T.

A topological space *is a pair $\langle \mathcal{U}, T \rangle$. The members of T are called* open *sets.*

In a topological space $\langle \mathcal{U}, T \rangle$, a subset $\mathtt{X}$ of $\mathcal{U}$ is called a *closed* set if its complement $\mathtt{X}^c$ is an open set, i.e., if $\mathtt{X}^c$ belongs to T. By applying the DeMorgan laws, we obtain the following dual properties of closed sets:

1) $\mathcal{U}$ and $\emptyset$ are closed sets,

2) the intersection of any number of closed sets is a closed set,

3) the union of any two closed sets is a closed set.

If the particular topology T on a set $\mathcal{U}$ is clear or not important, then $\mathcal{U}$ can also denote the topological space.

Closely related to the concept of an open set is that of a neighborhood.

DEFINITION 3 (NEIGHBORHOOD, NEIGHBORHOOD SYSTEM). — *Let $\mathcal{U}$ be a topological space and $p \in \mathcal{U}$ be a point in $\mathcal{U}$.*

– $N \subset \mathcal{U}$ is said to be a neighborhood *of p if there is an open subset $O \subset \mathcal{U}$ such that $p \in O \subset N$.*

– The family of all neighborhoods of p is called the neighborhood system *of p, denoted as $\mathcal{N}_p$.*

A neighborhood system $\mathcal{N}_p$ has the property that every finite intersection of members of $\mathcal{N}_p$ belongs to $\mathcal{N}_p$. Based on the notion of neighborhood it is possible to define some important notions.

DEFINITION 4 (INTERIOR, EXTERIOR, BOUNDARY, CLOSURE). — *Let $\mathcal{U}$ be a topological space, $\mathtt{X} \subset \mathcal{U}$ be a subset of $\mathcal{U}$ and $p \in \mathcal{U}$ be a point in $\mathcal{U}$.*

– p is said to be an interior point *of $\mathtt{X}$ if there is a neighborhood N of p contained in $\mathtt{X}$. The set of all interior points of $\mathtt{X}$ is called the* interior *of $\mathtt{X}$, denoted $i(\mathtt{X})$.*

– p is said to be an exterior point *of $\mathtt{X}$ if there is a neighborhood N of p that contains no point of $\mathtt{X}$. The set of all exterior points of $\mathtt{X}$ is called the* exterior *of $\mathtt{X}$, denoted $e(\mathtt{X})$.*

– p is said to be a boundary point *of $\mathtt{X}$ if every neighborhood N of p contains at least one point in $\mathtt{X}$ and one point not in $\mathtt{X}$. The set of all boundary points of $\mathtt{X}$ is called the* boundary *of $\mathtt{X}$, denoted $b(\mathtt{X})$.*

– The closure *of $\mathtt{X}$, denoted $c(\mathtt{X})$, is the smallest closed set which contains $\mathtt{X}$.*

The closure of a set is the union of its interior and its boundary. Every open set is its own interior, every closed set is its own closure.

DEFINITION 5 (REGULAR OPEN, REGULAR CLOSED). — *Let $\mathtt{X}$ be a subset of a topological space $\mathcal{U}$.*

– $\mathtt{X}$ is said to be regular open *if $\mathtt{X}$ is the interior of its closure, i.e., $\mathtt{X} = i(c(\mathtt{X}))$.*

– $\mathtt{X}$ is said to be regular closed *if $\mathtt{X}$ is the closure of its interior, i.e., $\mathtt{X} = c(i(\mathtt{X}))$.*

Topological spaces can be categorized according to the degree that points or closed sets can be separated by open sets. Different possibilities are given by the *separation axioms* T_i. A topological space $\mathcal{U}$ that satisfies axiom T_i is called a T_i space. Three of these separation axioms which are important for this work are the following:

T_1 : Given any two distinct points $p, q \in \mathcal{U}$, each point belongs to an open set which does not contain the other point.

T_2 : Given any two distinct points $p, q \in \mathcal{U}$, there exist disjoint open sets $O_p, O_q \subseteq \mathcal{U}$ containing p and q respectively.

T_3 : If X is a closed subset of $\mathcal{U}$ and p is a point not in X, there exist disjoint open sets $O_X, O_p \subseteq \mathcal{U}$ containing X and p respectively.

A *connected space* $\mathcal{U}$ is a topological space which cannot be partitioned into two disjoint open sets, i.e., if $\mathcal{U}$ is the union of two non-empty subsets A and B, then either the closure of A intersected with B or the closure of B intersected with A is non-empty. A topological space is *regular*, if it satisfies axioms T_2 and T_3. Two subsets of a topological space are called *separated* if the closure of one subset is disjoint from the closure of the other subset. A subset of a topological space is *connected* (or *internally connected* as it is called in the RCC community) if it cannot be written as a union of two separated sets.

It is possible to use any topological space which is a model for the RCC axioms as specified in [RAN 92b]. Gotts [GOT 96a] has shown that every regular connected topological space is a model for the RCC axioms (see also Section 7). So, whenever we refer to a topological space in the remainder of the paper, we mean a regular connected topological space.

3. Modal Encoding & Canonical Models

After making a brief introduction to modal logic, we will introduce the modal encoding of RCC-8 and a canonical model for this encoding.

3.1. *Modal Logic & Kripke Semantics*

Propositional modal logic [FIT 93, CHE 80] extends classical propositional logic by additional unary *modal operators* $\Box_i$. A common semantic interpretation of modal formulas is the *Kripke semantics* which is based on a *Kripke frame* $\mathcal{F} = \langle W, \mathcal{R} \rangle$ consisting of a set of *worlds* W and a set $\mathcal{R}$ of *accessibility relations* between the worlds, where $R \subseteq W \times W$ for every accessibility relation $R \in \mathcal{R}$. A different accessibility relation $R_{\Box_i} \in \mathcal{R}$ is assigned to every modal operator $\Box_i$. If $u, v \in W$, $R \in \mathcal{R}$, and uRv holds, we say that v is *R-accessible* from u or v is an *R-successor* of u.

A *Kripke model* $\mathcal{M} = \langle W, \mathcal{R}, \pi \rangle$ uses an additional valuation π that assigns each world and each propositional atom a truth value $\{true, false\}$. Using a Kripke model, a modal formula can be interpreted with respect to the set of worlds, the accessibility relations, and the valuation. For example, a propositional atom a holds in a world w of the Kripke model $\mathcal{M}$ (written as $\mathcal{M}, w \Vdash a$) if and only if $\pi(w, a) = true$. An

RCC-8 *constraint*	*Model Constraints*	*Entailment Constraints*
$\mathsf{DC}(x,y)$	$\neg(\mathsf{X} \wedge \mathsf{Y})$	$\neg\mathsf{X}, \neg\mathsf{Y}$
$\mathsf{EC}(x,y)$	$\neg(\mathbf{I}\mathsf{X} \wedge \mathbf{I}\mathsf{Y})$	$\neg(\mathsf{X} \wedge \mathsf{Y}), \neg\mathsf{X}, \neg\mathsf{Y}$
$\mathsf{PO}(x,y)$	—	$\neg(\mathbf{I}\mathsf{X} \wedge \mathbf{I}\mathsf{Y}), \mathsf{X} \rightarrow \mathsf{Y}$, $\mathsf{Y} \rightarrow \mathsf{X}, \neg\mathsf{X}, \neg\mathsf{Y}$
$\mathsf{TPP}(x,y)$	$\mathsf{X} \rightarrow \mathsf{Y}$	$\mathsf{X} \rightarrow \mathbf{I}\mathsf{Y}, \mathsf{Y} \rightarrow \mathsf{X}, \neg\mathsf{X}, \neg\mathsf{Y}$
$\mathsf{TPP}^{-1}(x,y)$	$\mathsf{Y} \rightarrow \mathsf{X}$	$\mathsf{Y} \rightarrow \mathbf{I}\mathsf{X}, \mathsf{X} \rightarrow \mathsf{Y}, \neg\mathsf{X}, \neg\mathsf{Y}$
$\mathsf{NTPP}(x,y)$	$\mathsf{X} \rightarrow \mathbf{I}\mathsf{Y}$	$\mathsf{Y} \rightarrow \mathsf{X}, \neg\mathsf{X}, \neg\mathsf{Y}$
$\mathsf{NTPP}^{-1}(x,y)$	$\mathsf{Y} \rightarrow \mathbf{I}\mathsf{X}$	$\mathsf{X} \rightarrow \mathsf{Y}, \neg\mathsf{X}, \neg\mathsf{Y}$
$\mathsf{EQ}(x,y)$	$\mathsf{X} \rightarrow \mathsf{Y}, \mathsf{Y} \rightarrow \mathsf{X}$	$\neg\mathsf{X}, \neg\mathsf{Y}$

Table 2. *Encoding of the* RCC-8 *base relations in modal logic.*

arbitrary modal formula is interpreted according to its inductive structure. A modal formula $\Box_i\varphi$, e.g., holds in a world w of the Kripke model $\mathcal{M}$, i.e., $\mathcal{M}, w \Vdash \Box_i\varphi$, if and only if φ holds in all $R_{\Box_i}$-successors of w. $\mathcal{M}, w \Vdash \neg\Box_i\varphi$ if and only if there is an $R_{\Box_i}$-successor of w where φ does not hold. The operators $\neg$, $\wedge$ and $\vee$ are interpreted in the same way as in classical propositional logic.

