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A Spatial Odyssey of the Interval Algebra: 1. Directed Intervals

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Abstract

Allen's well-known Interval Algebra has been developed for temporal representation and reasoning, but there are also interesting spatial applications where intervals can be used. A prototypical example are traffic scenarios where cars and their regions of influence can be represented as intervals on a road as the underlying line. There are several differences of temporal and spatial intervals which have to be considered when developing a spatial interval algebra. In this paper we analyze the first important difference: as opposed to temporal intervals, spatial intervals can have an intrinsic direction with respect to the underlying line. We develop an algebra for qualitative spatial representation and reasoning about directed intervals, identify tractable subsets, and show that path-consistency is sufficient for deciding consistency for a particular subset which contains all base relations.

1 Introduction

Qualitative spatial representation and reasoning has become more and more important in recent years. The best-known approach in this field is the Region Connection Calculus RCC8 [Randell *et al.*, 1992] which describes topological relationships between n-dimensional spatial regions of arbitrary shape. For some applications, however, it is sufficient to use spatial regions with more restricted properties. The block algebra [Balbiani *et al.*, 1999], for instance, considers only spatial regions which are n-dimensional blocks whose sides are parallel to the defining axes. The most restricted spatial regions are (one-dimensional) intervals. A prototypical spatial application of intervals are traffic scenarios. Vehicles usually move only along given ways (also sea-/airways). Therefore, when looking at vehicles on one particular way, vehicles and their regions of influence (such as safety margin, braking distance, or reaction distance) could be represented as intervals on a line which represents the possibly winded way. Similar to the well-known Interval Algebra [Allen, 1983] developed for temporal intervals, it seems useful to develop a spatial interval algebra for spatial intervals.

There are several differences between spatial and temporal intervals which have to be considered when extending the Interval Algebra towards dealing with spatial applications. (1) spatial intervals can have different directions, either the same or the opposite direction as the underlying line. (2) ways usually have more than one lane where vehicles can move, i.e., it should be possible to represent that intervals are on different lanes and that one interval is, e.g., left of, right of, or beside another interval. (3) it is interesting to represent intervals on way networks instead of considering just isolated ways. (4) intervals such as those corresponding to regions of influence often depend on the speed of vehicles, i.e., it should be possible to represent dynamic information. This is also necessary for predicting future positions of vehicles which is an important task in traffic control. As for temporal intervals it is also important to represent qualitative or metric information on the length of intervals and on the distance between intervals.

We start this spatial odyssey of the Interval Algebra by analyzing the first important difference between spatial and temporal intervals, namely, direction of intervals. We define the directed intervals algebra which consists of 26 jointly exhaustive and pairwise disjoint *base relations*, identify tractable subsets, and show that path-consistency decides consistency for a particular subset which contains all base relations.

2 Directed Intervals

The Interval Algebra (IA) describes the possible relationships between convex intervals on a directed line. The default application of the Interval Algebra is temporal, so the directed line is usually considered to be the timeline. The 13 IA base relations (before $\prec$, after $\succ$, meets m, met-by mi, overlaps o, overlapped-by oi, equals $\equiv$, during d, includes di, starts s, started-by si, finishes f, and finished-by fi) describe a combination of topological relations (disconnected, externally connected, partial overlap, equal, non-tangential proper part, tangential proper part, and the converse of the latter two) and order relations ($<$,$>$). The topological distinctions are exactly those which are made by RCC8. Therefore, RCC8 is often considered as the spatial counterpart of the Interval Algebra. Or, from another point of view, what distinguishes the Interval Algebra from RCC8 and what makes it its key

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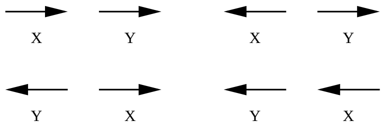

Figure 1: Four structurally different instantiations of the relation "x behind y" with directed intervals

feature is the given direction of the (one-dimensional) line. This given direction naturally imposes a direction also on the intervals: an interval can have the same or the opposite direction as the underlying line. However, because of its original temporal interpretation (no event can end before it starts), direction of intervals has never been considered in AI. Actually, directed intervals have been studied in the large field of Interval Arithmetics, but work in this field is completely different from the qualitative and constraint-based approaches studied in AI. When using the Interval Algebra for spatial applications, direction of intervals has to be taken into account. This leads to the obvious question: can the large body of work and the large number of results obtained on the Interval Algebra such as algorithms and complexity results also be applied to a spatial interpretation of the Interval Algebra, or is it necessary to completely start from scratch again?

Before answering this question, consider the example of Figure 1 which illustrates the differences of having directed intervals from having only intervals of the same direction. Since all four combinations of the directions of the two intervals are possible, there are four structurally different instantiations of every relation instead of just one. Therefore, it is possible that inconsistent instances of the Interval Algebra become consistent when allowing directed intervals.

