Qualitative Representation and Reasoning over Direction Relations across Different Frames of Reference

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Abstract

Direction relations are some of the most commonly used spatial relations in human communication. A number of formalisms have been proposed to capture direction information, but all of them suffer from the same problem: people use direction relations as if they are binary relations (e.g. A is to the left of B) when they are in fact ternary relations (e.g. A is to the left of B when viewed from C). This implicit third piece of information that is required, but typically missing, is often referred to as the frame of reference (FoR). Given only binary direction relations without knowing the FoR means that we cannot do proper spatial reasoning (e.g. A is to the left of B and A is to the right of B are both consistent), we cannot integrate direction information from different sources, not even from the same source, and it can be difficult to understand what exactly someone means.

In this paper we provide 1) the spatial constraint language $\mathcal{DAF}_m$ that can represent and compose direction relations across different FoRs; and 2) the foundations for deciding the overall consistency. We hope our model of representing and reasoning can bring research on direction relations to another level and will finally make it possible to properly use them in a comprehensive way.

Introduction

Representing spatial information and being able to reason about it is crucial in the field of Artificial Intelligence (Cohn and Renz 2008). Direction relations are very common in practical spatial descriptions and therefore have been studied extensively in the last two decades. Numerous formalisms have been created to model and reason about direction relations (Frank 1991; Freksa 1992; Ligozat 1993; Isli et al. 2001; Renz and Mitra 2004; Clementini and Billen 2006; Moratz 2006; Moratz and Ragni 2008; Lee et al. 2013). They are defined in different categories of **frames of reference** (**FoR**), which provides a coordination system (Frank 1998) based on which direction relations (and distance relations (Moratz and Ragni 2008)) can be described. Several different groups of terms have been used to represent the roles of spatial entities in direction relation descriptions. For example, **figure**, **ground**, and **viewpoint** (Levinson 1996); **locatum**, **relatum** (Tenbrink and Kuhn 2011), and **observer**. The latter group of terms are used throughout the paper, where locatum is the entity that is to be located by relating to the position of a relatum; observers provide perspectives and they might or might not appear in a description (Tenbrink and Kuhn 2011). According to Levinson's taxonomy (Levinson 1996), there are three categories of FoRs (**absolute**, **intrinsic**, and **relative**). Their difference can be explained with a practical route description example "go south at the next intersection until you see a bus station to your left; facing to the bus station you can see the library is to the right of the station" (Figure 1a):

- In absolute FoRs, an invariant direction (Levinson 1996) (e.g. north) is used as the reference. For example, the direction "you" (relatum) go is "south" because it is opposite to north (Figure 1b). Examples of formalisms defined in absolute FoRs are Cardinal Direction Algebra ($\mathcal{CDA}$) (Frank 1991) and $\mathcal{STAR}_m$ (Renz and Mitra 2004).
- In intrinsic FoRs, direction relations among spatial entities are described according to the "inherent features" (Levinson 1996) of relatums. For example, "a bus station" (locatum) is "to your left" based on the direction "you" (relatum) are facing to (Figure 1c). Examples of formalisms defined in intrinsic FoRs are Oriented Point Relation Algebra ($\mathcal{OPRA}_m$) (Moratz 2006); and $\mathcal{SV}_m$ (Lee et al. 2013).
- In relative FoRs, the line that connects the observer and the relatum is applied as the baseline. For example, "the library" (locatum) is described as "to the right of the station" based on the fact that "you" (observer) look at "the station" (relatum) (Figure 1d). Examples of formalisms defined in relative FoRs are Single Cross & Double Cross (Freksa 1992); flipflop calculus (Ligozat 1993); and Ternary Point Configuration Calculus (TPCC) (Moratz and Ragni 2008).

Most existing direction relation formalisms are defined in a specific category of FoR (absolute, intrinsic or relative). However, in practice direction relations could be expressed in different categories of FoRs (e.g. the above route description example), which means that by using any single existing formalism it is almost impossible to express, integrate and reason about direction information from different sources.

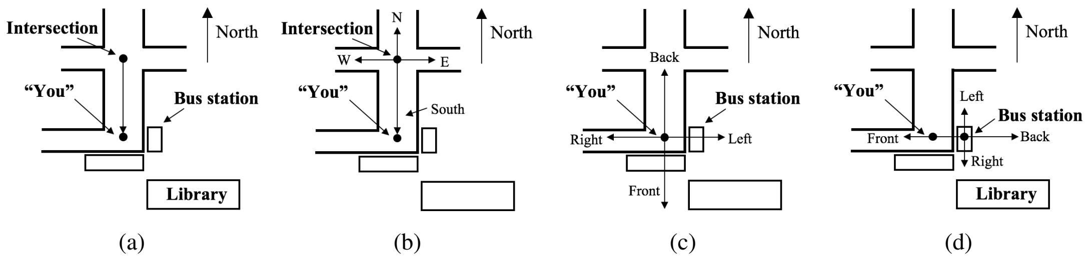


Figure 1: (a) Start at the intersection and then go south until the library is located by relating to the bus station.(b) Absolute FoR: based on north, the direction "you" go is south. (c) Intrinsic FoR: based on the direction "you" are facing to the bus station is to the left of "you". (d) Relative FoR: based on the direction from the bus station to "you" the library is to the right of the station.

This problem is critical because for example a set of direction relations that is consistent in absolute FoRs (Figure 2a) might turn out to be invalid when combining with another set of direction relations described in relative FoRs (Figure 2b).

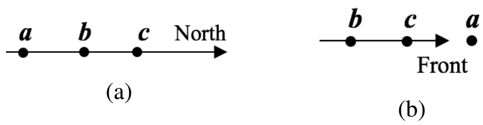


Figure 2: (a) "b is to the north of a and c is to the north of b", which is inconsistent with (b) "from a's perspective, c is in front of b", because based on the former, b is between a and c while based on latter b is not.

To solve this problem, in this paper we provide 1) an inclusive representation that can express and integrate direction information across different FoRs; 2) methods of across-FoR compositions; and 3) the foundations for deciding the overall consistency.

Related work

Direction relations are very common and numerous formalisms have been created to model and reason about direction relations in the last two decades. For example, cone-based and projection-based approaches for cardinal directions (Frank 1991); the flipflop calculus (Ligozat 1993); cardinal direction relations with arbitrary granularity ($\mathcal{STAR}_m$) (Renz and Mitra 2004); relative direction relations between oriented points ($\mathcal{OPRA}_m$) (Moratz 2006); ternary relative direction relations between points (TPCC) (Moratz and Ragni 2008); and observer orientation-based relative directions ($\mathcal{SV}_m$) (Lee et al. 2013).

However, previous research on the problem of representing and reasoning about direction relations across different FoRs is limited and we are the first to solve this problem in a general sense. For example, in (Hernandez 1994) Hernandez described how to transform direction relations in intrinsic, extrinsic, and deictic FoRs into implicit FoRs; in (Clementini 2013) Clementini provides a framework that can convert relations in different FoRs into his 5-intersection model. However, in both of these two works across-category composition and reasoning have not been discussed.

Our model could be the most inclusive direction relation formalism so far because it is compatible to direction information expressed in most of the previous direction relation formalisms. For example, $\mathcal{STAR}_m$ (Renz and Mitra 2004), $\mathcal{OPRA}_m$ (Moratz 2006) and TPCC (Moratz and Ragni 2008) can be regarded as special cases of our model because similar space division strategy is applied while each of them just cover one category of FoRs.