Different modal operators can be distinguished according to their different accessibility relations. In this paper we are using a so-called S4-operator and an S5-operator. The accessibility relation of an S4-operator is reflexive and transitive, the accessibility relation of an S5-operator is reflexive, transitive, and Euclidean (i.e., if uRv and uRw holds, then vRw holds as well).

3.2. *Modal Encoding of RCC8*

The encoding of RCC-8 in propositional modal logic was introduced by Bennett [BEN 96b] and extended in [REN 99b]. In both cases the encoding is restricted to regular closed regions. The encoding is based on a set of *model* and *entailment constraints* for each base relation, where model constraints must be true and entailment constraints must not be true. Bennett encoded these constraints in modal logic by transforming every spatial variable to a propositional atom and introducing an S4-operator $\mathbf{I}$ which he interpreted as an interior operator [BEN 96b]. In order to distinguish between spatial variables and the corresponding propositional atoms we will write propositional atoms as X, Y. Table 2 displays the constraints for the eight base relations. In order to combine them to a single modal formula, Bennett introduced an S5-operator[1] $\Box$, where $\Box\varphi$ is written for every model constraint φ and $\neg\Box\psi$ for every entailment constraint ψ [BEN 96b]. All constraints of a single base relation are then combined conjunctively to a single modal formula. In order to represent unions of base relations, the modal formulas of the corresponding base relations are combined

1. Bennett called this a *strong* S5-operator as all worlds are $R_\Box$-accessible from each other, i.e., $R_\Box = W \times W$.

disjunctively. In this way every RCC-8 constraint $R(x, y)$ can be mapped to a modal formula $m_1(R(x, y))$. Additional constraints $m_2(x)$ are necessary to guarantee that only regular closed regions are used [REN 99b]: every region must be equivalent to the closure of its interior, and the complement of a region must be equivalent to its interior.

$$m_2(x) = \Box(\mathsf{X} \leftrightarrow \neg\mathbf{I}\neg\mathbf{I}\mathsf{X}) \wedge \Box(\neg\mathsf{X} \leftrightarrow \mathbf{I}\neg\mathsf{X}).$$

So, any set of RCC-8 constraints Θ can be written as a single modal formula $m(\Theta)$

$$m(\Theta) = \bigwedge_{R(X,Y)\in\Theta} m_1(R(x, y)) \quad \wedge \bigwedge_{X\in Reg(\Theta)} m_2(x),$$

where $Reg(\Theta)$ is the set of spatial variables of Θ.

3.3. *A Canonical Model of RCC-8*

The modal encoding of RCC-8 can be interpreted by Kripke models. As the modal encoding of RCC-8 is equivalent to a set of RCC-8 constraints [BEN 96b, NUT 99], a canonical model of RCC-8 is a structure that allows a Kripke model for the modal encoding of any consistent set of RCC-8 constraints Θ. In order to obtain a canonical model, we distinguish different levels of worlds of W [REN 99b]. A world w is of *level 0* if there is no world $v \neq w$ with $vR_{\mathbf{I}}w$. A world w is of *level l* if there is a world v of level $l-1$ with $vR_{\mathbf{I}}w$ and there is no world $u \neq w$ of a level higher than $l-1$ with $uR_{\mathbf{I}}w$. Based on this hierarchy of worlds, we will define the canonical model of RCC-8.

DEFINITION 6 (RCC-8-STRUCTURE, RCC-8-CLUSTER, RCC-8-MODEL). — *An* RCC-8-structure of size n $\mathcal{S}^n_{RCC8} = \langle W, \{R_\Box, R_{\mathbf{I}}\}, \pi\rangle$ *has the following properties:*

1) W *contains only worlds of level* 0 *and* 1.

2) For every world u *of level* 0 *there are exactly* $2n$ *worlds* v *of level* 1 *with* $uR_{\mathbf{I}}v$. *These* $2n+1$ *worlds form an* RCC-8-cluster *of size* $2n+1$ *(cf. Figure 2).*

3) For every world v *of level* 1 *there is exactly one world* u *of level* 0 *with* $uR_{\mathbf{I}}v$.

4) For all worlds $w, v \in W$: $wR_{\mathbf{I}}w$ *and* $wR_\Box v$.

$\mathcal{S}^n_{RCC8}$ *contains* RCC-8-*clusters of size* $2n+1$ *with all possible valuations*[2] *with respect to* $R_{\mathbf{I}}$. *The* RCC-8-structure $\mathcal{S}_{RCC8} = \bigcup_{n\geq 1} \mathcal{S}^n_{RCC8}$ *is the union of all* RCC-8-*structures of size* n. *A set of* RCC-8-*clusters* $\mathcal{M} = \langle W, \{R_\Box, R_{\mathbf{I}}\}, \pi\rangle \subset \mathcal{S}^n_{RCC8}$ *is an* RCC-8-model *of* $m(\Theta)$ *if* $\mathcal{M}, w \Vdash m(\Theta)$ *for a world* $w \in W$ *and* n *is the number of variables in* Θ. *In a* polynomial RCC-8-model *the number of worlds is polynomially bounded by the number of regions* n.

In [REN 99b] it was proven that if $m(\Theta)$ is satisfiable, there is a polynomial RCC-8-model $\mathcal{M}$ with $\mathcal{M}, w \Vdash m(\Theta)$ that uses $O(n^2)$ different worlds of level 0 –

2. As the number of spatial variables is countable, the number of RCC-8-clusters with different valuations is also countable.

one world of level 0 for every entailment constraint. So the RCC-8-structure $\mathcal{S}_{RCC8}$ is a canonical model[3] of the modal encoding of RCC-8. In order to obtain a "topological" canonical model for the topological calculus RCC-8, we give in the next section a topological interpretation of RCC-8-models.

4. Topological Interpretation of the Canonical Model

The modal encoding of RCC-8 was obtained by introducing a modal operator $\mathbf{I}$ corresponding to the topological interior operator and transferring the topological properties and axioms to modal logic. In this section we present a way of topologically interpreting RCC-8-models such that all parts of the models can be interpreted consistently on a topological level.

Because $\mathbf{I}$ is an S4-operator and because of the additional constraints $m_2(x)$, exactly one of the following formulas is true for every world w of $\mathcal{M}$ and every propositional atom X (see Figure 2).

1) $\mathcal{M}, w \Vdash \mathbf{I}\mathsf{X}$
2) $\mathcal{M}, w \Vdash \mathbf{I}\neg\mathsf{X}$
3) $\mathcal{M}, w \Vdash \mathsf{X} \wedge \neg\mathbf{I}\mathsf{X}$

Consider a particular world w. Then the set of all spatial variables can be divided into three disjoint sets of spatial variables according to which of the three possible formulas is true in w (see Figure 2). Let $\mathcal{I}_w$, $\mathcal{E}_w$, and $\mathcal{B}_w$ be the sets of spatial variables where

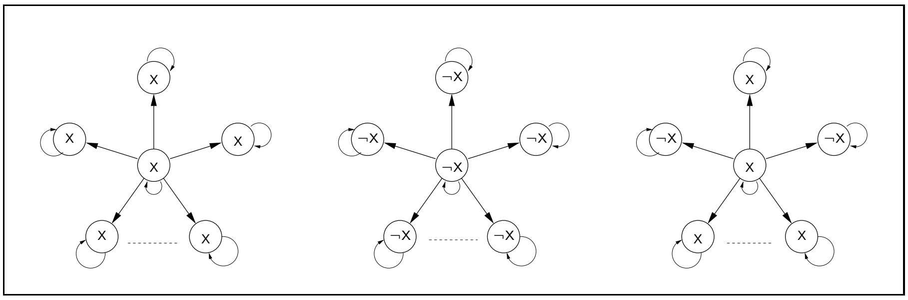

Figure 2. *Three possible* RCC-8-*clusters of the* RCC-8-*structure* $\mathcal{S}_{RCC8}$.

the first, the second, and the third formula is true in w, respectively, i.e., $\mathcal{M}, w \Vdash \mathbf{I}\mathsf{X} \wedge \mathbf{I}\neg\mathsf{Y} \wedge (\mathsf{Z} \wedge \neg\mathbf{I}\mathsf{Z})$ for all $x \in \mathcal{I}_w$, $y \in \mathcal{E}_w$, and $z \in \mathcal{B}_w$.

When looking at points in a topological space, for every region there are three different kinds of points: interior points, exterior points, and boundary points of a

3. The RCC-8-structure does not cover all possible Kripke models of $m(\Theta)$. The goal of a canonical model is just to provide a model for any consistent sentence of a calculus, not to cover all possible models.

region. If a point is an interior or exterior point of a region, there is a neighborhood of the point such that all points of the neighborhood are inside or outside the region, respectively. If a point is a boundary point of a region, every neighborhood contains points inside and points outside the region (see Definition 4).