3 The Directed Intervals Algebra

A straightforward way for dealing with directed intervals would be to add additional constraints on the direction of intervals to constraints over the Interval Algebra and treat the two types of constraints separately while propagating information from one type to the other (similar to what has been done in [Gerevini and Renz, 1998].) We say that an interval has *positive direction* if it has the same direction as the underlying line and *negative direction* otherwise. So possible direction constraints could be unary constraints like "x has positive/negative direction" or binary constraints like "x and y have the same/opposite direction". This approach, however, is not possible since the Interval Algebra loses its property of being a relation algebra when permitting directed intervals. This can be easily seen when considering the "behind" relation of Figure 1. The converse of "x behind y" is "y is behind or in front of x", whose converse is "x is behind or in front of y", i.e., applying the converse operation ($\cdot^{\smile}$) twice leads to a different relation than the original relation. This is a contradiction to one of the requirements of relation algebras ($R^{\smile\smile} = R$) [Ladkin and Maddux, 1994]. This contradiction does not occur when we refine the "behind" relation into two disjoint sub-relations "behind$_=$" and "behind$_{\neq}$" where the subscript indicates that both intervals have the same ($=$) or opposite ($\neq$) direction. The converse of both relations is "in-front-of$_=$" and "behind$_{\neq}$", respectively. Applying the converse operation again leads to the original relations.

Directed Intervals Base Relation	Symbol	Pictorial Example
x behind$_=$ y / y in-front-of$_=$ x	b$_=$ / f$_=$	-x-> -y->
x behind$_{\neq}$ y	b$_{\neq}$	<-x- -y->
x in-front-of$_{\neq}$ y	f$_{\neq}$	-x-> <-y-
x meets-from-behind$_=$ y / y meets-in-the-front$_=$ x	mb$_=$ / mf$_=$	–x–> –y–>
x meets-from-behind$_{\neq}$ y	mb$_{\neq}$	<–x– –y–>
x meets-in-the-front$_{\neq}$ y	mf$_{\neq}$	–x–> <–y–
x overlaps-from-behind$_=$ y / y overlaps-in-the-front$_=$ x	ob$_=$ / of$_=$	—x—> —y—>
x overlaps-from-behind$_{\neq}$ y	ob$_{\neq}$	<—x— —y—>
x overlaps-in-the-front$_{\neq}$ y	of$_{\neq}$	—x—> <—y—
x contained-in$_=$ y / y extends$_=$ x	c$_=$ / e$_=$	–x–> —y—>
x contained-in$_{\neq}$ y / y extends$_{\neq}$ x	c$_{\neq}$ / e$_{\neq}$	<–x– —y—>
x contained-in-the-back-of$_=$ y / y extends-the-front-of$_=$ x	cb$_=$ / ef$_=$	–x–> —y—>
x contained-in-the-back-of$_{\neq}$ y / y extends-the-back-of$_{\neq}$ x	cb$_{\neq}$ / eb$_{\neq}$	<–x– —y—>
x contained-in-the-front-of$_=$ y / y extends-the-back-of$_=$ x	cf$_=$ / eb$_=$	–x–> —y—>
x contained-in-the-front-of$_{\neq}$ y / y extends-the-front-of$_{\neq}$ x	cf$_{\neq}$ / ef$_{\neq}$	<–x– —y—>
x equals$_=$ y	eq$_=$	—x—> —y—>
x equals$_{\neq}$ y	eq$_{\neq}$	—x—> <—y—

Table 1: The 26 base relations of the directed intervals algebra

Since a relation algebra must be closed under composition, intersection, and converse, we have to make the same distinction also for all other IA relations. This leads us to the definition of the directed intervals algebra (DIA). It consists of the 26 base relations given in Table 1, which result from refining each IA relation into two sub-relations specifying either same or opposite direction of the involved intervals, and of all possible unions of the base relations. This gives a total number of 2^{26} DIA relations. Converse relations are given in the same table entry. If a converse relation is not explicitly given, the corresponding relation is its own converse. We denote the set of 26 DIA base relations as $\mathcal{B}$. Then DIA $= 2^{\mathcal{B}}$. Complex relations which are the union of more than one base relation $R_1, \ldots, R_k$ are written as $\{R_1, \ldots, R_k\}$. The union of all base relations, the universal relation, is denoted $\{*\}$.

A DIA base relation $R = I_d$ consist of two parts, the interval part I which is a spatial interpretation of the Interval Algebra and the direction part d which gives the mutual direction of both intervals, either $=$ or $\neq$. If a complex relation R consist of base relations with the same direction part d, we can combine the interval parts and write $R = \{I^1, \ldots, I^k\}_d$ instead of $R = \{I^1_d, \ldots, I^k_d\}$. We write R_e (resp. R_n) in

R	$\prec$	$\succ$	m	mi	o	oi	s	si	d	di	f	fi	$\equiv$
R^r	$\succ$	$\prec$	mi	m	oi	o	f	fi	d	di	s	si	$\equiv$
$\mathsf{dia}(R)$	b	f	mb	mf	ob	of	cb	ef	c	e	cf	eb	eq

Table 2: IA base relations R, their reverses R^r, and their spatial interpretations $\mathsf{dia}(R)$

order to refer to the union of the interval parts of every subrelation of a complex relation R where the direction part is $\{=\}$ (resp. $\{\neq\}$.) In this way, every DIA relation R can be written as $R = \{R_e\}_= \cup \{R_n\}_{\neq}$. DIA_I denotes the set of 2^{13} possible interval parts of DIA relations.