Representation

Spatial entities

Spatial entities can be modeled as points e.g. in (Frank 1991; Ligozat 1993; Isli et al. 2001; Renz and Mitra 2004; Moratz 2006; Moratz and Ragni 2008; Lee et al. 2013); or as regions e.g. in (Randell, Cui, and Cohn 1992; Clementini and Billen 2006). Point representation of spatial entities is more popular for theoretical analysis because irrelevant features could be ignored while regions are more close to real-life spatial objects. In our model, spatial entities are oriented points (Mossakowski and Moratz 2012) in 2-D Euclidean plane: given an oriented point $a = (x_a, y_a, \phi_a) \in D_{op}$ where $D_{op} = \mathbb{R}^2 \times \Theta$ and $\Theta = [0, 2\pi)$, $(x_a, y_a) \in \mathbb{R}^2$ decides the position of a and the value of $\phi_a \in \Theta$ decides the orientation of a.

Frames of reference

As described in the introduction FoRs can be absolute, intrinsic, or relative (which is also called as the **category** of a FoR). Each FoR consists of three elements: the centroid (which is usually the relatum), the **anchor** and sectors. The anchor is the orientation of a FoR (Frank 1998) and the baseline from which the direction relation between a locatum and a relatum is determined. Since there are three categories of FoR, correspondingly absolute/intrinsic/relative anchors can be defined in Definition 1.

Definition 1 (Absolute/Intrinsic/Relative anchor)**.** *The absolute/intrinsic/relative anchor is the anchor used in absolute/intrinsic/relative FoRs to decide the direction relations between oriented points.*

The absolute anchor is fixed and can be expressed as the orientation of a fake oriented point ***0*** *whose position is unknown and its orientation is* $\phi_0 = 0$.

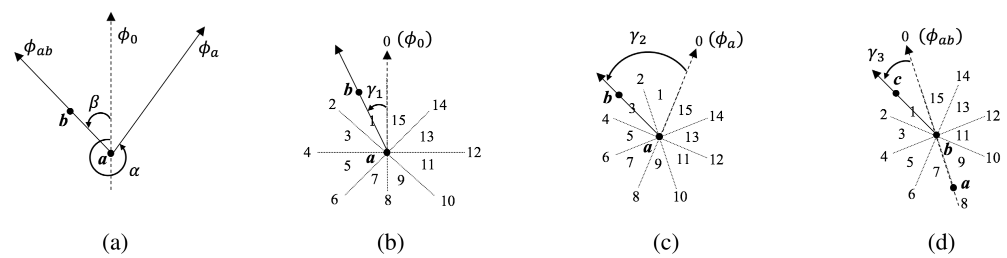


Figure 3: (a) The value of an intrinsic anchor ϕ_a or a relative anchor ϕ_{ab} is decided by the value of the anticlockwise angle deviated from the absolute anchor ϕ_0 (α for ϕ_a and β for ϕ_{ab}). (b) In an absolute FoR, the 2-D plane is divided into 16 sectors with the relatum a as the centroid and the absolute anchor ϕ_0 decides the direction of Sector 0. The absolute direction relation between a and b is decided by the value of the angle that is formed by vector **ab** anticlockwise deviating from ϕ_0. (c) In an intrinsic FoR, the 2-D plane is divided into 16 sectors with the relatum a as the centroid and the intrinsic anchor ϕ_a (i.e. the orientation of a) decides the direction of Sector 0. The intrinsic direction relation between a and b is decided by the value of the angle that is formed by vector **ab** anticlockwise deviating from ϕ_a. (d) In a relative FoR, the 2-D plane is divided into 16 sectors with the relatum b as the centroid and the relative anchor ϕ_{ab} (i.e. the orientation from the observer a to the relatum b) decides the direction of Sector 0. The relative direction relation between a, b and c is decided by the value of the angle that is formed by vector **bc** anticlockwise deviating from ϕ_{ab}.

The intrinsic anchor is the orientation (or the intrinsic front) of a relatum. For example, given a relatum a, its orientation ϕ_a might be used as an intrinsic anchor.

The relative anchor is the perspective decided by the observer looking into the relatum. For example, given an observer a and a relatum b, the orientation from a to b can be used as a relative anchor (denoted as ϕ_{ab}).

The value of an intrinsic or relative anchor is decided by the value of the anticlockwise angle deviated from the absolute anchor (e.g. $\phi_a = \alpha$ and $\phi_{ab} = \beta$ in Figure 3a). All angle values throughout this paper are by default taken modulo 2π. Sectors are decided by the strategy of space division and will be explained in the following section.

Space division and sectors

Our space division model is similar to $\mathcal{OPRA}_m$ (Moratz, Dylla, and Frommberger 2005; Mossakowski and Moratz 2012): given a certain anchor, the 2-D plane is divided into $4m$ sectors by m straight lines ($m \in \mathbb{N}$). The value of m decides the granularity of space division (Lee et al. 2013). Three $m = 4$ examples are in Figure 3b-3d. These sectors are indexed anticlockwise from 0 to $4m - 1$. The anchor of a FoR is in the same direction as Sector 0. Sectors with even indexes are linear sectors (also called as even sectors) while sectors with odd indexes are planar sectors (also called as odd sectors) (Moratz, Dylla, and Frommberger 2005). All results of integer operations in the remainder of this paper are by default taken modulo $4m$. The domain of Sector i (D_i) is defined as follows (Mossakowski and Moratz 2012):

$$D_i = \begin{cases} \frac{2\pi i}{4m}, & \text{if } i \text{ is even;} \\ (\frac{2\pi(i-1)}{4m}, \frac{2\pi(i+1)}{4m}), & \text{if } i \text{ is odd.} \end{cases}$$

For example, in Figure 3b $D_0 = 0$ and $D_1 = (0, \frac{\pi}{4})$.

The spatial constraint language $\mathcal{DAF}_m$

Direction relations **A**cross different **F**rames of references (or $\mathcal{DAF}_m$) is an inclusive constraint language that can express and integrate direction information across different FoRs in a general sense. It is defined over the domain of oriented points (D_{op}) and m is the granularity parameter (Lee et al. 2013). Its $16m + 5$ basic relations consist of 5 same position relations, $4m$ absolute direction relations, $4m$ intrinsic direction relations, $4m$ relative direction relations and $4m$ anchor relations, which are defined as follows.

Same position relations are defined before direction relations because it is almost meaningless to discuss direction relations if spatial entities are in the same position. **Binary same position relation** (Definition 2) can be regarded as a special case of direction descriptions in absolute or intrinsic FoRs that are related to two spatial entities. **Ternary same position relations** (Definition 3) should also be defined to correspond to direction descriptions in relative FoRs that cover three spatial entities.

Definition 2 (Binary same position relation)**.** *Given two oriented points $a, b \in D_{op}$, they are in the binary same position relation (denoted as S^2_{ab}) iff $x_a = x_b \wedge y_a = y_b$. The negation of S^2_{ab} is $\neg S^2_{ab}$.*

Definition 3 (Ternary same position relations)**.** *Given three oriented points a,b and $c \in D_{op}$, they are in the ternary same position relation (denoted as S^3_{abc}) iff S^2_{ab} and S^2_{bc}.*

A degenerate case of ternary same position relation is ternary 12 same position relation (i.e. only the first two oriented points are in the same position). It is denoted as S^{12}_{abc} iff S^2_{ab} and $\neg S^2_{bc}$.

Similarly, they are in the ternary 23 same position relation (denoted as S^{23}_{abc}) iff $\neg S^2_{ab}$ and S^2_{bc}; they are in the ternary 13 same position relation (denoted as S^{13}_{abc}) iff S^2_{ac} and $\neg S^2_{bc}$.

In a certain FoR, direction relations between oriented points are determined by the value of the anticlockwise included angle between the anchor and the vector from the relatum to the locatum (e.g. γ_1 in Figure 3b, γ_2 in Figure 3c, and γ_3 in Figure 3d). Specifically, direction relations in absolute FoRs (Definition 4) are always determined according to the absolute anchor ϕ_0 (Figure 3b); direction relations in intrinsic FoRs (Definition 5) are decided by the orientation of the relatum (e.g. ϕ_a in Figure 3c); while in relative FoRs direction relations (Definition 6) are determined by the perspective of the observer looking into the relatum (e.g. ϕ_{ab} in Figure 3d).