There seems to be a correspondence between worlds and points of a topological space, and between the accessibility relation $R_{\mathbf{I}}$ and topological neighborhoods. In the following lemma we further investigate this correspondence by deriving topological constraints from the modal formulas.

LEMMA 7. — *Let x and y be two spatial variables of Θ. Depending on which sets $\mathcal{I}_w, \mathcal{E}_w$, or $\mathcal{B}_w$ they are contained in for a world w, the following relations between x and y are impossible. This has some topological consequences on possible instantiations* $\mathtt{X}, \mathtt{Y}$*:*

x	y	*Impossible Relations* $R(x, y)$	*Consequences*
$\mathcal{I}_w$	$\mathcal{I}_w$	DC, EC	$i(\mathtt{X}) \cap i(\mathtt{Y}) \neq \emptyset$
$\mathcal{I}_w$	$\mathcal{E}_w$	TPP, NTPP, EQ	$i(\mathtt{X}) \cap e(\mathtt{Y}) \neq \emptyset$
$\mathcal{I}_w$	$\mathcal{B}_w$	DC, EC, TPP,	
		NTPP, EQ	$i(\mathtt{X}) \cap b(\mathtt{Y}) \neq \emptyset$
$\mathcal{E}_w$	$\mathcal{E}_w$	–	–
$\mathcal{E}_w$	$\mathcal{B}_w$	TPP^{-1}, NTPP^{-1}, EQ	$e(\mathtt{X}) \cap b(\mathtt{Y}) \neq \emptyset$
$\mathcal{B}_w$	$\mathcal{B}_w$	DC, NTPP, NTPP^{-1}	$b(\mathtt{X}) \cap b(\mathtt{Y}) \neq \emptyset$[4]

Proof. Most entries in the table follow immediately from the encoding of the relations in modal logic. The only more difficult entry is the relation $\mathsf{EC}(x, y)$ in the third line of the table. This relation is not possible because of the property $\Box(\mathsf{Y} \rightarrow \neg\mathbf{I}\neg\mathbf{I}\mathsf{Y})$ which states that for any world w that satisfies Y there is a world v with $wR_{\mathbf{I}}v$ that satisfies $\mathbf{I}$Y. As v also satisfies $\mathbf{I}$X, the model constraint of $\mathsf{EC}(x, y)$ is violated, so this relation is not possible. The topological consequences result from distinguishing the impossible from the possible relations. ■

It can be seen that when, e.g., $\mathbf{I}$X and $\mathbf{I}$Y hold in a world w, then X and Y must have a common interior. So, there is a common interior point of X and Y where w can be mapped to. In the following theorem we give a mapping of every world to a point in the topological space.

THEOREM 8. — *Let Θ be a consistent set of* RCC-8 *constraints, $m(\Theta)$ be the modal encoding of Θ, $\mathcal{M} = \langle W, \{R_\Box, R_{\mathbf{I}}\}, \pi\rangle$ be an* RCC-8*-model of $m(\Theta)$, and $\mathcal{U}$ a topological space. Then there is a function $p : W \mapsto \mathcal{U}$ that maps each world $w \in W$ to a point $p(w) \in \mathcal{U}$ and a function $N : W \mapsto 2^{\mathcal{U}}$ that assigns each world $w \in W$ a neighborhood $N(w)$ of $p(w)$ such that $p(w)$ is in the interior of* X *if $\mathcal{M}, w \Vdash \mathbf{I}$*X

4. If $\mathsf{PO}(x, y)$ holds, X and Y do not necessarily have a common boundary point if one of them is not internally connected. However, assuming $b(\mathtt{X}) \cap b(\mathtt{Y}) \neq \emptyset$ in this case does not contradict any RCC-8 constraint, since RCC-8 is not expressive enough to distinguish different kinds of partial overlap.

holds, $p(w)$ *is in the exterior of* X *if* $\mathcal{M}, w \Vdash \mathbf{I}\neg \mathsf{X}$ *holds,* $p(w)$ *is on the boundary of* X *if* $\mathcal{M}, w \Vdash \mathsf{X} \wedge \neg \mathbf{I}\mathsf{X}$ *holds, and* $p(u) \in N(w)$ *if and only if* $wR_{\mathbf{I}}u$ *holds.*[5]

Proof. Let w be a world of W and $\mathcal{I}_w, \mathcal{E}_w$, and $\mathcal{B}_w$ be the corresponding sets of spatial variables. We assume that there is a realization of Θ such that there is at least one point in the topological space that is in the interior of every X, in the exterior of every Y, and on the boundary of every Z simultaneously ($x \in \mathcal{I}_w$, $y \in \mathcal{E}_w$, $z \in \mathcal{B}_w$). It follows from Lemma 7 that this is true for every pair of regions. As RCC-8 permits only binary constraints between spatial variables and regions are allowed to be internally disconnected, this assumption holds. We further assume that p maps w to one of these points.

Because of Definition 4, there must be neighborhoods $N_X(w)$ and $N_Y(w)$ of $p(w)$ for every $x \in \mathcal{I}_w$ and every $y \in \mathcal{E}_w$ such that $N_X(w)$ is in the interior of X and $N_Y(w)$ is disjoint with Y. Also, for every $z \in \mathcal{B}_w$, every neighborhood $N_Z(w)$ of $p(w)$ contains points inside and outside Z. All these neighborhoods are members of the neighborhood system of $p(w)$, so their intersection $N(w)$ is also a neighborhood of $p(w)$ where all $R_{\mathbf{I}}$-successors of w can be mapped to. ■

Using the above defined functions p and N, $\mathcal{M}, w \Vdash \mathbf{I}\mathsf{X}$ can be interpreted as "there is a neighborhood $N(w)$ of $p(w)$ such that all points of $N(w)$ are in X". This obeys the intended interpretation of $\mathbf{I}$ as an interior operator, as $\mathcal{M}, w \Vdash \mathsf{X}$ means that $p(w)$ is in X and $\mathcal{M}, w \Vdash \mathbf{I}\mathsf{X}$ means that $p(w)$ is in the interior of X.

The function N, as defined in Theorem 8, can be replaced by any function $N' : W \mapsto 2^{\mathcal{U}}$, with $N'(w) \subseteq N(w)$ for all $w \in W$, if $N'(w)$ is a member of the neighborhood system of $p(w)$. p has to be changed accordingly. In particular, we will regard in the following all neighborhoods as d-dimensional spheres where d is the dimension of the underlying topological space.

In order to make the following argumentation more readable, a world mapped to an interior point of X is denoted *interior world* of x, a world mapped to an exterior point of X *exterior world* of x, and a world mapped to a boundary point of X *boundary world* of x. Accordingly, a region is called *interior, exterior* or *boundary region* of a world. In particular, a world w with $\mathcal{M}, w \Vdash \mathbf{I}\mathsf{X}$ is an interior world of x, a world w with $\mathcal{M}, w \Vdash \mathbf{I}\neg\mathsf{X}$ is an exterior world of x, and a world w with $\mathcal{M}, w \Vdash \mathsf{X} \wedge \neg\mathbf{I}\mathsf{X}$ is a boundary world of x.

5. RCC-8 Models and the Dimension of Space

In the previous section we have shown how the RCC-8-models introduced in Section 3.3 can be mapped to topological space, but we still have no information about the dimension of the topological space. In this section we investigate the influence

5. The properties for $R_\Box$ ($p(u) \in \mathcal{U}$ if $wR_\Box u$ holds and $p(w) \in \mathcal{U}$) can be omitted as we already defined N and p such that only points of $\mathcal{U}$ are used.

of dimension on the possibility to map the RCC-8-models to the topological space, i.e., which dimension is required in order to find a realization of a consistent set of RCC-8 constraints Θ. We will start with proving that for any RCC-8-model there is a realization in two-dimensional space. It is sufficient to prove this only for sets of base relations as every realization of Θ uses only base relations.[6] For this proof it is important to keep in mind that regions do not have to be internally connected, i.e., they might consist of different disconnected pieces. It will turn out that our proof leads to realizations in any dimension $d \geq 1$. Finally, for three- and higher-dimensional space we will prove that every consistent set Θ can also be realized with internally connected regions.

For the following analysis we restrict regions to be sets of d-dimensional polytopes. Sets are required since regions might consist of several disconnected pieces where each piece is a single polytope. This restriction will be lifted later and the results can be generalized to arbitrary regular regions.

Let Θ be a consistent set of RCC-8 constraints and $\mathcal{M}$ be an RCC-8-model of $m(\Theta)$, the modal encoding of Θ. Suppose that only two-dimensional regions are permitted, i.e., the topological space is a two-dimensional plane $\mathcal{U}$. All worlds of $\mathcal{M}$ are mapped to points of $\mathcal{U}$ as specified in Theorem 8. The general intuition of the proof is that every RCC-8-cluster, i.e., every world of level 0 together with its $R_{\mathbf{I}}$-successors is mapped to an independent neighborhood such that each neighborhood can be placed on an arbitrary but distinct position on the plane. Each neighborhood will then be extended to different closed sets that form the pieces of the spatial regions. In the following we will study the requirements neighborhoods have to meet in order to guarantee two-dimensional realizations.