It is important to note that the spatial interpretation of the Interval Algebra was chosen in a way that the interval part of a relation xI_dy only depends on the direction of y and not on the direction of x. Therefore, if the direction of x is reversed, written as $\overline{x}$, then only the direction part changes, i.e., $xI_dy = \overline{x}I_{\neg d}y$. This would not be the case in a straightforward spatial interpretation of the original temporal relations. For instance, IA relations like "x started-by y" or "x finished-by y" depend on the direction of x. Instead, we interpret these relations spatially as "x extends-the-front/back-of y" and "x contained-in-the-front/back-of y". This interpretation is independent of the direction of x. When all intervals have the same direction, both interpretations are equivalent. In order to transform the spatial and the temporal interval relations (independent of the direction of the intervals) into each other, we introduce two mutually inverse functions $\mathsf{dia} : \mathsf{IA} \mapsto \mathsf{DIA}_I$ and $\mathsf{ia} : \mathsf{DIA}_I \mapsto \mathsf{IA}$, i.e., $\mathsf{dia}(\mathsf{ia}(R)) = R$ and $\mathsf{ia}(\mathsf{dia}(R)) = R$. The mapping is given in Table 2.

All relations of the directed intervals algebra are invariant with respect to the direction of the underlying line, i.e., when reversing the direction of the line, all relations remain the same. This is obviously not the case for the Interval Algebra, e.g., if x is before y and one reverses the direction of the timeline, then x is after y. In order to transform DIA relations into the corresponding IA relations and *vice versa*, we introduce a unary *reverse* operator $(\cdot^r)$ on relations R such that R^r specifies the relation which results from R when reversing the direction of the underlying line. For all relations $R \in \mathsf{DIA}$ we have that $R^r = R$. For IA relations, the reverse relation is given in Table 2. The reverse of a complex relation is the union of the reverses of the involved base relations. The reverse of the composition $(\circ)$ of two relations is equivalent to the composition of the reverses of the two involved relations, i.e., $(R \circ S)^r = R^r \circ S^r$. Applying the reverse operator twice results in the original relation, i.e., $R^{rr} = R$. Using the reverse operator we can also specify what happens with a relation xI_dy if only the direction of y is changed. Then the topological relation of the intervals stays the same, but the order changes, i.e., "front" becomes "behind"/"back" and *vice versa*. The mutual direction also changes. This can be expressed in the following way: $xI_dy = x\,\mathsf{dia}(\mathsf{ia}(I)^r)_{\neg d}\,\overline{y}$.

We now have all requirements for computing the composition $(\circ)$ of DIA relations using composition of IA relations (denoted here by $\circ_{ia}$) as specified by Allen [1983].

Theorem 3.1 *Let R_p, S_q be* DIA *base relations.*

1. *If $q = \{=\}$, then $R_p \circ S_q = \mathsf{dia}(\mathsf{ia}(R) \circ_{ia} \mathsf{ia}(S))_p$*
2. *If $q = \{\neq\}$, then $R_p \circ S_q = \mathsf{dia}(\mathsf{ia}(R)^r \circ_{ia} \mathsf{ia}(S))_{\neg p}$*

Proof. Assume that xR_py and yS_qz holds. If $p = q = \{=\}$ and x, y, z have positive direction, it is clear that the interval part of the composition of the DIA relations is the same as the composition of the IA relations (with respect to the different interpretations.) The result of the composition is the same if x, y, z have negative direction, since DIA relations are invariant with respect to the direction of the underlying line. R only depends on the direction of y and S only depends on the direction of z. Therefore, reversing the direction of x (i.e., $p = \{\neq\}, q = \{=\}$) does not change the result of the interval part of the composition, only the resulting direction part. This proves the first rule.

Assume that $p = q = \{\neq\}$ and x, z have positive direction while y has negative direction. If we reverse the direction of y, which changes the relations to $x\,\mathsf{dia}(\mathsf{ia}(R)^r)_=\,\overline{y}$ and to $\overline{y}S_=z$, then we can apply the first composition rule. This results in $R_{\neq} \circ S_{\neq} = \mathsf{dia}(\mathsf{ia}(R)^r)_= \circ S_= =_{(1.)} \mathsf{dia}(\mathsf{ia}(\mathsf{dia}(\mathsf{ia}(R)^r)) \circ_{ia} \mathsf{ia}(S))_= = \mathsf{dia}(\mathsf{ia}(R)^r \circ_{ia} \mathsf{ia}(S))_=$, the second composition rule. As in the first case, this rule does not change when we reverse the direction of x (i.e., $p = \{=\}, q = \{\neq\}$) or the direction of all three intervals. This proves the second rule. ■

The composition of complex relations is as usual the union of the composition of the contained base relations. It follows from the closedness of the Interval Algebra that DIA is closed under composition, intersection, converse, and reverse.