Definition 4 (Absolute direction relations)**.** *Given $a, b \in D_{op}$ and $i \in \{0, 1, ..., 4m-1\}$, a and b are in an absolute direction relation $i^{\mathcal{A}}$ (denoted as A^i_{ab}) iff $\phi_{ab} - \phi_0 \in D_i$ and $\neg S^2_{ab}$. Note that ϕ_{ab} represents the value of the included angle between the vector from a to b and ϕ_0.*

Definition 5 (Intrinsic direction relations)**.** *Given $a, b \in D_{op}$ and $i \in \{0, 1, ..., 4m-1\}$, a and b are in an intrinsic direction relation $i^{\mathcal{I}}$ (denoted as I^i_{ab}) iff $\phi_{ab} - \phi_a \in D_i$ and $\neg S^2_{ab}$.*

Definition 6 (Relative direction relations)**.** *Given $a, b, c \in D_{op}$ and $i \in \{0, 1, ..., 4m-1\}$, a, b, c are in a relative direction relation $i^{\mathcal{R}}$ (denoted as R^i_{abc}) iff $\phi_{bc} - \phi_{ab} \in D_i$ and $\neg S^2_{ab}$ and $\neg S^2_{bc}$.*

For example, A^1_{ab} means in the absolute FoR a and b are in the absolute direction relation $1^{\mathcal{A}}$ (Figure 3b); in Figure 3c from a's point of view b is in Sector 3, which can be denoted as I^3_{ab}; while in Figure 3d, R^1_{abc} can be used to indicate from a's point of view c is in Sector 1 of b.

With these definitions, we can easily show that

Lemma 1. *The basic relations of $\mathcal{DAF}_m$ are not **JEPD**.*

It can be proved by the fact that S^{13}_{abc} is a subset of R^{2m}_{abc}. For example, in Figure 3d if c is removed to the same position as a and b remains unchanged, it is not difficult to conclude that S^{13}_{abc} is a special case of R^8_{abc}.

Except for the same position relations and direction relations, the relations between anchors should also be modeled because as will be seen in the following sections it is the prerequisite of reasoning about direction relations across different FoRs. To be uniform with the relations defined above, the relations between anchors are also expressed by the relations between oriented points (Definition 7).

Definition 7 (Anchor relations)**.** *Given $a, b \in D_{op}$ and $i \in \{0, 1, ..., 4m-1\}$, a and b are in an anchor relation i^{ϕ} (denoted as Φ^i_{ab}) iff $\phi_b - \phi_a \in D_i$.*

Φ^i_{a0} denotes that an intrinsic anchor ϕ_a and the absolute anchor ϕ_0 are in the anchor relation i^{ϕ} (recall that absolute anchor is the orientation of a fake oriented point **0**). A relative anchor relates to two oriented points. To express its relation with other anchors, a dedicated new oriented point with the same orientation as the relative anchor can be created. For example, given a relative anchor ϕ_{ab} that is decided by an observer a and a relatum b, we can create a new oriented point b' whose orientation is $\phi_{b'} = \phi_{ab}$.

A set of relations can indicate the indeterminacy about the relation between oriented points. For example, the absolute direction relation between a and b can be $\{0^{\mathcal{A}}, 2^{\mathcal{A}}\}$ (denoted as $A^{\{0,2\}}_{ab}$), which means the absolute direction relation between a and b can be $0^{\mathcal{A}}$ or $2^{\mathcal{A}}$. Note that $\{1^c, ..., 5^c\}$ is different from $\{5^c, ..., 1^c\}$ where $\mathcal{C} \in \{\mathcal{A}, \mathcal{I}, \mathcal{R}, \phi\}$ indicates relation category. The former means $\{1^c, 2^c, 3^c, 4^c, 5^c\}$ while the latter means $\{5^c, ..., (4m-1)^c, 0^c, 1^c\}$. **Non-informative** (or **universal**) absolute direction relation is a special case of sets of absolute direction relations and is defined as follows. Non-informative intrinsic/relative direction relations and anchor relations are defined analogously.

Definition 8 (Non-informative/universal absolute direction relations)**.** *Two oriented points a and b are in the non-informative/universal absolute direction relation $U^{\mathcal{A}}$ (denoted as A^U_{ab}) if there is no information about their relation in absolute FoRs. Namely $U^{\mathcal{A}} = \{0^{\mathcal{A}}, 1^{\mathcal{A}}, ..., (4m-1)^{\mathcal{A}}\}$ or $A^U_{ab} = A^{\{0,1,...,4m-1\}}_{ab}$.*

Each **instance** of $\mathcal{DAF}_m$ (Θ) consists of a finite set of same position relations, absolute/intrinsic/relative direction relations, and anchor relations over the domain of oriented points.

The converse and composition of binary and ternary same position relations are trivial by definition. For example,
1. S^2_{ab} iff S^2_{ba}. 2. (S^2_{ab} and S^2_{bc})$\Rightarrow S^2_{ac}$. 3. $S^{12}_{abc} \Rightarrow S^{23}_{cab}$.
4. S^3_{abc} iff S^3_*, where $*$ is the permutations of abc.

However, converse and composition of direction relations and anchor relations should be defined according to corresponding relation semantics (Renz and Mitra 2004) as will be introduced in the following sections.

Converse of Direction Relations

In this section the converse of absolute/intrinsic/relative direction relations will be discussed separately.

Proposition 1 (Converse of absolute direction relations)**.** *Given two oriented points $a, b \in D_{op}$ and their absolute direction relation A^i_{ab} where $i \in \{0, 1, ..., 4m-1\}$, the converse of A^i_{ab} is $A^{(i+2m)}_{ba}$ (denoted as $A^{i\smile}_{ab}$).*

Proof. By definition, A^i_{ab} iff $\phi_{ab} - \phi_0 \in D_i$. Assume A^j_{ba} then $\phi_{ba} - \phi_0 \in D_j$. Because $\phi_{ba} = \phi_{ab} + \pi$ (e.g. $\phi_{ab} = \alpha$ and $\phi_{ba} = \beta = \alpha + \pi$ in Figure 4a), $\phi_{ba} - \phi_0 = \phi_{ab} - \phi_0 + \pi \in D_i + \pi$. Then it is not difficult to infer $j = i + 2m$ based on the definition of sector domains. Note that angle values throughout this paper are by default taken modulo 2π and integer operations are by default taken modulo $4m$. □

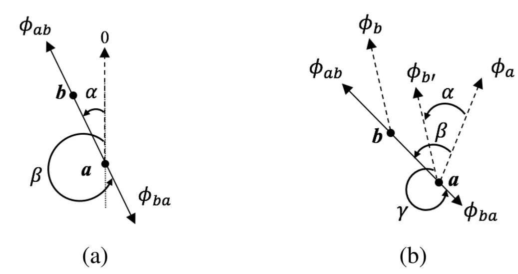


Figure 4: (a) $\beta = \alpha + \pi$ where $\alpha = \phi_{ab}$ and $\beta = \phi_{ba}$ (b) $\gamma = \beta + \pi - \alpha$ where $\alpha = \phi_b - \phi_a$, $\beta = \phi_{ab} - \phi_a$, and $\gamma = \phi_{ba} - \phi_b$.