For every spatial variable x_i ($1 \leq i \leq n$) and every world w of level 0, we define a *region vector* $r_i^w = (r_{i,1}^w, \ldots, r_{i,2n}^w)$ that represents the affiliation of the $2n$ $R_{\mathbf{I}}$-successors of w to X_i, i.e., $r_{i,j}^w = 1$ if $\mathcal{M}, v_j \Vdash \mathsf{X}_i$ and $r_{i,j}^w = 0$ if $\mathcal{M}, v_j \nVdash \mathsf{X}_i$ where v_j is the jth $R_{\mathbf{I}}$-successor of w. Since in the two-dimensional case the neighborhood $N(w)$ is a circle, we suppose that the points $p(v_j)$ corresponding to the $R_{\mathbf{I}}$-successors v_j of w are ordered clock-wise within the circle according to j. If $p(w)$ is a boundary point of $\mathtt{X}_i$, some values of r_i^w are 1 and some are 0. Otherwise all values of r_i^w are either 1 (if $p(w)$ is contained in $\mathtt{X}_i$) or 0 (if $p(w)$ is not contained in $\mathtt{X}_i$).

LEMMA 9. — *If for every world w of level 0 there is a permutation P_w of the values of r_i^w such that $(r_{i,P_w(1)}^w, \ldots, r_{i,P_w(2n)}^w)$ is a bitonic sequence[7] for all $1 \leq i \leq n$, then the neighborhoods $N(w)$ can be placed in a two-dimensional plane such that all spatial relations are satisfied within the neighborhoods.*

6. The relation EQ can be omitted as any pair of spatial variables x and y with $\mathsf{EQ}(x, y)$ can be combined to a single spatial variable.
7. The values of a *bitonic* sequence are in a form $0^e 1^f 0^g$ or $1^e 0^f 1^g$ for $e, f, g \geq 0$ [COR 90, p. 642].

Proof. If r_i^w is a bitonic sequence and $p(w)$ is a boundary point of $\mathtt{X}_i$, then the mappings of the worlds of level 1 corresponding to the values of r_i^w can be separated into points inside $\mathtt{X}_i$ and points outside $\mathtt{X}_i$ by at most two line segments meeting at $p(w)$ (see Figure 3).

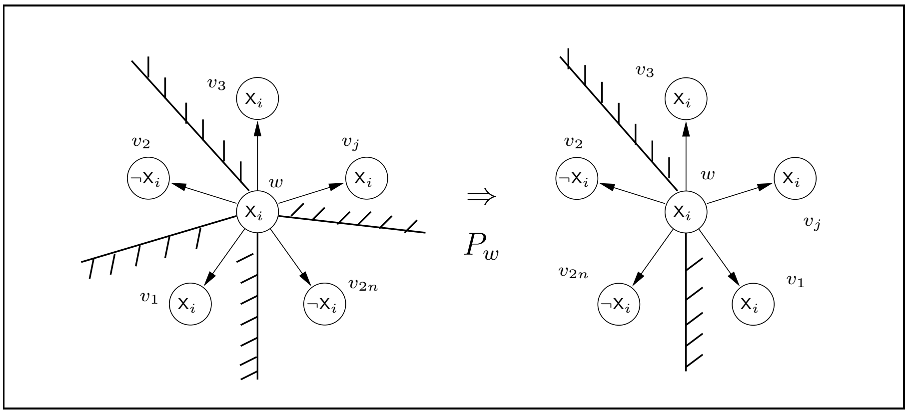

Figure 3. *Permutation P_w of the $R_\mathbf{I}$-successors of a world w. The solid line indicates the boundary of* $\mathtt{X}$*, the hashed region the interior of* $\mathtt{X}$*.*

These line segments can be regarded as the part of the boundary of $\mathtt{X}_i$ which is inside $N(w)$. So, neighborhoods can be separated in an interior and an exterior part of a region by a one-dimensional boundary. Therefore all neighborhoods can be placed in a two-dimensional plane. As the permutation of the $R_\mathbf{I}$-successors has no influence on the relations between the regions, all spatial relations between the regions hold within the neighborhoods. ■

Actually, a permutation as described in the previous lemma is not necessary to guarantee two-dimensional realizations. A region might look as shown on the left of Figure 3, but in this case we restrict the shape and the internal connection of the regions by the neighborhoods we are using which is not at all desirable. However, a permutation as described in Lemma 9 is necessary for one-dimensional realizations and realizations with internally connected regions.

Since a permutation P_w is only necessary for boundary worlds, we will in the following try to keep the number of boundary worlds as small as possible. Therefore, we consider only RCC-8-models for which all boundary worlds are explicitly forced to be boundary worlds by the constraints. In order to do so, we have to take a closer look at which worlds are introduced as boundary worlds of some regions by the entailment constraints, and which worlds are forced to be boundary worlds of regions by the constraints. As a world w of level 0 is forced to be a boundary world of x if $\mathcal{M}, w \Vdash \mathsf{X}$ and $\mathcal{M}, v \nVdash \mathsf{X}$ hold for a world v with $wR_\mathbf{I}v$, we have to find out which of the model and entailment constraints force $\mathcal{M}, w \Vdash \mathsf{X}$ if $\mathcal{M}, v \nVdash \mathsf{X}$ holds or force $\mathcal{M}, v \nVdash \mathsf{X}$ if $\mathcal{M}, w \Vdash \mathsf{X}$ holds.

PROPOSITION 10. — *Boundary worlds are introduced only by the following relations (see Table 2):*

1) $\mathsf{EC}(x,y)$*:* $\neg\Box(\neg(\mathsf{X}\wedge\mathsf{Y}))$ *introduces a boundary world of* x *and* y *because of* $\Box(\neg(\mathbf{I}\mathsf{X}\wedge\mathbf{I}\mathsf{Y}))$.

2) $\mathsf{TPP}(x,y)$*:* $\neg\Box(\mathsf{X}\rightarrow\mathbf{I}\mathsf{Y})$ *introduces a boundary world of* x *and* y *because of* $\Box(\mathsf{X}\rightarrow\mathsf{Y})$.

3) $\mathsf{TPP}^{-1}(x,y)$*:* $\neg\Box(\mathsf{Y}\rightarrow\mathbf{I}\mathsf{X})$ *introduces a boundary world of* x *and* y *because of* $\Box(\mathsf{Y}\rightarrow\mathsf{X})$.

Apart from the above worlds that are introduced as boundary worlds of particular regions, worlds can also be forced to be boundary worlds of other regions.

PROPOSITION 11. — *A world* w *is forced to be a boundary world of* x *only with the following constraints:*

1) $\Box(\mathsf{X}\rightarrow\mathsf{Y})$*: If* w *is a boundary world of* y *and* X *is true in* w*, then* w *must also be a boundary world of* x*.*

2) $\Box(\mathsf{Y}\rightarrow\mathsf{X})$*: If* w *is a boundary world of* y *and* $\neg\mathsf{X}$ *is true in an* $R_{\mathbf{I}}$*-successor of* w*, then* w *must also be a boundary world of* x*.*

3) $\Box(\neg(\mathbf{I}\mathsf{X}\wedge\mathbf{I}\mathsf{Y}))$*: If* X *and* Y *are true in* w*, then* w *must be a boundary world of* x *and* y*.*

For the constraints $\Box(\mathsf{X}\rightarrow\mathsf{Y})$ and $\Box(\mathsf{Y}\rightarrow\mathsf{X})$, w must already be a boundary world of some other region, so w must be introduced by one of the relations $\mathsf{EC}(x,y)$, $\mathsf{TPP}(x,y)$, or $\mathsf{TPP}^{-1}(x,y)$. If w is forced to be a boundary world of x and y with the constraint $\Box(\neg(\mathbf{I}\mathsf{X}\wedge\mathbf{I}\mathsf{Y}))$, then X and Y must both be true in w. This can only be forced when there is a $z_1\in Reg(\Theta)$ with $\mathsf{TPP}(z_1,x)$ and Z_1 is true in w, a $z_2\in Reg(\Theta)$ with $\mathsf{TPP}(z_2,y)$ and Z_2 is true in w, and w is a boundary world of z_1 and z_2 introduced by $\mathsf{EC}(z_1,z_2)$. So, in any case when a world is forced to be a boundary world of some region it must already be a boundary world of other regions introduced as described in Proposition 10.