4 Reasoning over Directed Intervals

The main reasoning problem in spatial and temporal reasoning is the consistency problem $\mathsf{CSPSAT}(\mathcal{S})$ where $\mathcal{S}$ is a set of relations over a relation algebra [Renz and Nebel, 1999].

Instance: A set $\mathcal{V}$ of variables over a domain $\mathcal{D}$ and a finite set Θ of binary constraints xRy ($R \in \mathcal{S}$ and $x, y \in V$.)

Question: Is there a consistent instantiation of all n variables in Θ with values from $\mathcal{D}$ which satisfies all constraints?

The consistency problem of the directed intervals algebra, CSPSAT(DIA), is clearly NP-hard since the consistency problem of the Interval Algebra is already NP-hard. On the other hand it is not clear whether the consistency problem is tractable if only the DIA base relations are used.

Additional to the DIA relations, we also give the possibility of explicitly specifying the direction of intervals. We maintain them in a set Δ which contains unary direction constraints of the form (x, d) where x is a variable over a directed interval and $d \subseteq \{+, -\}$ gives the direction of x, either positive $\{+\}$, negative $\{-\}$, or indefinite $\{+, -\}$. Unary direction constraints and DIA constraints interact in two ways.

Proposition 4.1 *Given two intervals x, y, the* DIA *constraint xRy with $R = \{R^1_{m_1}, \ldots, R^k_{m_k}\}$, and the unary direction constraints (x, d_1) and (y, d_2). These constraints interact in the following way:*

1. *If all m_i ($i = 1 \ldots k$) are equivalent, then (a) $d_1 = d_1 \cap d_2$ and $d_2 = d_1 \cap d_2$ if $m_1 = \{=\}$ and (b) $d_1 = d_1 \cap \neg d_2$ and $d_2 = \neg d_1 \cap d_2$ if $m_1 = \{\neq\}$.*
2. *If d_1 and d_2 are both definite, then (a) $R = \{R_e\}_=$ if $d_1 = d_2$ and (b) $R = \{R_n\}_{\neq}$ if $d_1 \neq d_2$.*

If all information is propagated from Θ to Δ and from Δ to Θ we write the resulting sets as Θ_Δ and Δ_Θ. If the empty constraint occurs during this propagation, then Θ is inconsistent.

There are several ways of deciding consistency of a given set of constraints over a set of relations $\mathcal{S}$. The most common way is to use backtracking over a tractable subset of $\mathcal{S}$ which contains all base relations and enforce *path-consistency* as forward-checking (this is done by applying for each triple of constraints xRy, ySz, xTz the operation $T := T \cap (R \circ S)$; if the empty relation is not contained, the resulting set is path-consistent) [Ladkin and Reinefeld, 1997]. Before we can use this method for deciding CSPSAT(DIA), we must prove that the consistency problem is tractable for the DIA base relations and preferably that path-consistency decides consistency for these relations. In order to prove this, we need a different method for deciding consistency and we have to show that this method is polynomial for the set of DIA base relations.

For the Interval Algebra most tractability proofs were carried out using the endpoint encoding of the IA relations (e.g. [Nebel and Bürckert, 1995]) which describes the qualitative relations between the four endpoints of the two involved intervals. For instance, the "before" relation can be encoded as $X_e < Y_s$ plus the default relations $X_s < X_e$ and $Y_s < Y_e$ which hold for all non-directed intervals (X_s, Y_s denote the start points and X_e, Y_e the end points of the intervals X, Y.) It is also possible to specify an endpoint encoding of the DIA relations. Since spatial intervals can have different directions, the default relations do not hold anymore. Furthermore, we have to take into consideration that DIA relations are invariant with respect to the reverse operation. Therefore, it is the most compact way to use the "betweenness" predicate for specifying an endpoint encoding of DIA relations. $between(X_e, X_s, Y_s)$ means that X_s is between X_e and Y_s, no matter which direction the intervals have. Using this predicate, the relation $\mathsf{b}_{\neq}$, for instance, can be encoded as $between(X_s, Y_s, Y_e) \wedge between(X_e, X_s, Y_s)$. Since the BETWEENNESS problem is NP-hard [Garey and Johnson, 1979], this encoding does not seem to be helpful for proving any tractability results. We will therefore refrain from specifying the endpoint formulas of the DIA base relations.

Another possibility of deciding the DIA consistency problem is to transform a set of DIA constraints Θ into an equivalent set of IA constraints Θ' and decide consistency of Θ'. In order to make such a transformation, the direction of every interval must be known. Then it is possible to reverse the direction of certain intervals such that all intervals have the same direction and transform the updated DIA constraints into IA constraints. We call this the *normal form* of a set of DIA constraints Θ and a set of definite unary direction constraints Δ for each interval involved in Θ. The normal form (written as $\mathsf{nf}(\Theta, \Delta)$) is obtained as follows.