Proposition 2 (Converse of intrinsic direction relations)**.** *Given two oriented points $a, b \in D_{op}$, their intrinsic direction relation I^i_{ab} and their anchor relation Φ^j_{ab} ($i, j \in$*

$\{0, 1, ..., 4m-1\}$), *the converse of* I^i_{ab} *is* I^k_{ba} *(denoted as* $I^{i\smile}_{ab}$*) where* $k = \Omega_{i,j,i+2m-j}$ *and* $\Omega_{x,y,z} =$

$$\begin{cases} z, & \textit{if } x \textit{ or } y \textit{ are even;} \\ \{z-1, z, z+1\}, & \textit{if } x \textit{ and } y \textit{ are odd.} \end{cases}$$

Proof. By definition, I^i_{ab} iff $\phi_{ab} - \phi_a \in D_i$ and Φ^j_{ab} iff $\phi_b - \phi_a \in D_j$. Assume I^k_{ba} then $\phi_{ba} - \phi_b \in D_k$. Since $\phi_{ba} - \phi_b = (\phi_{ab} + \pi) - (\phi_b - \phi_a + \phi_a) = (\phi_{ab} - \phi_a) + \pi - (\phi_b - \phi_a)$, $\phi_{ba} - \phi_b \in D_i + \pi - D_j$. For example, in Figure 4b, $\alpha = \phi_b - \phi_a, \beta = \phi_{ab} - \phi_a, \gamma = \phi_{ba} - \phi_b$ and $\gamma = \beta + \pi - \alpha$.

Let $z = i+2m-j$. If i and j are both even, $D_i + \pi - D_j = \frac{2\pi z}{4m}$ then $k = z$; If one of i and j is even and the other is odd, $D_i + \pi - D_j = (\frac{2\pi(z-1)}{4m}, \frac{2\pi(z+1)}{4m})$ then k is also equal to z; If i and j are both odd, $D_i + \pi - D_j = (\frac{2\pi(z-2)}{4m}, \frac{2\pi(z+2)}{4m})$ then $k = \{z-1, z, z+1\}$. □

Different from absolute/intrinsic direction relations, relative direction relations are ternary relations. According to (Zimmermann and Freksa 1996; Scivos and Nebel 2001), there are five converse relations for each ternary relation.

Proposition 3 (Converse of relative direction relations)**.** *Given three oriented points* $a, b, c \in D_{op}$ *that are in different positions and their relative direction relation* R^i_{abc} *where* $i \in \{0, 1, ..., 4m-1\}$*, following the terminology in (Zimmermann and Freksa 1996),*

1. ***Inversion*** *of* R^i_{abc} *is* $R^{i\smile inv}_{abc} = R^j_{bac}$ *where* $j =$

$$\begin{cases} 2m, & \textit{if } i = 0; \\ \{0, 2m\}, & \textit{if } i = 2m; \\ \{2m+1, ..., i+2m-\Delta_i\}, & \textit{if } 0 < i < 2m; \\ \{i-2m+\Delta_i, ..., 2m-1\}, & \textit{if } 2m < i < 4m. \end{cases}$$

where Δ_i *is defined as*

$$\Delta_i = \begin{cases} 1, & \textit{if } i \textit{ is even;} \\ 0, & \textit{if } i \textit{ is odd.} \end{cases}$$

2. ***Short cut*** $R^{i\smile sc}_{abc} = R^j_{acb}$*. Interestingly, the transition result of* $R^{i\smile sc}_{abc}$ *are the same as that of* $R^{i\smile inv}_{abc}$*.*
3. ***Short cut inverse*** $R^{i\smile sci}_{abc} = R^j_{cab}$ *if* R^{-j}_{bac}*.*
4. ***Homing*** $R^{i\smile hm}_{abc} = R^j_{bca}$ *if* R^{-j}_{acb}*.*
5. ***Homing inverse*** $R^{i\smile hmi}_{abc} = R^{-i}_{cba}$*.*

Proof. It is trivial to prove Rule 5 because by definition R^i_{abc} iff $\phi_{bc} - \phi_{ab} \in D_i$, R^j_{cba} iff $\phi_{ba} - \phi_{cb} \in D_j$ and $\phi_{bc} - \phi_{ab} = (\phi_{cb} + \pi) - (\phi_{ba} + \pi) = -(\phi_{ba} - \phi_{cb})$. Rule 3 and 4 can be proved analogously.

For Rule 1, by definition R^i_{abc} iff $\phi_{bc} - \phi_{ab} \in D_i$, R^j_{bac} iff $\phi_{ac} - \phi_{ba} \in D_j$. If $i = 0$ or $2m$ the proof is trivial. If $0 < i < 2m$, based on the geometric properties of triangle abc (Figure 5a), $0 < \beta < \alpha$ where $\alpha = \phi_{bc} - \phi_{ab}$ and $\beta = \phi_{ac} - \phi_{ba} - \pi$. Then $\phi_{ac} - \phi_{ba} \in (\pi, \phi_{bc} - \phi_{ab} + \pi)$. If i is even, $D_j = (\frac{2\pi(2m)}{4m}, \frac{2\pi((i+2m-1)-1)}{4m})$ so $j = \{2m+1, ..., i+2m-1\}$; If i is odd, $D_j = (\frac{2\pi(2m)}{4m}, \frac{2\pi((i+2m)+1)}{4m})$ so $j = \{2m+1, ..., i+2m\}$.

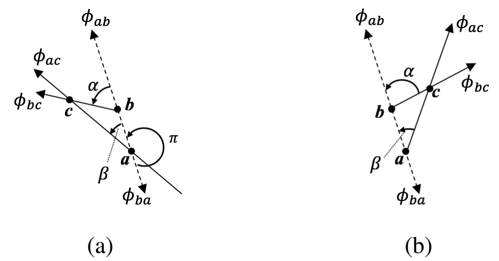


Figure 5: (a) when R^i_{abc} and $0 < i < 2m$, $0 < \beta < \alpha$ where $\alpha = \phi_{bc} - \phi_{ab}$ and $\beta = \phi_{ac} - \phi_{ba} - \pi$. (b) when R^i_{abc} and $2m < i < 4m$, $0 < \beta < \alpha$ where $\alpha = 2\pi - (\phi_{bc} - \phi_{ab})$ and $\beta = \pi - (\phi_{ac} - \phi_{ba})$.

If $2m < i < 4m$, similarly, as shown in Figure 5b, $0 < \beta < \alpha$ where $\alpha = 2\pi - (\phi_{bc} - \phi_{ab})$ and $\beta = \pi - (\phi_{ac} - \phi_{ba})$. Then $\phi_{ac} - \phi_{ba} \in (\phi_{bc} - \phi_{ab} - \pi, \pi)$. If i is even, $D_j = (\frac{2\pi((i-2m+1)-1)}{4m}, \frac{2\pi(2m)}{4m})$ so $j = \{i-2m+1, ..., 2m-1\}$; if i is odd, $D_j = (\frac{2\pi((i-2m)-1)}{4m}, \frac{2\pi(2m)}{4m})$ so $j = \{i-2m, ..., 2m-1\}$. Rule 2 can be proved analogously. □

Composing Direction Relations

Definition 9 (Direction relation composition)**.** *Direction relation composition is to infer new direction relation by composing direction relations from the same or different categories of FoRs. Let* $\circ^{\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3}$ *denote the direction relation composition operator where* $\mathcal{C}_x \in \{\mathcal{A}, \mathcal{I}, \mathcal{R}\}$ *indicates the category of the corresponding direction relation of* x^{th} *element of the operation and* $x \in \{1, 2, 3\}$*. If* $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3$*,* $\circ^{\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3}$ *is an **in-category composition**; if not, it is an **across-category composition**.*

For example, given three direction relations $i^{\mathcal{C}_1}$, $j^{\mathcal{C}_2}$, $k^{\mathcal{C}_3}$ where $i, j, k \in \{0, 1, ..., 4m-1\}$, $i^{\mathcal{C}_1} \circ^{\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3} j^{\mathcal{C}_2} = k^{\mathcal{C}_3}$ indicates that $k^{\mathcal{C}_3}$ can be inferred by composing $i^{\mathcal{C}_1}$ and $j^{\mathcal{C}_2}$. In theory, there are at least 3 forms of in-category composition (i.e. $\circ^{\mathcal{AAA}}$, $\circ^{\mathcal{III}}$, and $\circ^{\mathcal{RRR}}$) and 24 forms of across-category composition.