We will now have a look at how regions must be related in order to force a world to be a boundary world of these regions using the constraints of Proposition 11. Suppose that w is a boundary world of x and y introduced by either $\mathsf{EC}(x,y)$ or $\mathsf{TPP}(x,y)$.[8] We will write $x|y$ in order to express that we can either use x or y but always the same. With one of the following constraints it can be forced that w is also a boundary world of $z\neq x,y$ (v is an $R_{\mathbf{I}}$-successor of w):

$$\begin{array}{ll}\Box(\mathsf{Z}\rightarrow(\mathsf{X}|\mathsf{Y}))\text{ and }\mathcal{M},w\Vdash\mathsf{Z} & (\rightsquigarrow\mathsf{TPP}(z,x|y))\\ \Box(\neg(\mathbf{I}\mathsf{Z}\wedge\mathbf{I}(\mathsf{X}|\mathsf{Y})))\text{ and }\mathcal{M},w\Vdash\mathsf{Z} & (\rightsquigarrow\mathsf{EC}(z,x|y))\\ \Box((\mathsf{X}|\mathsf{Y})\rightarrow\mathsf{Z})\text{ and }\mathcal{M},v\Vdash\neg\mathsf{Z} & (\rightsquigarrow\mathsf{TPP}^{-1}(z,x|y))\end{array}$$

$\mathcal{M},w\Vdash\mathsf{Z}$ is forced with the following constraint:

8. We omit $\mathsf{TPP}^{-1}(x,y)$ as $\mathsf{TPP}^{-1}(x,y)=\mathsf{TPP}(y,x)$ and the order is not important.

$$\Box(\mathsf{U} \rightarrow \mathsf{Z}) \text{ and } \mathcal{M}, w \Vdash \mathsf{U} \qquad\qquad (\rightsquigarrow \mathsf{TPP}(u,z))$$

$\mathcal{M}, v \Vdash \neg\mathsf{Z}$ is forced with the following constraints:

$$\Box(\mathsf{Z} \rightarrow \mathsf{U}) \text{ and } \mathcal{M}, v \Vdash \neg\mathsf{U} \qquad (\rightsquigarrow \mathsf{TPP}^{-1}(u,z))$$
$$\Box(\neg(\mathbf{I}\mathsf{Z} \wedge \mathbf{I}\mathsf{U})) \text{ and } \mathcal{M}, v \Vdash \mathsf{U} \qquad (\rightsquigarrow \mathsf{EC}(u,z))$$

When we compose these relations, we obtain the possible relations between u and $x|y$.

$R(u,z)$	$S(z,x\|y)$	$(R \circ S)(u,x\|y)$
TPP	TPP	TPP, NTPP
TPP	EC	DC, EC
TPP^{-1}	TPP^{-1}	$\mathsf{TPP}^{-1}, \mathsf{NTPP}^{-1}$
EC	TPP^{-1}	DC, EC

As w is a boundary world of x and y, $\mathsf{DC}(u,x|y)$ and $\mathsf{NTPP}(u,x|y)$ are not possible together with $\mathcal{M}, w \Vdash \mathsf{U}$, and $\mathsf{NTPP}^{-1}(u,x|y)$ is not possible together with $\mathcal{M}, v \Vdash \neg\mathsf{U}$. In order to force $\mathcal{M}, w \Vdash \mathsf{U}$, there must be a sequence of spatial variables u_i with $\mathsf{TPP}(u_1,u)$, $\mathsf{TPP}(u_{i+1},u_i)$, until there is a u_m that is equal to x or y, so $\mathsf{TPP}(x,z)$ or $\mathsf{TPP}(y,z)$ must hold. In order to force $\mathcal{M}, v \Vdash \neg\mathsf{U}$, there must be a sequence of spatial variables u_i with $\mathsf{TPP}^{-1}(u_1,u)$, $\mathsf{TPP}^{-1}(u_{i+1},u_i)$, and $\mathsf{EC}(u_j,u_{j-1})$ and $\mathcal{M}, v \Vdash \mathsf{U}_j$ must hold. In order to force $\mathcal{M}, v \Vdash \mathsf{U}_j$, there must be a sequence of TPP-related spatial variables, as described above, until one of them is equal to x or y, so $\mathsf{EC}(z,x)$ or $\mathsf{EC}(z,y)$ must hold. This results in only three different possibilities of how w is forced to be a boundary world of z if w was introduced as a boundary world of x and y.

a. $\mathsf{TPP}(x,y), \mathsf{TPP}(x,z)$, and $\mathsf{TPP}(z,y)$ hold.

b. $\mathsf{EC}(x,y), \mathsf{TPP}(x,z)$, and $\mathsf{EC}(z,y)$ hold.

c. $\mathsf{EC}(x,y), \mathsf{TPP}(y,z)$ and $\mathsf{EC}(z,x)$ hold.

As different spatial variables z_i, z_j, for which w is forced to be a boundary world of, all have the boundary world w in common, only the relations EC, PO, TPP, or TPP^{-1} can hold between them.

We have shown that only those worlds are boundary worlds which are introduced as boundary worlds of some regions by the entailment constraints, and, further, that other regions are only forced to be boundary regions of these worlds when they are related in a particular way. This will be used in the following lemma.

LEMMA 12. — *Let $\mathcal{M}$ be an* RCC-8*-model. Then two different types of $R_{\mathbf{I}}$-successors are sufficient for every world w of level 0.*

Proof. If w is not a boundary world of some region, all $R_{\mathbf{I}}$-successors of w satisfy exactly the same formulas as w. Otherwise, w is introduced as a boundary world by either $\mathsf{EC}(x,y)$ or $\mathsf{TPP}(x,y)$ (see Proposition 10). Let w be forced to be a boundary world of the spatial variables z_i. For $\mathsf{EC}(x, y)$, some of the $R_{\mathbf{I}}$-successors of w satisfy X but not Y, and some satisfy Y but not X, the others neither satisfy X nor Y. For all z_i with $\mathsf{TPP}(x, z_i)$ and $\mathsf{EC}(z_i, y)$ and all z_j with $\mathsf{TPP}(y, z_j)$ and $\mathsf{EC}(z_j, x)$, all $R_{\mathbf{I}}$-successors of w satisfy Z_i if they satisfy X and satisfy Z_j if they satisfy Y. So all $R_{\mathbf{I}}$-successors of w that satisfy X satisfy the same formulas, and all $R_{\mathbf{I}}$-successors of w that satisfy Y satisfy the same formulas. For the $R_{\mathbf{I}}$-successors v of w which do not satisfy X or Y, there are only two requirements: if $\mathsf{TPP}(z_{k'}, z_k)$ holds then Z_k must be true in v whenever $\mathsf{Z}_{k'}$ is true in v; if $\mathsf{EC}(z_k, z_{k'})$ holds then Z_k and $\mathsf{Z}_{k'}$ must not both be true in v. However, there is no constraint that forces the existence of these worlds, so it can be assumed that all $R_{\mathbf{I}}$-successors of w satisfy either X or Y. As the respective worlds all satisfy the same formulas, two different kinds of $R_{\mathbf{I}}$-successors of the boundary world w introduced by $\mathsf{EC}(x, y)$ are sufficient.

For $\mathsf{TPP}(x, y)$, all $R_{\mathbf{I}}$-successors of w that satisfy X also satisfy Y, all $R_{\mathbf{I}}$-successors of w that do not satisfy Y also do not satisfy X, and some $R_{\mathbf{I}}$-successors of w satisfy Y but not X. For all z_i with $\mathsf{TPP}(x, z_i)$ and $\mathsf{TPP}(z_i, y)$, all $R_{\mathbf{I}}$-successors of w satisfy Z_i if they satisfy X. For the $R_{\mathbf{I}}$-successors of w that satisfy Y but not X, there is only one requirement, namely, that Z_k must be true whenever $\mathsf{Z}_{k'}$ is true in these worlds for any two spatial variables $z_k, z_{k'}$ with $\mathsf{TPP}(x, z_k)$, $\mathsf{TPP}(z_{k'}, y)$ and $\mathsf{TPP}(z_k, z_{k'})$. However, there is again no constraint that forces the existence of these worlds, so it can be assumed that all $R_{\mathbf{I}}$-successors of w satisfy X if they satisfy Y. ■

Whether a boundary world w is introduced by $\mathsf{EC}(x, y)$ or by $\mathsf{TPP}(x, y)$, in both cases two different kinds of $R_{\mathbf{I}}$-successors are sufficient. Thus, by grouping together the respective $R_{\mathbf{I}}$-successors for every world w of level 0 of $\mathcal{M}$, we can always find a permutation of the worlds of level 1 such that r^w is a bitonic sequence for all regions.

Instead of having $2n$ $R_{\mathbf{I}}$-successors for every world of level 0 from which we know that they belong to only two different types, it is sufficient two use only two $R_{\mathbf{I}}$-successors for every world of level 0. This leads to a very simple canonical model which is defined in the same way as in Definition 6 except that we have exactly two worlds of level 1 instead of $2n$ worlds.

DEFINITION 13 (REDUCED RCC-8-STRUCTURE/ -CLUSTER/ -MODEL). — *A reduced* RCC-8-structure $\mathcal{S}^*_{RCC8} = \langle W, \{R_\Box, R_{\mathbf{I}}\}, \pi\rangle$ *has the following properties:*

1) W contains only worlds of level 0 and 1.

2) For every world u of level 0 there are exactly two worlds v of level 1 with $uR_{\mathbf{I}}v$. These three worlds form a reduced RCC-8-cluster *(see Figure 4a).*

3) For every world v of level 1 there is exactly one world u of level 0 with $uR_{\mathbf{I}}v$.