Proposition 4.2 *Given a set of* DIA *constraints* Θ *and a set* Δ *of definite unary direction constraints for each interval involved in* Θ*. The normal form* $\mathsf{nf}(\Theta, \Delta)$ *is obtained by applying the following procedure.*

1. *For each constraint* $xR_dy \in \Theta_\Delta$ *do*
2. *If y has negative direction, add* $x\ \mathsf{ia}(R)^r\ y$ *to* $\mathsf{nf}(\Theta, \Delta)$
3. *If y has positive direction, add* $x\ \mathsf{ia}(R)\ y$ *to* $\mathsf{nf}(\Theta, \Delta)$

Lemma 4.3 *Given a set of* DIA *constraints* Θ *and a set* Δ *of definite unary direction constraints for each interval involved in* Θ*.* $\mathsf{nf}(\Theta, \Delta)$ *can be computed in time* $O(n^2)$*.*

Proof. Θ_Δ can be computed in time $O(n^2)$, since all constraints of Δ are definite and information has to be propagated only from every pair of intervals to the corresponding constraint in Θ using rule 2 of Proposition 4.1. Θ_Δ is transformed to $\mathsf{nf}(\Theta, \Delta)$ in time $O(n^2)$, since each of the $O(n^2)$ constraints is transformed separately in constant time. ■

Lemma 4.4 *Given a set of* DIA *constraints* Θ *and a set* Δ *of definite unary direction constraints for each interval involved in* Θ*.* Θ_Δ *is consistent if and only if* $\mathsf{nf}(\Theta, \Delta)$ *is consistent.*

Proof. Suppose that Θ_Δ is consistent and that $\mathcal{I}$ is an instantiation of Θ_Δ. The direction of each interval of $\mathcal{I}$ is as specified in Δ and the relation between each pair of intervals x, y is a base relation R'_d which is a sub-relation of $xR_dy \in \Theta_\Delta$. We can now reverse the direction of all intervals of $\mathcal{I}$ with negative directions, resulting in $\mathcal{I}^+$. Since all DIA relations xRy only depend on the direction of y, the relations between the intervals of $\mathcal{I}^+$ are now $x\ \mathsf{dia}(\mathsf{ia}(R')^r)_=\ y$ if the direction of y was negative in $\mathcal{I}$ and $xR'_=y$ if the direction of y was positive in $\mathcal{I}$. Transforming these relations into IA relations results for every pair of intervals in sub-relations of $\mathsf{nf}(\Theta, \Delta)$. Thus, $\mathcal{I}^+$ is a consistent instantiation of $\mathsf{nf}(\Theta, \Delta)$. The opposite direction can be proved similarly. Suppose that $\mathsf{nf}(\Theta, \Delta)$ is consistent and that $\mathcal{J}$ is an instantiation of it where all intervals are considered to have positive direction. Let Θ^+ be the set of constraints between all intervals of $\mathcal{J}$ using DIA base relations. Reversing the direction of all intervals which must have negative direction according to Δ results in $\mathcal{J}^\pm$ and adopting the constraints of Θ^+ results in $\Theta^\pm$. Since applying the reverse operator twice gives the original relation, each constraint of $\Theta^\pm$ is a sub-constraint of a constraint of Θ_Δ. Thus, $\mathcal{J}^\pm$ is a consistent instantiation of Θ_Δ. ■

Using the normal form, we can now decide consistency of a set of DIA constraints Θ by computing or guessing a set Δ containing the direction of all intervals, computing $\mathsf{nf}(\Theta, \Delta)$, and deciding consistency of $\mathsf{nf}(\Theta, \Delta)$ using the methods developed for the Interval Algebra. Since there are 2^n different direction combinations of n directed intervals, it is in general NP-hard to find a suitable set Δ for which $\mathsf{nf}(\Theta, \Delta)$ is consistent or to show that there is no such set. If, however, we can show that for a given set $\mathcal{S}$ of DIA relations all possible candidate sets Δ can be identified in polynomial time and if $\mathsf{nf}(\Theta, \Delta)$ contains only relations of a tractable subset of the Interval Algebra, then CSPSAT($\mathcal{S}$) is tractable. Using this method, we identify several tractable subsets of the directed intervals algebra in the following section.

5 Tractable Subsets of DIA

The first set we analyze is the set of DIA base relations $\mathcal{B}$.

Lemma 5.1 CSPSAT*($\mathcal{B} \cup \{*\}$) is tractable.*

Proof. Consistency of a set Θ of constraints over $\mathcal{B} \cup \{*\}$ can be decided in polynomial time by using the following steps.

1. Transform Θ into a graph $G_\Theta = (V, E)$ where V is the set of variables involved in Θ and E contains an (undirected) edge (x, y) if $xRy \in \Theta$ where $R \in \mathcal{B}$.