In-category composition

Proposition 4 ($\circ^{\mathcal{AAA}}$)**.** *Given two absolute direction relations* A^i_{ab} *and* A^j_{bc}*, their in-category composition is* $A^i_{ab} \circ^{\mathcal{AAA}} A^j_{bc} = A^k_{ac}$ *where* $k =$

$$\begin{cases} i, \textit{ if } j - i = 0; \\ U, \textit{ if } j - i = 2m, \textit{ and } i \textit{ is odd;} \\ \{i, j\}, \textit{ if } j - i = 2m \textit{ and } i \textit{ is even;} \\ \{i+\Delta_i, ..., j-\Delta_j\}, \textit{ if } 0 < j - i < 2m; \\ \{j+\Delta_j, ..., i-\Delta_i\}, \textit{ if } 2m < j - i < 4m. \end{cases}$$

Proof. The proof is similar to that of Rule 1 in Proposition 3. By definition A^i_{ab} iff $\phi_{ab} - \phi_0 \in D_i$, A^j_{bc} iff $\phi_{bc} - \phi_0 \in D_j$ and A^k_{ac} iff $\phi_{ac} - \phi_0 \in D_k$. If $j - i = 0$ or $2m$ the proof is trivial. If $0 < j - i < 2m$, based on the geometric properties of triangle abc (Figure 6a), $0 < \beta < \alpha$ where $\alpha = \phi_{bc} - \phi_{ab}$ and $\beta = \phi_{ac} - \phi_{ab}$. Then $\phi_{ac} - \phi_0 \in (\phi_{ab} -$

$\phi_0, \phi_{bc} - \phi_0)$. The rest of the proof can refer to that of Rule 1 in Proposition 3. If $2m < i < 4m$ (Figure 6b), $0 < \beta < \alpha$ where $\alpha = 2\pi - (\phi_{bc} - \phi_{ab})$ and $\beta = 2\pi - (\phi_{ac} - \phi_{ab})$ and the proof is similar. □

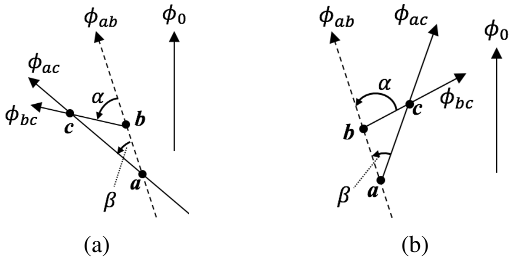


Figure 6: (a) when A^i_{ab}, A^j_{bc} and $0 < j - i < 2m$, $0 < \beta < \alpha$ where $\alpha = \phi_{bc} - \phi_{ab}$ and $\beta = \phi_{ac} - \phi_{ab}$. (b) when A^i_{ab}, A^j_{bc} and $2m < j - i < 4m$, $0 < \beta < \alpha$ where $\alpha = 2\pi - (\phi_{bc} - \phi_{ab})$ and $\beta = 2\pi - (\phi_{ac} - \phi_{ab})$.

In contrast, an intrinsic direction relation I^i_{ab} cannot be directly composed with I^j_{bc} because they might be defined in different FoRs along with different anchors (i.e. ϕ_a and ϕ_b). To achieve $\circ^{\mathcal{III}}$, intrinsic direction relations should be translated into the same FoR and then refer to the composition rules of $\circ^{\mathcal{AAA}}$. Direction relation translation is defined as follows and corresponding translation rules are in Proposition 5.

Definition 10 (Direction relation translation)**.** *Direction relation translation is to translate a direction relation decided in one FoR into that in another based on the relation between the anchors of the FoRs. It can be regarded as a composition of a direction relation and an anchor relation and is denoted as* $\circ^{\mathcal{T}}$.

Proposition 5 ($\circ^{\mathcal{T}}$)**.** *Given three oriented points* $a, b, c \in D_{op}$ *and* $i, j, k \in \{0, 1, ..., 4m-1\}$,

1. $A^i_{ab} \circ^{\mathcal{T}} \Phi^j_{0a} = I^k_{ab}$. *2.* $I^i_{ab} \circ^{\mathcal{T}} \Phi^j_{a0} = A^k_{ab}$.
3. $I^i_{ab} \circ^{\mathcal{T}} \Phi^j_{ab} = I^{k+2m}_{ba}$. *4.* $A^i_{bc} \circ^{\mathcal{T}} \Phi^j_{0b'} = R^k_{abc}$.
5. $R^i_{abc} \circ^{\mathcal{T}} \Phi^j_{b'0} = A^k_{bc}$. *6.* $I^i_{bc} \circ^{\mathcal{T}} \Phi^j_{bb'} = R^k_{abc}$.
7. $R^i_{bc} \circ^{\mathcal{T}} \Phi^j_{b'b} = I^k_{bc}$.

where $\phi_{b'} = \phi_{ab}$ *and* $k = \Omega_{i,j,i-j}$. *Note that* $\circ^{\mathcal{T}}$ *is commutative (e.g.* $A^i_{ab} \circ^{\mathcal{T}} \Phi^j_{0a} = \Phi^j_{0a} \circ^{\mathcal{T}} A^i_{ab}$*).*

Proof. For Rule 1, by definition A^i_{ab} iff $\phi_{ab} - \phi_0 \in D_i$, Φ^j_{0a} iff $\phi_a - \phi_0 \in D_j$, and I^k_{ab} iff $\phi_{ab} - \phi_a \in D_k$. Since $\phi_{ab} - \phi_a = (\phi_{ab} - \phi_0) - (\phi_a - \phi_0)$ (Figure 7a), $D_k = D_i - D_j$ and then $k = \Omega_{i,j,i-j}$. Rule 2-7 can be proved analogously and the proof of commutativeness is trivial. □

Proposition 6 ($\circ^{\mathcal{III}}$)**.** *Given two intrinsic direction relations* I^i_{ab}, I^j_{bc} *and* Φ^k_{ab} *the anchor relation between* a *and* b*, to achieve* $I^i_{ab} \circ^{\mathcal{III}} I^j_{bc}$, I^j_{bc} *should be first translated into* $I^{j'}_{b'c}$ *where* $S^2_{bb'}$, $\Phi^0_{b'a}$ *and* $j' = j + k$ *(Figure 7b). Then* $I^i_{ab} \circ^{\mathcal{III}} I^{j'}_{b'c}$ *can refer to the result of* $A^i_{ab} \circ^{\mathcal{AAA}} A^{j'}_{b'c}$ *because* I^i_{ab} *and* $I^{j'}_{b'c}$ *are defined in the same FoR.*

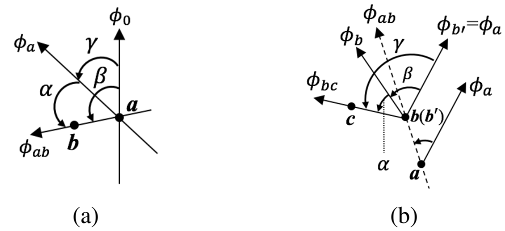


Figure 7: (a) $\alpha = \beta - \gamma$ where $\alpha = \phi_{ab} - \phi_a$, $\beta = \phi_{ab} - \phi_0$ and $\gamma = \phi_a - \phi_0$. (b) $\gamma = \alpha + \beta$ where $\alpha = \phi_{bc} - \phi_b$, $\beta = \phi_b - \phi_a$ and $\gamma = \phi_{bc} - \phi_a$.

As discussed in (Isli et al. 2001; Scivos and Nebel 2001), there are many possible ways to compose two ternary relations. However, it does not mean all the composition results are **informative** (Definition 11). In our representation, for example, the composition of two ternary direction relations with only one common spatial entity is always **non-informative** (Scivos and Nebel 2004) (e.g. R^i_{abc} and R^j_{cde}). With two common spatial entities, there might be two types of informative composition which differ at which entity is omitted (Definition 12 and 13). For example, both R^k_{abd} and R^l_{acd} might be inferred from composing R^i_{abc} with R^j_{bcd}.