4) For all worlds $w, v \in W_{\mathcal{S}}$: $wR_{\mathbf{I}}w$ and $wR_\Box v$.

$\mathcal{S}^*_{RCC8}$ *contains* RCC-8-*clusters with all possible valuations. The corresponding models are denoted as* reduced RCC-8-models*.*

It follows from Lemma 12 that the reduced RCC-8-structure is a canonical model for RCC-8. Note that the reduced RCC-8-structure is equivalent to an RCC-8-

structure of size 1 and that a reduced RCC-8-cluster is equivalent to an RCC-8-cluster of size 3.

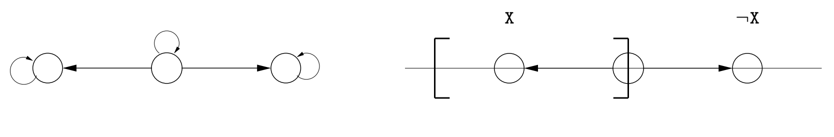

Figure 4. *(a) shows a reduced* RCC-8-*cluster of the reduced* RCC-8-*structure. (b) shows how a neighborhood can be placed in one-dimensional space. The two brackets indicate a one-dimensional region* X *where the neighborhood represents a boundary point of* X*.*

We can now apply Lemma 9 and place all neighborhoods independently on the plane while all relations between spatial regions hold within the neighborhoods. Thereby, neighborhoods corresponding to non-boundary worlds are homogeneous in the sense that all points within one of these neighborhoods have the same topological properties. Neighborhoods corresponding to a boundary world w consist of two homogeneous parts corresponding to the two $R_{\mathbf{I}}$-successors of w. These two parts are divided by the common boundary of the boundary regions of w (see Figure 5a).

In order to obtain a realization, we have to find regions such that the relations between them hold in the whole plane and not just within the neighborhoods. Since regions do not have to be internally connected, it is possible to compose every region out of pieces resulting from the corresponding neighborhoods, i.e., for every neighborhood a region is affiliated with, we generate a piece of that region. As the neighborhoods are open sets and regions as well as their pieces must be regular closed sets, we have to *close* every neighborhood, i.e., find a closed set $\mathtt{X}^w$ for every region X and every neighborhood $N(w)$ with $\mathcal{M}, w \Vdash \mathsf{X}$ such that all relations hold between the regions composed of the pieces. As all neighborhoods are independent of each other, we only have to make sure that the relations of the different pieces corresponding to a single neighborhood do not violate the relations of the compound regions. This can be done independently for every neighborhood.

Consider a particular neighborhood $N(w)$. If w is not a boundary world, then only the relations PO, TPP, NTPP, and their converse are possible between the regions affiliated with $N(w)$, since they share $N(w)$ as their common interior. For closing the neighborhood $N(w)$, all pieces must fulfill the "part of" relations whereas the PO relations cannot be violated as long as the corresponding pieces have a common interior.

One possibility to fulfill the "part of" relations is using a hierarchy of the regions, defined as follows.

DEFINITION 14 (HIERARCHY OF REGIONS). — *A* hierarchy of regions H_Θ *is a mapping of regions to levels, where a region* X *is of level* $H_\Theta(\mathtt{X}) = 1$ *if there is no region* Y *which is part of* X*. A region* X *is of level* $H_\Theta(\mathtt{X}) = k$ *if there is a region* Y *of level* $H_\Theta(\mathtt{Y}) = k - 1$ *which is part of* X *and if there is no region* Z *which is part of* X *and has a higher level than* $H_\Theta(\mathtt{Z}) = k - 1$.[9]

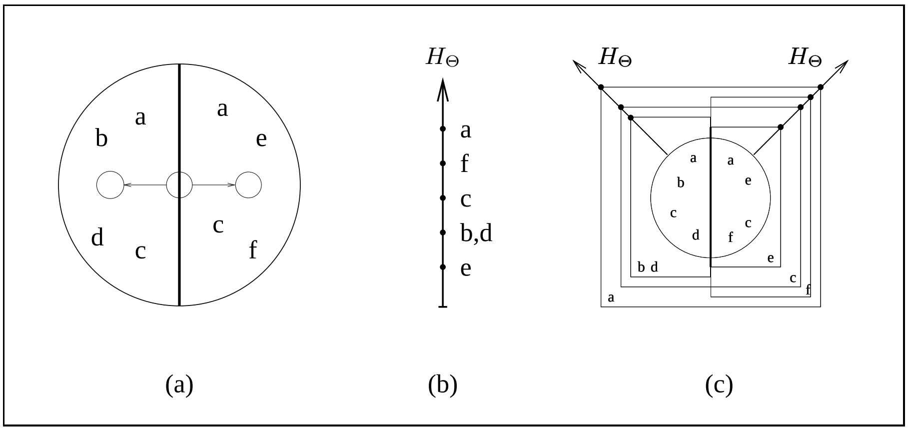

Figure 5. *(a) shows the two-dimensional neighborhood of a boundary world which is divided into two parts by the common boundary of the boundary regions* b, d, e*, and* f*. (b) shows a possible hierarchy* H_Θ *of regions. In (c) the neighborhood is closed with respect to* H_Θ*.*

The pieces of all regions affiliated with $N(w)$ must then be chosen according to H_Θ, i.e., pieces of regions of the same level are equal for this particular neighborhood and are non-tangential proper part of all pieces of regions of a higher level. We choose the single pieces to be rectangles.

If w is a boundary world, the boundary regions of w are only affiliated with one part of $N(w)$ and their pieces must share the common boundary. Therefore, both parts of $N(w)$ must be closed separately according to H_Θ (see Figure 5c). In the same way as for two-dimensional space, neighborhoods can be placed in any higher dimensional space and closed therein according to H_Θ. As the three points corresponding to a world w of level 0 and its two $R_{\mathbf{I}}$-successors can always be aligned, $N(w)$ can also be placed on a line. Thus, all neighborhoods can be placed independently in a one-dimensional space and closed as intervals according to H_Θ (see Figure 4b).

THEOREM 15. — *Every consistent set of* RCC-8 *constraints can be realized in any dimension* $d \geq 1$ *where regions are (sets of) d-dimensional polytopes.*

9. This corresponds to the finish time of depth-first search for each vertex of a graph G_Θ where regions are vertices V_Θ and "part of" relations are directed edges E_Θ, computable in time $O(V_\Theta + E_\Theta)$ [COR 90, p.477ff]

So far all regions consist of as many pieces as there are neighborhoods affiliated with them, i.e., $O(n^2)$ many pieces for every region. We can further show that for three- and higher-dimensional space all regions can also be realized as internally connected. For this we construct a $d+1$-dimensional realization of internally connected regions by connecting all pieces of the same regions of a d-dimensional realization of internally disconnected regions.

THEOREM 16. — *Every consistent set Θ of* RCC-8 *constraints can be realized with internally connected regions in any dimension $d \geq 3$ where regions are polytopes.*

Proof. Suppose that Θ is consistent. With the following construction we obtain a three-dimensional realization of internally connected regions starting from a two-dimensional realization. We begin with constructing a particular two-dimensional realization in the plane determined by the x- and y-axes. (1a) Place all neighborhoods on a circle with center C such that the common boundary of each neighborhood corresponding to a boundary world points to C (see Figure 6a). (1b) Close all neighborhoods according to the hierarchy H_Θ such that all pieces of regions are rectangles. We now extend this two-dimensional realization to a three-dimensional realization. The third dimension is determined by the z-axis. (2a) Proceed from the two-dimensional realization according to H_Θ by first choosing pairwise distinct intervals on the positive z-axis for every region with $H_\Theta = 1$, i.e., for the regions that do not contain any other region. (2b) Build a pipe parallel to the z-axis for every piece of these regions starting from the plane (z=0) up to the endpoint of the corresponding interval. (2c) Connect the pipes of the same region within the range of the corresponding interval using pipes pointing to the center (see Figure 6b). (3) Next the regions with $H_\Theta = 2$

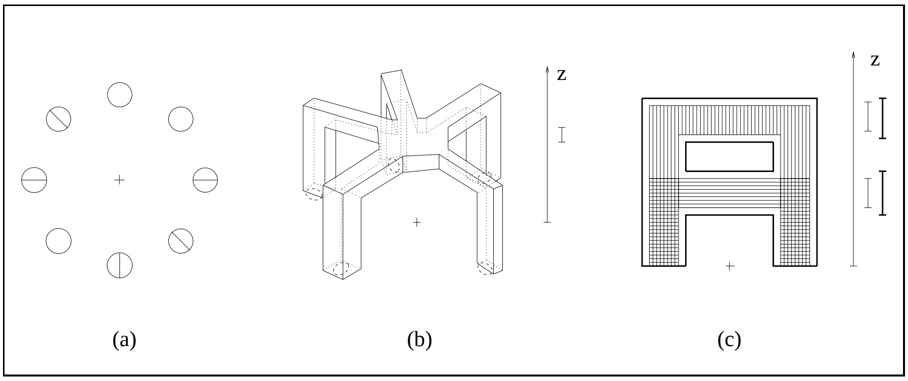

Figure 6. *Construction of the three-dimensional realization. (a) placing the two-dimensional neighborhoods on a circle. (b) connecting the pieces of a region on a particular level. (c) connecting the pipes of a region (bold line) that contains the vertically and the horizontally hashed regions.*

are connected, i.e., those regions that only contain already connected regions. To do this, (3a) choose intervals on the z-axis for these regions such that the intervals contain all intervals of the contained regions but do not overlap with any other interval.