2. Split V into disjoint subsets $V = V_1 \cup \ldots \cup V_k$ such that for each pair of variables $x, y \in V_i$ there is a path from x to y in E and for each pair of variables $x \in V_i, y \in V_j$ $(i \neq j)$ there is no path from x to y in E.
3. Generate a set of direction constraints Δ by selecting one variable x_i for each V_i and adding $(x_i, \{+\})$ to Δ.
4. Compute $\mathsf{nf}(\Theta, \Delta_\Theta)$ and decide its consistency.

It is clear that each of the four steps can be computed in polynomial time. For each pair of variables of different sets V_i, V_j there are only constraints involving the universal relation, i.e., $\Theta(V_i)$ and $\Theta(V_j)$ which specify subsets of Θ containing all constraints involving only variables of V_i or V_j, respectively, are completely independent of each other. There is a path from each variable of V_i to every other variable of V_i and each path consists of constraints where each constraint involves only DIA relations with the same direction part. Therefore, it is sufficient to have the direction of only one variable $x_i \in V_i$ given in order to compute the direction of all variables $y \in V_i$. If the path contains an odd number of constraints involving DIA relations of the type $R_{\neq}$, the direction of y is opposite to the direction of x_i. Otherwise they have the same direction. Thus, Δ_Θ contains definite unary direction constraints for all variables of Θ. If there are conflicting paths, then Δ_Θ is inconsistent. Since DIA relations are invariant with respect to changing the direction of the underlying line, it does not matter for consistency purposes if we select the direction of x as positive or negative. $\mathsf{nf}(\Theta, \Delta_\Theta)$ contains only relations of a tractable subset of IA. It follows from Lemma 4.4 that its consistency is equivalent to the consistency of Θ. ■

In the above proof it is not important that all non-universal relations are base relations, only that all non-universal relations consist of DIA base relations with the same direction part. Therefore, we can easily extend the above result.

Theorem 5.2 *Let $\mathcal{S}$ be a tractable subset of the Interval Algebra which is closed under the reverse operator. Then $\mathcal{S}^{\pm} = \{\mathsf{dia}(R)_= | R \in \mathcal{S}\} \cup \{\mathsf{dia}(R)_{\neq} | R \in \mathcal{S}\} \cup \{*\}$ is a tractable subset of the directed intervals algebra.*

Proof. We can apply the same proof as given for Lemma 5.1. But only if all IA relations contained in the normal form are contained in a tractable subset of the Interval Algebra. This is clearly the case if $\mathcal{S}$ is closed under the reverse operator which is used in the transformation into the normal form. ■

ORD-Horn (also denoted $\mathcal{H}$) is the only maximal tractable subset of the Interval Algebra which contains all IA base relations (and for which path-consistency decides consistency) [Nebel and Bürckert, 1995]. Using a machine-assisted comparison of the ORD-Horn relations we found that they are closed under the reverse operator. This is not true for some of the maximal tractable subclasses identified in [Drakengren and Jonsson, 1998] which do not contain all IA base relations.

All tractability results we have given so far rely on given mutual directions of intervals. For some applications this is a realistic assumption, but what happens if this is not given in all cases, if some constraints involve relations with different direction parts such as $x\{\mathsf{b}_=, \mathsf{c}_{\neq}, \mathsf{mb}_{\neq}\}y$? Assume that we have given a set Θ of constraints over arbitrary DIA relations.

x_d	y_d	z_d	xRy	ySz	xTz
$+$	$+$	$+$	$\mathsf{ia}(R_e)$	$\mathsf{ia}(S_e)$	$\mathsf{ia}(T_e)$
$+$	$+$	$-$	$\mathsf{ia}(R_e)$	$\mathsf{ia}(S_n)^r$	$\mathsf{ia}(T_n)^r$
$+$	$-$	$+$	$\mathsf{ia}(R_n)^r$	$\mathsf{ia}(S_n)$	$\mathsf{ia}(T_e)$
$+$	$-$	$-$	$\mathsf{ia}(R_n)^r$	$\mathsf{ia}(S_e)^r$	$\mathsf{ia}(T_n)^r$
$-$	$+$	$+$	$\mathsf{ia}(R_n)$	$\mathsf{ia}(S_e)$	$\mathsf{ia}(T_n)$
$-$	$+$	$-$	$\mathsf{ia}(R_n)$	$\mathsf{ia}(S_n)^r$	$\mathsf{ia}(T_e)^r$
$-$	$-$	$+$	$\mathsf{ia}(R_e)^r$	$\mathsf{ia}(S_n)$	$\mathsf{ia}(T_n)$
$-$	$-$	$-$	$\mathsf{ia}(R_e)^r$	$\mathsf{ia}(S_e)^r$	$\mathsf{ia}(T_e)^r$

Table 3: Transformation of DIA constraints over a triple of variables x, y, z (depending on their directions x_d, y_d, z_d) into IA relations of the normal form.