Definition 11 (Informative/non-informative operation)**.** *An operation is informative if the operation result could include informative relations; an operation is non-informative if the operation result always includes only non-informative relations.*

Definition 12 (Type-1 of $\circ^{\mathcal{RRR}}$)**.** *Given four oriented points in a fixed order of* a, b, c *and* d*, type-1 ternary relative direction relation composition (denoted as* $\circ^{31}$*) is to infer the relation among* a*,* b *and* d *by composing the relation among* a*,* b *and* c *with the relation among* b*,* c *and* d*. Composition rules are given in Proposition 7.*

Definition 13 (Type-2 of $\circ^{\mathcal{RRR}}$)**.** *Given four oriented points in a fixed order of* a, b, c *and* d*, type-2 ternary relative direction relation composition (denoted as* $\circ^{32}$*) is to infer the relation among* a*,* c*, and* d *by composing the relation among* a*,* b*, and* c *with the relation among* b*,* c*, and* d*. Composition rules are given in Proposition 8.*

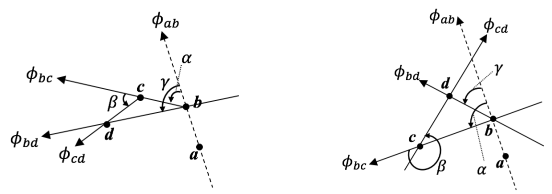
Proposition 7 ($\circ^{31}$)**.** *Given two ternary direction relations* R^i_{abc} *and* R^j_{bcd}*, their type-1 ternary composition* $R^i_{abc} \circ^{31} R^j_{bcd} = R^k_{abd}$ *where* $k =$

$$\begin{cases} i, \text{ if } j = 0; \\ \{i, i+2m\}, \text{ if } j = 2m; \\ \{i+\Delta_i, ..., i+j-\Psi_{i,j}\}, \text{ if } 0 < j < 2m; \\ \{i+j+\Psi_{i,j}, ..., i-\Delta_i\}, \text{ if } 2m < j < 4m. \end{cases}$$

where $\Psi_{i,j}$ *is defined as*

$$\Psi_{i,j} = \begin{cases} 1, \text{ if both } i \text{ and } j \text{ are even;} \\ -1, \text{ if both } i \text{ and } j \text{ are odd;} \\ 0, \text{ otherwise.} \end{cases}$$

Proof. By definition R^i_{abc} iff $\phi_{bc} - \phi_{ab} \in D_i$, R^j_{bcd} iff $\phi_{cd} - \phi_{bc} \in D_j$ and R^k_{abd} iff $\phi_{bd} - \phi_{ab} \in D_k$. If $j = 0$ or $2m$ the proof is trivial. If $0 < j < 2m$, based on the geometric properties of triangle bcd (Figure 8a), $0 < \gamma - \alpha < \beta$ where $\alpha = \phi_{bc} - \phi_{ab}$, $\beta = \phi_{cd} - \phi_{bc}$ and $\gamma = \phi_{bd} - \phi_{ab}$. Then $\phi_{bd} - \phi_{ab} \in (\phi_{bc} - \phi_{ab}, \phi_{bc} - \phi_{ab} + \phi_{cd} - \phi_{bc})$. The rest of the proof is similar to that of Rule 1 in Proposition 3. If $2m < i < 4m$, $0 < \alpha - \gamma < 2\pi - \beta$ where $\alpha = \phi_{bc} - \phi_{ab}$, $\beta = \phi_{cd} - \phi_{bc}$ and $\gamma = \phi_{bd} - \phi_{ab}$ (Figure 8b) and the rest of the proof can refer to the case when $0 < j < 2m$. □



(a) (b)

Figure 8: (a) when R^j_{bcd} and $0 < j < 2m$, $0 < \gamma - \alpha < \beta$ where $\alpha = \phi_{bc} - \phi_{ab}$, $\beta = \phi_{cd} - \phi_{bc}$ and $\gamma = \phi_{bd} - \phi_{ab}$. (b) when R^j_{bcd} and $2m < j < 4m$, $0 < \alpha - \gamma < 2\pi - \beta$ where $\alpha = \phi_{bc} - \phi_{ab}$, $\beta = \phi_{cd} - \phi_{bc}$ and $\gamma = \phi_{bd} - \phi_{ab}$.

Proposition 8 ($\circ^{32}$)**.** *Given two ternary direction relations R^i_{abc} and R^j_{bcd}, their type-2 ternary composition $R^i_{abc} \circ^{32} R^j_{bcd} = R^k_{acd}$ where $k =$*

$$\begin{cases} j, \text{ if } i = 0; \\ \{j, j + 2m\}, \text{ if } i = 2m; \\ \{j + \Delta_j, ..., i + j - \Psi_{i,j}\}, \text{ if } 0 < i < 2m; \\ \{i + j + \Psi_{i,j}, ..., j - \Delta_j\}, \text{ if } 2m < i < 4m. \end{cases}$$

The proof is analogous to that of $\circ^{31}$.

Across-category composition

Across-category composition also allows inference of new informative direction relations. In theory there are at least 24 forms of across-category composition, and most of them rely on translating direction relations in one FoR into that in another FoR (**translation-dependent composition**) because across-category composition covers direction relations from different FoRs. However, there are 5 forms of across-category composition that do not depend on direction relation translation (**translation-independent composition**). For example, $\circ^{\mathcal{AAR}}$, $\circ^{\mathcal{ARA}}$ and their composition rules are given in Proposition 9 and 10 respectively.

Proposition 9 ($\circ^{\mathcal{AAR}}$)**.** *Given two absolute direction relations A^i_{ab} and A^j_{bc}, R^k_{abc} can be inferred from $A^i_{ab} \circ^{\mathcal{AAR}} A^j_{bc}$ where $k = \Omega_{i,j,j-i}$.*

The proof is trivial given the fact that $\phi_{bc} - \phi_{ab} = (\phi_{bc} - \phi_0) - (\phi_{ab} - \phi_0)$.

Proposition 10 ($\circ^{\mathcal{ARA}}$)**.** *Given an absolute direction relations A^i_{ab} and a relative direction relation R^j_{abc}, A^k_{bc} can be inferred from $A^i_{ab} \circ^{\mathcal{ARA}} R^j_{abc}$ where $k = \Omega_{i,j,i+j}$.*

The proof is trivial given the fact that $\phi_{bc} - \phi_0 = (\phi_{bc} - \phi_{ab}) + (\phi_{ab} - \phi_0)$. The rules of $\circ^{\mathcal{RAA}}$, $\circ^{\mathcal{IRI}}$ and $\circ^{\mathcal{RII}}$ are analogous.

Different from translation-independent compositions, translation-dependent compositions are based on one or more direction relation translations (Definition 10), for example $\circ^{\mathcal{AIA}}$ should be first transformed into $\circ^{\mathcal{AAA}}$ (Proposition 11).

Proposition 11 ($\circ^{\mathcal{AIA}}$)**.** *Given an absolute direction relation A^i_{ab} and an intrinsic direction relation I^j_{bc}, their across-category composition $A^i_{ab} \circ^{\mathcal{AIA}} I^j_{bc}$ can be transformed into $A^i_{ab} \circ^{\mathcal{AAA}} A^{j'}_{bc}$ by translating I^j_{bc} into $A^{j'}_{bc}$ according to Rule 2 of $\circ^{\mathcal{T}}$ (Proposition 5).*

Similar strategy can be used in $\circ^{\mathcal{AII}}$, $\circ^{\mathcal{AAI}}$, $\circ^{\mathcal{IIA}}$, $\circ^{\mathcal{IAA}}$, $\circ^{\mathcal{IAI}}$, $\circ^{\mathcal{IIR}}$ and in across-category compositions that cover all three categories (there are 6 of them and for example $\circ^{\mathcal{AIR}}$). The remaining 6 forms of across-category compositions (i.e $\circ^{\mathcal{RRA}}$, $\circ^{\mathcal{ARR}}$, $\circ^{\mathcal{RAR}}$, $\circ^{\mathcal{RRI}}$, $\circ^{\mathcal{IRR}}$ and $\circ^{\mathcal{RIR}}$) that have not been discussed so far include exactly two relative direction relations and all of them are non-informative.