(3b) Build a pipe for every piece up to the endpoint of the corresponding interval with the largest z-value, and (3c) connect the pipes of every region within the range of all corresponding intervals (see Figure 6c). (4) Repeat step 3 successively for every level of H_Θ until all regions are connected. (5) Finally, close all neighborhoods on the negative z-axis according to H_Θ.

Obviously, with this construction all regions are internally connected. Furthermore all internally connected three-dimensional regions hold the same base relations as the two-dimensional realizations from which we started the construction. This is because all intervals on the z-axis are either contained in each other or are distinct, they have no common boundary points. All intervals corresponding to region X are contained in the intervals of region Y if and only if $\mathsf{NTPP}(x, y)$ or $\mathsf{TPP}(x, y)$. When two regions are disconnected they remain disconnected as they are not affiliated with the same neighborhoods. Two externally connected regions remain externally connected because every neighborhood was placed on the circle such that the common boundary points to its center. Therefore, if two of these regions are both affiliated with the same neighborhood, their pipes are externally connected and the horizontal connection of the single pipes is distinct. All other requirements of relations as, e.g., a common boundary point are already met by the pipes.

With a similar construction, a $d + 1$-dimensional realization of internally connected regions can be obtained from a d-dimensional realization of internally disconnected regions. In this case, the d-dimensional neighborhoods must be placed on a d-dimensional sphere and the intervals must be chosen at the $d + 1$ dimensional axis. All constructions kept the polytopic shape of the regions, so every region can be realized as a (d-dimensional) polytope. ■

The restriction of regions to be polytopes can immediately be generalized to an arbitrary shape of regions.

6. Applicability of the Canonical Model

In the previous sections we reported about the existence of (reduced) RCC-8-models and how they can be mapped to topological spaces of different dimensions. In this section we study how RCC-8-models can be determined and how a realization can be generated from them. As there is a (reduced) RCC-8-model $\mathcal{M}$ for every consistent set of spatial relations Θ, and as it is always possible to generate a realization of $\mathcal{M}$, RCC-8 models are suitable for representing spatial regions with respect to their relations. RCC-8-models represent the characteristic points and information about their neighborhoods of a possible realization. As in the previous section we assume that Θ contains only constraints over the base relations.

6.1. *Determination of RCC-8 models*

Given a set of RCC-8 constraints Θ which contains only base relations, we have to find a reduced RCC-8-model $\mathcal{M}$ for the modal encoding of RCC-8 such that only those worlds are boundary worlds of regions which are forced to be by the constraints. The Kripke frame of $\mathcal{M}$, i.e., the number of worlds and their accessibility relations are already known from the entailment constraints, but we have to find a valuation for every world and every region. For some worlds and some regions the valuation is already given by the constraints, for some it can be inferred using the constraints, for others it can be chosen. In order to make the inference step as easy as possible, we use the propositional encoding of RCC-8 with respect to a Kripke frame where every world w and every spatial variable x is transformed to a propositional atom X_w which is true if and only if X holds in w [REN 99b]. The valuation of $\mathcal{M}$ can then be obtained from the satisfying assignment of the propositional formula. Although the encoding of the reduced RCC-8-models is not a Horn formula,[10] unit-resolution plus additional choices is sufficient for finding a satisfying assignment. As all clauses of the propositional encoding use worlds of the same RCC-8-cluster, the inference step is independent for every cluster. From Proposition 10 it is known which RCC-8-clusters contain a boundary world. Suppose that an RCC-8-cluster contains a boundary world, then the valuation of the two regions which introduced the boundary world can be chosen in all worlds of the RCC-8-cluster according to the relation of the two regions. The valuations of the other regions are either determined by unit-resolution or can be chosen according to their other valuations: If the valuation of a particular region in some world of the RCC-8-cluster is true, then the other valuations are also chosen as true, otherwise all valuations are chosen as false. If an RCC-8-cluster does not contain a boundary world, all worlds of the RCC-8-cluster have the same valuation. If the valuation of a region is not determined by unit-resolution it is chosen as false. With these choices a satisfying assignment is always found, even though the propositional formula is not Horn. As there are $O(n^2)$ worlds and n regions, there are $O(n^4)$ clauses [REN 99b], so a reduced RCC-8-model can be determined in time $O(n^4)$.

6.2. *Generating a realization*

Suppose we have given a reduced RCC-8-model of a consistent set of RCC-8 constraints Θ. We have to distinguish the tasks of generating a realization of internally connected and disconnected regions. A realization of disconnected regions in d-dimensional space can be obtained by placing the $O(n^2)$ different neighborhoods in the d-dimensional space and close each neighborhood as specified in Section 5. For this, the hierarchy H_Θ of regions must be known, which can be computed in time $O(n + P_\Theta)$ where $P_\Theta \in O(n^2)$ is the number of "part of" relations in Θ (see

10. This is because of the constraint $\Box(\mathsf{X} \rightarrow \neg\mathbf{I}\neg\mathbf{I}\mathsf{X})$ which is transformed to $\bigwedge_{w\in W_0}(\neg\mathsf{X}_w \vee \mathsf{X}^1_w \vee \mathsf{X}^2_w)$ in the notation of [REN 99b].

Definition 14). Let $A_\Theta \in O(n)$ be the maximal number of regions affiliated with a neighborhood, then the closure of a neighborhood can be computed in time $O(A_\Theta)$.

THEOREM 17. — *Given a reduced* RCC-8 *model of a set of* RCC-8 *relations* Θ*, a realization of* Θ *in* d*-dimensional space* ($d \geq 1$) *can be generated in time* $O(n^2 A_\Theta)$ *when regions are allowed to be disconnected.*

In order to generate a realization of internally connected regions we can use the construction of the proof to Theorem 16. For every region we have to find the corresponding intervals on the z-axis. The number of intervals of a particular region `X` is equal to the number of regions with $H_\Theta = 1$ that are contained in `X`. Let $I_\Theta \in O(n)$ be the maximal number of regions with $H_\Theta = 1$ that are contained in a region.

THEOREM 18. — *Given a reduced* RCC-8 *model of a set of* RCC-8 *relations* Θ*, a realization of* Θ *with internally connected regions in* d*-dimensional space* ($d \geq 3$) *can be generated in time* $O(n^2 A_\Theta I_\Theta)$.

If $P_\Theta \in O(n^2)$ is the maximal number of neighborhoods affiliated with a region, every region can be realized as a polytope with $O(P_\Theta I_\Theta)$ vertices.

7. Discussion

In this paper we identified a canonical model of RCC-8 based on Kripke semantics. In order to obtain a "topological" canonical model, we gave a topological interpretation of the Kripke models such that regions can be represented by points in the topological space and information about the neighborhood of these points with respect to the spatial relations holding between the regions. Using this canonical model, we proved that every consistent set of RCC-8 constraints has a realization in any dimension when regions are not forced to be internally connected, which is the case for regions as used by RCC-8. If regions are forced to be internally connected, we proved that a realization can always be found for three- and higher dimensional space. Furthermore, we give for the first time algorithms for generating realizations of either internally connected or disconnected regions.

There is some work on identifying canonical models for the RCC axioms, i.e., determining what mathematical structures fulfill all the RCC axioms, as, e.g., every region has a non-tangential proper part [RAN 92b]. Gotts [GOT 96a] found that every connected and regular topological space is a model for the RCC axioms. Stell and Worboys [STE 97] identified a whole class of models based on Heyting structures. Both approaches only describe models for the RCC axioms, i.e., what kind of regions can be used at all. When additional constraints expressing relationships between regions are added, these results do not say anything about models anymore.[11] They are

11. Consider the hypothetical case of a set of RCC-8 constraints which is realizable in two- but not in one-dimensional space. Then, a one-dimensional space is still a model of the RCC axioms, but not of RCC-8.

also by no means constructive, as they do not provide a way of effectively representing regions or generating realizations.

Previous approaches to dealing with dimension and internal connectedness of regions tried to specify predicates and suitable axioms in order to restrict dimension and connectedness of regions [BEN 96a, GOT 94]. As all regular regions have the same dimension as the underlying space, using our results it is not necessary for consistency purposes to specify the dimension of regions explicitly if internally disconnected regions are permitted. If internally connected regions are required, these predicates only have an influence on the consistency of a set of spatial relations in one- or two-dimensional applications. In three- and higher-dimensional space all regions may be either internally connected or disconnected. Forcing internal connectedness of regions in two-dimensional space leads to difficult computational problems as there are no algorithms for dealing with this task. As Grigni et al. [GRI 95] pointed out, a well-known open problem in graph theory which is NP-hard but not known to be decidable [KRA 91a, KRA 91b] can be reduced to the consistency problem for two-dimensional internally connected regions.