One way of obtaining a possible candidate set Δ of definite unary directions for each interval in Θ is to look at each triple of variables (x, y, z) of Θ separately and check all 2^3 different combinations of directions (x_d, y_d, z_d) of (x, y, z) (x_d can be either $+$ or $-$). If enforcing path-consistency to the normal form gives the empty relation for a particular choice of (x_d, y_d, z_d), then this choice makes Θ inconsistent. If we combine all such inconsistent triples $t_i = \{(x, x_d), (y, y_d), (z, z_d)\}$, then Θ is inconsistent if $\Phi = \bigvee_i (l_x \wedge l_y \wedge l_z)$ is satisfiable (l_x is a placeholder for $\neg x_d$ if $(x, -) \in t_i$ and for x_d if $(x, +) \in t_i$, analogously for l_y and l_z.) A possible candidate set Δ can be obtained by computing a model of the complement of this formula, namely, $\Psi = \neg\Phi = \bigwedge_i (\neg l_x \vee \neg l_y \vee \neg l_z)$. This formula is an instance of 3SAT and, thus, NP-hard to compute. Eventually, since we did not propagate information between different triples, we have to check all possible models of Ψ.

Because of its NP hardness, this way of generating a candidate set Δ does not seem to be helpful. However, we can show that it leads to a tractability proof for a restricted but interesting set of DIA relations, namely, those DIA relations $\mathcal{B}_{\mathcal{A}}$ which correspond to the set $\mathcal{A}$ of 13 base relations of the Interval Algebra, i.e., $\mathcal{B}_{\mathcal{A}} = \{\{\mathsf{dia}(R)_=, \mathsf{dia}(R)_{\neq}\} | R \in \mathcal{A}\}$.

Theorem 5.3 *Let Θ be a set of DIA constraints over the variables $x_1, \ldots, x_n$ which contains a constraint $x_i R x_j$ with $R \in \mathcal{B}_{\mathcal{A}}$ if and only if $i < j$. Consistency of Θ can be decided in polynomial time.*

Proof. For each triple of variables x_i, x_j, x_k of Θ with $i \neq j \neq k, i < k$ we check for all 2^3 possible directions if the normal form of the triple is consistent or not. Depending on whether $i < j$, either $x_i R x_j$ or $x_j R x_i$ is given in Θ. Equivalently, either $x_j S x_k$ or $x_k S x_j$ is given in Θ. Since $i < k$, $x_i T x_k \in \Theta$. Obviously, it is not possible that $x_j R x_i$ and $x_k S x_j$ are both in Θ. Therefore, we compare in the normal form either (1) $R \circ S$, (2) $R^{\smile} \circ S$, or (3) $R \circ S^{\smile}$ with T in order to check consistency of the triple. In all three cases, the result is invariant with respect to the direction of one of the three variables and depends only on the direction of the other two variables. We can verify this using Table 3. In the first case, the only thing which changes in Table 3 when varying the direction of x_i (which is x_d in the table) are the subscripts of R and T: R_e changes to R_n, T_e changes to T_n and *vice versa*. Since all relations used in Θ are of the form $\{I_=, I_{\neq}\}$, I_e and I_n are always equivalent, i.e., the result of comparing $R \circ S$ with T is invariant with respect to the direction of x_i. In the second case, the result is invariant with respect to the direction of x_j (which is y_d in the table.) When varying y_d in the table, S_e changes to S_n, $\mathsf{ia}(R_e^\smile)$ changes to $\mathsf{ia}(R_n^\smile)^r$, and $\mathsf{ia}(R_n^\smile)$ changes to $\mathsf{ia}(R_e^\smile)^r$ and *vice versa*. Since S_e is always equal to S_n, the change of S does not change the result. The change of $R^\smile$ is more difficult. Using a case analysis of the 13 different cases, we were able to prove that $\mathsf{ia}(R_e^\smile) = \mathsf{ia}(R_n^\smile)^r$ and that $\mathsf{ia}(R_n^\smile) = \mathsf{ia}(R_e^\smile)^r$ if R is of type $\{I_=, I_{\neq}\}$. As an example consider the relation $R = \{\mathsf{b}_=, \mathsf{b}_{\neq}\}$ whose converse is $R^\smile = \{\mathsf{f}_=, \mathsf{b}_{\neq}\}$. $\mathsf{ia}(\mathsf{f}) = \{\succ\} = \mathsf{ia}(\mathsf{b})^r$ and $\mathsf{ia}(\mathsf{b}) = \{\prec\} = \mathsf{ia}(\mathsf{f})^r$. In the third case, the result is invariant with respect to the direction of x_i which can be proved equivalently to the first case.