Anchor Relation Operations

As can be seen in the previous section, without the information about the relation between anchors, it is almost impossible to connect direction information described in one FoR to that in another (except for absolute direction relations as they are defined in the same absolute anchor). Anchor relations might be directly described in natural languages. For example, "the chair is facing north" (Levinson 1996), where the orientation of the chair (ϕ_c) is north (ϕ_0) and can be expressed as Φ^0_{c0}. The relation between two anchors can also be inferred if the direction relation between two oriented points is described according to both of them (Proposition 13) or by composing another two anchor relations (Proposition 14).

Proposition 12 (Converse of anchor relations)**.** *Given two oriented points $a, b \in D_{op}$ and their anchor relation Φ^i_{ab} where $i \in \{0, 1, ..., 4m - 1\}$, the converse of Φ^i_{ab} is Φ^{-i}_{ba} (denoted as $\Phi^{i\smile}_{ab}$).*

Proof. By definition, Φ^i_{ab} iff $\phi_b - \phi_a \in D_i$. Assume Φ^j_{ba} then $\phi_a - \phi_b \in D_j$. It is not difficult to infer $i + j = 0$ from $\phi_a - \phi_b = -(\phi_b - \phi_a)$ (Figure 9a). □

(a) (b)

Figure 9: (a) $\alpha + \beta = 0$ where $\alpha = \phi_b - \phi_a$ and $\beta = \phi_a - \phi_b$. (b) $\gamma = \alpha + \beta$ where $\alpha = \phi_b - \phi_a$, $\beta = \phi_c - \phi_b$ and $\gamma = \phi_c - \phi_a$.

Proposition 13 (Anchor relation inference)**.** *Given three oriented points $a, b, c \in D_{op}$ and $i, j, k \in \{0, 1, ..., 4m-1\}$,*

1. A^i_{ab} and $I^j_{ab} \Rightarrow \Phi^k_{0a}$ 2. I^i_{ab} and $I^j_{ba} \Rightarrow \Phi^k_{0a}$
3. A^i_{bc} and $R^j_{abc} \Rightarrow \Phi^k_{0b'}$ 4. I^i_{bc} and $R^j_{abc} \Rightarrow \Phi^k_{bb'}$ where $\phi_{b'} = \phi_{ab}$ and $k = \Omega_{i,j,i-j}$.

Proof. For Rule 1, the proof is trivial given the fact that $\phi_a - \phi_0 = (\phi_{ab} - \phi_0) - (\phi_{ab} - \phi_a)$ (Figure 7a). Rule 2-4 can be proved analogously. □

Proposition 14 (Anchor relation composition)**.** *Let $\circ^\phi$ denote anchor relation composition. Given Φ^i_{ab} and Φ^j_{bc} two anchor relations, $\Phi^i_{ab} \circ^\phi \Phi^j_{bc} = \Phi^k_{ac}$ where $k = \Omega_{i,j,i+j}$.*

Proof. The proof is trivial given the fact that $\phi_c - \phi_a = (\phi_b - \phi_a) + (\phi_c - \phi_b)$ (Figure 9b). □

Operations over sets of basic relations

The rules of relation operation (i.e. converse and composition) in the previous sections are for basic relations. For sets of basic relations, the converse of a union of basic relations is equal to the union of the converse of all basic relations. Namely, $(\{i, j, ..., k\}^{\mathcal{C}})^{\smile} = (i^{\mathcal{C}})^{\smile} \cup (j^{\mathcal{C}})^{\smile} \cup ... \cup (k^{\mathcal{C}})^{\smile}$, where $\mathcal{C} \in \{\mathcal{A}, \mathcal{I}, \mathcal{R}, \phi\}$. $^{\smile}$ can also be converse of ternary relations e.g. $^{\smile inv}$.

The composition of two unions of basic relations is equal to the union of compositions of pairs of relations from these two sets. Namely, $\{i, j, ..., k\}^{\mathcal{C}_1} \circ^* \{u, v, ..., w\}^{\mathcal{C}_2} =$
$(i^{\mathcal{C}_1} \circ u^{\mathcal{C}_2}) \cup (i^{\mathcal{C}_1} \circ^* v^{\mathcal{C}_2}) \cup ... \cup (i^{\mathcal{C}_1} \circ^* w^{\mathcal{C}_2}) \cup$
$(j^{\mathcal{C}_1} \circ u^{\mathcal{C}_2}) \cup (j^{\mathcal{C}_1} \circ^* v^{\mathcal{C}_2}) \cup ... \cup (j^{\mathcal{C}_1} \circ^* w^{\mathcal{C}_2}) \cup$
...
$(k^{\mathcal{C}_1} \circ u^{\mathcal{C}_2}) \cup (k^{\mathcal{C}_1} \circ^* v^{\mathcal{C}_2}) \cup ... \cup (k^{\mathcal{C}_1} \circ^* w^{\mathcal{C}_2})$,
where $\circ^*$ can be $\circ^\phi$, $\circ^\mathcal{T}$ or $\circ^{c_1 c_2 c_3}$ and $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \in \{\mathcal{A}, \mathcal{I}, \mathcal{R}\}$.

Towards Constraint-based Reasoning

The core problem in qualitative spatial reasoning is deciding whether a set of spatial relations is **consistent** because most of the other problems can be reduced to it (Renz and Nebel 1999). The consistency problem can be transformed to a **constraint satisfaction problem** (CSP) because spatial relations can also be regarded as constraints between spatial entities (Renz and Nebel 2007). In our case, the most important reasoning problem over Θ (an instance of $\mathcal{DAF}_m$) is whether it is **across-category consistent**.

Definition 14 (Across-category consistency)**.** Θ *(an instance of $\mathcal{DAF}_m$) is across-category consistent or equivalently Θ is satisfiable iff there exists at least one set of oriented points that can satisfy all the constraints originated from its relations.*

Classical composition and path-consistency based methods cannot be used to check across-category consistency because the basic relations of $\mathcal{DAF}_m$ are not JEPD (Lemma 1) and some of the relations are not closed over converse or composition. However, $CSP(\mathcal{DAF}_m)$ is decidable (Theorem 1) because the constrains can be transformed into a set of inequalities (Moratz and Ragni 2008).

Theorem 1. *$CSP(\mathcal{DAF}_m)$ is **NP-hard** and in **PSPACE**.*

Proof. Given a instance of $\mathcal{DAF}_m$ and its set of relations (consisting of same position relations, absolute/intrinsic/relative direction relations and anchor relations), by definition same position relations and anchor relations are linear inequalities. For example, S^2_{ab} iff $x_a = x_b$ and $y_a = y_b$; Φ^i_{ab} iff $\phi_b - \phi_a \in D_i$. If i is even, $\phi_b - \phi_a = \frac{4m}{2\pi i}$; if i is odd, $\frac{4m}{2\pi(i-1)} < \phi_b - \phi_a < \frac{4m}{2\pi(i+1)}$.