It is certainly the better approach to have an additional connectedness predicate than forcing all regions to be internally connected which is done, e.g., by the similar calculus of Egenhofer [EGE 91], as there are many applications where regions are in fact disconnected. Within the area of geographical information systems, e.g., which offer a great variety of possible applications, many countries or other geographical entities are not internally connected regions. In areas like computer vision one often deals with two-dimensional projections of the three-dimensional space where many connected objects are perceived as disconnected objects due to occlusion. In robot navigation, maps are often two-dimensional cuttings of a three-dimensional space.

Acknowledgements

I would like to thank Bernhard Nebel and Tony Cohn for the fruitful discussions on the topic of the paper.

This research has been supported by DFG as part of the project FAST-QUAL-SPACE, which is part of the DFG special research effort on "Spatial Cognition". A preliminary version of this paper appears in the Proceedings of the 6th International Conference on Principles of Knowledge Representation and Reasoning [REN 98].

8. References

[BAU 91] BAUM J. D., *Elements of Point Set Topology,* Dover, 1991.

[BEE 92] VAN BEEK P., "Reasoning about Qualitative Temporal Information", *Artificial Intelligence*, vol. 58, num. 1-3, 1992, p. 297-321.

[BEN 94] BENNETT B., "Spatial Reasoning with Propositional Logic", DOYLE J., SANDEWALL E., TORASSO P., Eds., *Principles of Knowledge Representation and Reasoning: Pro-*

ceedings of the 4th International Conference, Bonn, Germany, May 1994, Morgan Kaufmann, p. 51-62.

[BEN 96a] BENNETT B., "Carving Up Space: steps towards construction of an absolutely complete theory of spatial regions", *Logics in Artificial Intelligence, Proceedings of JELIA'96*, vol. 1126 of *Lecture Notes in Computer Science*, 1996, p. 337-353.

[BEN 96b] BENNETT B., "Modal Logics for Qualitative Spatial Reasoning", *Bulletin of the IGPL*, vol. 4, num. 1, 1996, p. 23 – 45.

[CHE 80] CHELLAS B. F., *Modal Logic: An Introduction*, Cambridge University Press, Cambridge, UK, 1980.

[CLA 81] CLARKE B. L., "A Calculus of Individuals Based on Connection", *Notre Dame Journal of Formal Logic*, vol. 22, num. 3, 1981, p. 204–218.

[CLA 85] CLARKE B. L., "Individuals and Points", *Notre Dame Journal of Formal Logic*, vol. 26, num. 1, 1985, p. 61-75.

[COH 97] COHN A. G., "Qualitative Spatial Representation and Reasoning Techniques", BREWKA G., HABEL C., NEBEL B., Eds., *KI-97: Advances in Artificial Intelligence*, vol. 1303 of *Lecture Notes in Computer Science*, Freiburg, Germany, 1997, Springer-Verlag, p. 1-30.

[COR 90] CORMEN T. H., LEISERSON C. E., RIVEST R. L., *Introduction to Algorithms*, MIT Press, 1990.

[EGE 91] EGENHOFER M. J., "Reasoning about binary topological relations", GÜNTHER O., SCHEK H.-J., Eds., *Proceedings of the Second Symposium on Large Spatial Databases, SSD'91*, vol. 525 of *Lecture Notes in Computer Science*, p. 143-160, Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[FIT 93] FITTING M. C., "Basic Modal Logic", GABBAY D. M., HOGGER C. J., ROBINSON J. A., Eds., *Handbook of Logic in Artificial Intelligence and Logic Programming – Vol. 1: Logical Foundations*, p. 365-448, Oxford, Clarendon Press, 1993.

[GOT 94] GOTTS N. M., "How far can we C? Defining a 'doughnut' using connection alone", J. DOYLE E. S., TORASSO P., Eds., *Principles of Knowledge Representation and Reasoning: Proceedings of the 4th International Conference (KR94)*, San Francisco, 1994, Morgan Kaufmann, p. 246-257.

[GOT 96a] GOTTS N. M., "An Axiomatic Approach to Topology for Spatial Information Systems", report num. 96-25, 1996, University of Leeds, School of Computer Studies.

[GOT 96b] GOTTS N. M., "Topology from a single primitive relation: Defining Topological Properties and relations in terms of connection", report num. 96-23, 1996, University of Leeds, School of Computer Studies.

[GRI 95] GRIGNI M., PAPADIAS D., PAPADIMITRIOU C., "Topological Inference", *Proceedings of the 14th International Joint Conference on Artificial Intelligence*, Montreal, Canada, Aug. 1995, p. 901–906.

[KRA 91a] KRATOCHVÍL J., "String Graphs II: recognizing string graphs is NP-Hard", *Journal of Combinatorial Theory, Series B*, vol. 52, 1991, p. 67–78.

[KRA 91b] KRATOCHVÍL J., MATOUŠEK J., "String graphs requiring exponential representations", *Journal of Combinatorial Theory, Series B*, vol. 53, 1991, p. 1–4.

[LAD 94] LADKIN P. B., MADDUX R., "On Binary Constraint Problems", *Journal of the Association for Computing Machinery*, vol. 41, num. 3, 1994, p. 435-469.

[LEM 96] LEMON O., "Semantical Foundations of Spatial Logics", AIELLO L., DOYLE J., SHAPIRO S., Eds., *Principles of Knowledge Representation and Reasoning: Proceedings of the 5th International Conference*, Cambridge, MA, Nov. 1996, Morgan Kaufmann, p. 212-219.

[MAC 77] MACKWORTH A. K., "Consistency in networks of relations", *Artificial Intelligence*, vol. 8, 1977, p. 99-118.

[MON 74] MONTANARI U., "Networks of constraints: fundamental properties and applications to picture processing", *Information Science*, vol. 7, 1974, p. 95-132.

[MUN 74] MUNKRES J. R., *Topology; A First Course*, Prentice-Hall, Englewood Cliffs, NJ, 1974.

[NEB 95] NEBEL B., "Computational Properties of Qualitative Spatial Reasoning: First Results", WACHSMUTH I., ROLLINGER C.-R., BRAUER W., Eds., *KI-95: Advances in Artificial Intelligence*, vol. 981 of *Lecture Notes in Artificial Intelligence*, Bielefeld, Germany, 1995, Springer-Verlag, p. 233-244.

[NUT 99] NUTT W., "On the Translation of Qualitative Spatial Reasoning Problems into Modal Logics", BURGARD W., CHRISTALLER T., CREMERS A. B., Eds., *KI-99: Advances in Artificial Intelligence*, vol. 1701 of *LNAI*, Bonn, Germany, 1999, Springer-Verlag, p. 113-124.

[RAN 89] RANDELL D. A., COHN A. G., "Modelling Topological and Metrical Properties in Physical Processes", BRACHMAN R., LEVESQUE H. J., REITER R., Eds., *Principles of Knowledge Representation and Reasoning: Proceedings of the 1st International Conference*, Toronto, ON, May 1989, Morgan Kaufmann, p. 55-66.

[RAN 92a] RANDELL D. A., COHN A. G., "Exploiting Lattices in a Theory of Space and Time", *Computers and Mathematics with Applications*, vol. 23, num. 6-9, 1992, p. 459-476.

[RAN 92b] RANDELL D. A., CUI Z., COHN A. G., "A Spatial Logic based on Regions and Connection", NEBEL B., SWARTOUT W., RICH C., Eds., *Principles of Knowledge Representation and Reasoning: Proceedings of the 3rd International Conference*, Cambridge, MA, Oct. 1992, Morgan Kaufmann, p. 165-176.

[REN 98] RENZ J., "A canonical model of the Region Connection Calculus", COHN A., SCHUBERT L., SHAPIRO S., Eds., *Principles of Knowledge Representation and Reasoning: Proceedings of the 6th International Conference*, Trento, Italy, June 1998, p. 330 – 341.

[REN 99a] RENZ J., "Maximal Tractable Fragments of the Region Connection Calculus: A Complete Analysis", *Proceedings of the 16th International Joint Conference on Artificial Intelligence*, Stockholm, Sweden, Aug. 1999, p. 448-454.

[REN 99b] RENZ J., NEBEL B., "On the Complexity of Qualitative Spatial Reasoning: A Maximal Tractable Fragment of the Region Connection Calculus", *Artificial Intelligence*, vol. 108, num. 1-2, 1999, p. 69–123.

[REN 01] RENZ J., NEBEL B., "Efficient Methods for Qualitative Spatial Reasoning", *Journal of Artificial Intelligence Research*, vol. 15, 2001, p. 289-318.

[STE 97] STELL J., WORBOYS M., "The algebraic structure of sets of regions", *Proceedings of the 3rd International Conference on Spatial Information Theory (COSIT'97)*, vol. 1329 of *Lecture Notes in Computer Science*, 1997, p. 163-174.