Because consistency of every triple depends on the direction of only two of the three variables, the resulting formula Ψ (see above) is an instance of 2SAT and, thus, solvable in polynomial time. For any model of Ψ, the resulting set Δ of unary direction constraints leads to a consistent normal form $\mathsf{nf}(\Theta, \Delta)$. This is because $\mathsf{nf}(\Theta, \Delta)$ contains only IA base relations and because all triples are consistent (as it has been checked when Ψ was generated.) Therefore, enforcing path-consistency does not change any relation of $\mathsf{nf}(\Theta, \Delta)$. ■

Instead of transforming every set Θ of DIA constraints to the normal form in order to decide consistency, it would be nice to know if and for which sets of DIA relations path-consistency is sufficient for deciding consistency when applied directly to Θ. We show this for $\mathcal{H}^{\pm}$, the set of DIA relations which results from ORD-Horn (see Theorem 5.2).

Theorem 5.4 *Path-consistency decides* CSPSAT*($\mathcal{H}^{\pm}$).*

Proof. Consistency of a set Θ of DIA constraints over $\mathcal{H}^{\pm}$ can be decided in polynomial time by deciding consistency of its normal form Θ' as it is obtained by applying the steps given in the proof of Lemma 5.1. Assume that Θ is path-consistent, i.e., for every triple of variables x, y, z with $xRy, ySz, xTz \in \Theta$ we have that $T \subseteq R \circ S$. If Θ' is also path-consistent, then Θ is consistent. In order to show that Θ' is path-consistent, we have to show that $T \subseteq R \circ S$ implies $T' \subseteq R' \circ_{ia} S'$ for all triples x, y, z $(xR'y, yS'z, xT'z \in \Theta'.)$ Since all relations of $\mathcal{H}^{\pm}$ consist of DIA base relations with the same direction part, we can extend the composition rules given in Theorem 3.1 to all $\mathcal{H}^{\pm}$ relations. According to these rules, $T \subseteq R \circ S$ can be written as either $T_{\neq} \subseteq \mathsf{dia}(\mathsf{ia}(R_e)^r \circ_{ia} \mathsf{ia}(S_n))_{\neq}$, $T_{\neq} \subseteq \mathsf{dia}(\mathsf{ia}(R_n) \circ_{ia} \mathsf{ia}(S_e))_{\neq}$, $T_= \subseteq \mathsf{dia}(\mathsf{ia}(R_n)^r \circ_{ia} \mathsf{ia}(S_n))_=$, or $T_= \subseteq \mathsf{dia}(\mathsf{ia}(R_e) \circ_{ia} \mathsf{ia}(S_e))_=$ depending on the directions of x, y, z. The corresponding restrictions of T' can be derived using Table 3. If $z_d = \{+\}$, the restrictions of T' are equivalent to the results of applying ia to the interval parts of the above given restrictions of T. If $z_d = \{-\}$, we have to reverse T and the restrictions of T. Since $(U \circ_{ia} V)^r = U^r \circ_{ia} V^r$ and $U^{rr} = U$ for all IA relations U, V, the restrictions of T' are also equivalent to the results of applying ia to the interval parts of the restrictions of T (e.g. the first restriction $(\mathsf{ia}(R_e)^r \circ_{ia} \mathsf{ia}(S_n))^r = \mathsf{ia}(R_e) \circ_{ia} \mathsf{ia}(S_n)^r$ is the same as line 2 in Table 3.) Thus, $T \subseteq R \circ S$ implies $T' \subseteq R' \circ_{ia} S'$. ■

This result enables us to decide consistency of arbitrary sets Θ of DIA constraints by backtracking over the $\mathcal{H}^{\pm}$ relations and by using path-consistency as a forward-checking method and as a decision procedure for sub-instances of Θ which contain only relations of $\mathcal{H}^{\pm}$.

6 Discussion & Future Work

We extended the Interval Algebra for dealing with directed intervals which occur when interpreting intervals as spatial instead of temporal entities. Reasoning over the directed intervals algebra DIA is more difficult than over the Interval Algebra IA, but it is possible to transform a set of DIA constraints into an equivalent set of IA constraints if the mutual directions of all intervals are known. This enabled us to transfer some tractability results from IA to DIA. If the mutual directions are not known, a tractable subset of DIA can be identified if two potentially exponential nested problems can be shown to be tractable for the subset: (a) compute a set of mutual directions and (b) decide consistency of the resulting set of constraints. We proved this for a small but interesting subset of DIA, but the problem is mostly open. Consequently, no maximal tractable subset of DIA has been identified so far.

DIA can also be used instead of IA as a basis for defining a block algebra [Balbiani *et al.*, 1999]. Then it is possible to reason about n-dimensional blocks with intrinsic directions. Remotely related to directed intervals are line segments with arbitrary directions which were analyzed by Moratz [2000].

The spatial odyssey of the Interval Algebra is to be continued as follows: (1) extend DIA to deal with intervals on parallel lines and on networks of lines, (2) add qualitative and metric information on the length of intervals and on the distance between intervals, and finally (3) extend the algebra to deal with dynamic instead of just static information, e.g., intervals move on lines with a certain velocity and sometimes switch to accessible lines. These are the desired properties of a calculus for representing and reasoning about traffic scenarios, a prototypical application of spatial intervals. Hopefully, contact with applications will be made before 2010...

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