Absolute direction relations can be transformed into intrinsic direction relations. For example, A^i_{ab} can be transformed into $I^i_{a'b}$ where $S^2_{a'a}$ and $\Phi^0_{a'0}$ (Figure 10a). Intrinsic direction relations can be transformed into relative direction relations. For example, I^i_{ab} can be transformed into $R^i_{a'ab}$ where a' is an arbitrary point that is in the opposite direction of ϕ_a from a so that $\phi_{a'a} = \phi_a$ (Figure 10b). According to (Moratz and Ragni 2008), relative direction relations can be transformed into polynomial inequalities of power 2 with integer coefficients, which can be solved in polynomial space (Renegar 1992).

The NP-hardness of $CSP(\mathcal{DAF}_m)$ is based on the results in (Scivos and Nebel 2001). □

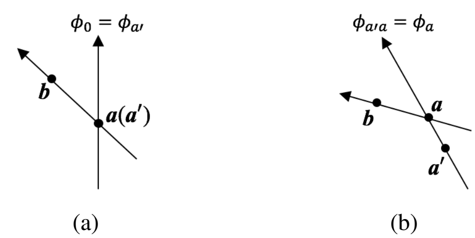


Figure 10: (a) An absolute direction relation A^i_{ab} can be transformed into an intrinsic direction relation $I^i_{a'b}$ where $S^2_{a'a}$ and $\Phi^0_{a'0}$. (b) An intrinsic direction relation I^i_{ab} can be transformed into a relative direction relation $R^i_{a'ab}$ where $\phi_a = \phi_{a'a}$.

Since path-consistency cannot be enforced in our case, another form of local consistency (i.e. converse consistency in Definition 15) is used as a prerequisite of across-category consistency.

Definition 15 (Converse consistency)**.** *A relation is converse consistent iff there is at least one consistent scenario that can satisfy its converse relation(s) and itself.*

Converse consistency can be used as a necessity of across-category consistency because if any relation of $\mathcal{DAF}_m$ is not converse consistent, it is not across-category consistent. For same position relations, absolute/intrinsic direction relations and anchor relations, converse consistency check is trivial because they are binary relations and there is only one form of converse relations. However, for relative direction relations, it is non-trivial because there are five converse relations and corresponding converse consistency check can refer to Proposition 15.

Proposition 15. *Given three relative direction relations R^i_{abc}, R^j_{cab}, and R^k_{bca} among oriented points a, b and c, they* *are converse consistent iff there exist at least one set of three oriented points* a*, b and c that can satisfy all of the following constraints:*

1. R^{-i}_{cba}*;* *2.* R^{-j}_{bac}*;* *3.* R^{-k}_{acb}*;*

4. Triangle$(2m-i, 2m-j, 2m-k)$ *where Triangle*(x,y,z) *is true iff there exist three angles* $\alpha \in D_x$*,* $\beta \in D_y$ *and* $\gamma \in D_z$ *that can form a triangle (Mossakowski and Moratz 2012).*

Proof. If a relative direction relation is converse consistent, there exists at least one set of three oriented points a, b and c that can satisfy this relative direction relation and all its converse relations. The proof of Constraint 1-3 can refer to Rule 4-6 in Proposition 3 (or as shown in Figure 11a, $\alpha+\beta=0$). For Constraint 4, based on the geometric properties of triangle abc in Figure 11a, γ_1, γ_2 and γ_3 can form a triangle where $\gamma_1 = \pi - (\phi_{bc} - \phi_{ab})$, $\gamma_2 = \pi - (\phi_{ab} - \phi_{ca})$ and $\gamma_3 = \pi - (\phi_{ca} - \phi_{bc})$. □

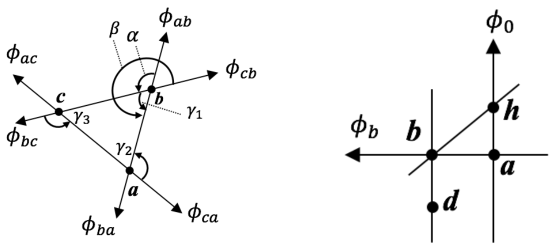


Figure 11: (a) $\alpha + \beta = 0$ and γ_1, γ_2 and γ_3 can form the triangle abc. (b) Relative positions of the hunters (**a** and **b**), the dog (**d**) and the hare (**h**). ϕ_0 indicates the north and ϕ_b is the orientation of b.

Case Study and Applications

Our case study is inspired by the hare-shooting story in (Ligozat 1993): there are two hunters Allen and Ben who dislike each other because Allen (a) insists on using absolute FoRs to describe direction information while Ben (b) prefers to describe direction relations in his own orientation. One day they go for hare-shooting with a "smart" hunting dog (d). Given the following information (Figure 11b), if Allen asks the dog to chase the hare (h) how can the dog know where the hare is? Assume these two hunters keep moving so that the strategy of first moving to one of the hunters cannot be used.

(1) Allen yells to Ben "Hey a hare in the north (of me)! Why do you face to west? It is in your northeast!".

(2) Ben keeps moving forwards and replies "Stop yelling at me! I am not interested at the hare if it is found by you. Why don't you ask the dog that is standing to my left to get it!?".

(3) The dog looks at Ben and finds out Allen is to the right of him.

It is almost impossible to solve this problem with the existing formalisms because most of them can just deal with direction information defined in one category of FoRs. However, useful information can be inferred by applying $\mathcal{DAF}_4$. The direction information in (1) can be expressed by A^0_{ah}, Φ^4_{0b}, and A^{14}_{bh}; that in (2) can be described by I^4_{bd}; while R^{12}_{dba} can express the direction information in (3). Our aim is to infer the direction relation between the hare and dog which can be denoted as A^i_{dh} where i is unknown. Based on the inference rules introduced in the previous sections, $i = \{13, ..., 3\}$ (induction details are as follows and refer to Figure 3b for a pictorial view of these sectors), which means the dog should move to a rough sense of north.

#1. $\Phi^{4\smile}_{0b} = \Phi^{12}_{b0}$ from (1).

#2. $I^4_{bd} \circ^{\mathcal{T}} \Phi^{12}_{b0} = A^8_{bd}$ from (2) and #1.

#3. $A^{14\smile}_{bh} = A^6_{hb}$ from (1).

#4. $A^0_{ah} \circ^{\mathcal{AAR}} A^6_{hb} = R^6_{ahb}$ from (1) and #3.

#5. $R^{6\smile sci}_{ahb} = R^{\{3,...,7\}}_{bah}$ from #4.

#6. $R^{12}_{dba} \circ^{31} R^{\{3,...,7\}}_{bah} = R^{\{13,...,3\}}_{dbh}$ from (3) and #5.

#7. $A^{8\smile}_{bd} = A^0_{db}$ from #2.

#8. $A^0_{db} \circ^{\mathcal{ARA}} R^{\{13,...,3\}}_{dbh} = A^{\{13,...,3\}}_{bh}$ from #7 and #6.

#9. $A^0_{db} \circ^{\mathcal{AAA}} A^{\{13,...,3\}}_{bh} = \mathbf{A}^{\{13,...,3\}}_{\mathbf{dh}}$ from #7 and #8.

In practice, sets of relations can be used to model inaccuracy of inaccurate direction information. For example, $A^{\{15,0,1\}}_{ah}$ can indicate h is roughly north of a.

Direction relations occur very frequently in human communication and therefore also on Social Media and (future) human-robot interaction. Unlike humans, current AI agents are unable to understand and deal with these expressions properly. Our spatial constraint language $\mathcal{DAF}_m$ is a breakthrough in solving this problem, which could stimulate applications of qualitative spatial reasoning in areas like robot navigation, navigation regulation, direction relation recognition, query-answering and so on (Wolter and Wallgrün 2013).

Conclusion

To the best of our knowledge, we have presented the first formalism that can represent and reason direction relations across different frames of reference. Direction relations in three different categories of FoRs have been unified and the implicit factor (i.e. anchor) that is critical in deciding direction relations has been modeled. We provided approaches to infer the anchors used and based on which direction relations in one FoR can be translated into another. We also discussed how across-category composition and inconsistency check can be done.

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