Gray Code Sequences of Partitions

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Combinatorics Day, 27 May 2004

Binary Reflected Gray Codes

Figure: Frank Gray (Artist's Impression)

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The reversing of sublists will be crucial later!

Combinatorial Gray Codes

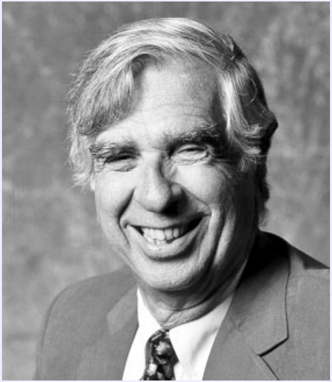

Figure: Herbert Wilf

Combinatorial Gray Codes

The General Problem

Given a class of combinatorial objects and a definition of what it means for two objects to be *close*, produce a list of all the objects in that class in such a way that succesive elements are close.

Combinatorial Gray Codes

First Solution

Find a neat recursive description of the class and use the reversing and gluing procedure.

Combinatorial Gray Codes

Second Solution

Define the *Gray Graph*:

Vertices: Objects

Edges: Two vertices are joined if they are close.

Find a Hamiltonian path through this graph.

Set Partitions

Figure: Partitioning the Cake Set

Set Partitions

Definition

Let X be a set. A *partition* of X is a family of *pairwise disjoint* subsets of X which together contain all of the elements of X. These subsets are called *blocks*.

Given a positive integer n, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$.

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- Gray code listing: Gideon Ehrlich (1973) - using Restricted Growth Tails!

Restricted Growth Tails

- Define the *Restricted growth tail* (RGT) of a partition of $[n]$ by:

Step 1:	Lex Order the Partition
Step 2:	Number the blocks $0, 1, \ldots$
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- If $(x_1, x_2, \dots, x_n)$ is a RGT then $0 \leq x_i \leq 1 + \max\{x_1, x_2, \dots, x_{i-1}\}$

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- Finding a Gray Ordering in the RGT rep was solved by Ehrlich - giving a gray code ordering for standard rep!

A Stirling Formula

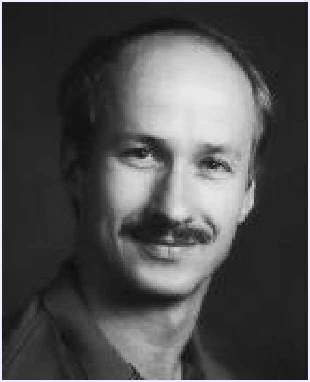

Figure: Frank Ruskey

A Stirling Formula

- $\left\{ {n \atop k} \right\}$ - the number of partitions of $[n]$ into blocks of size k.
- This is the *Stirling number of the second kind*

Lemma

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

Proof.

Done by Gordon in Lecture 6. But we make an observation. Let $x_1 x_2 \ldots x_n$ be the RGT of a partition into k blocks and $m = \max\{x_1, x_2, \ldots, x_{n-1}\}$. Then, if $x_n = k-1$ then $m = k-1$ or $k-2$ because $x_n \leq 1 + m$. If $0 \leq x_n < k-1$ then m $= k-1$, because the partition must have k blocks. □

Back and Forth

Using a slightly more complicated reversing and gluing procedure, Ruskey (1993) managed to use this formula to construct a Gray listing of RGT's of partitions into k blocks.

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- We construct two Gray lists: $\mathbf{S}(n, k, 0)$ and $\mathbf{S}(n, k, 1)$
- The construction of one depends on the other.
- Also need to consider k even and odd cases seperately.

The construction

First we do it for even k:

$\mathbf{S}(n,k,0)$ even k	$\mathbf{S}(n,k,1)$ even k
$\mathbf{S}(n-1,k-1,0).(k-1)\oplus$	$\mathbf{S}(n-1,k-1,1).(k-1)\oplus$
$\mathbf{S}(n-1,k,1).(k-1)\oplus$	$\overline{\mathbf{S}(n-1,k,1)}.(k-1)\oplus$
$\overline{\mathbf{S}(n-1,k,1)}.(k-2)\oplus$	$\mathbf{S}(n-1,k,1).(k-2)\oplus$
$\vdots$	$\vdots$
$\mathbf{S}(n-1,k,1).1\oplus$	$\overline{\mathbf{S}(n-1,k,1)}.1\oplus$
$\overline{\mathbf{S}(n-1,k,1)}.0$	$\mathbf{S}(n-1,k,1).0$

The construction

And now for odd k:

$\mathbf{S}(n,k,1)$ odd k	$\mathbf{S}(n,k,0)$ odd k
$\mathbf{S}(n-1,k-1,0).(k-1)\oplus$	$\mathbf{S}(n-1,k-1,1).(k-1)\oplus$
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$\overline{\mathbf{S}(n-1,k,1)}.0$	$\mathbf{S}(n-1,k,1).0$

A Little Game

Figure: Dr Evil

A Little Game

- Two integer partitions are close if they differ by the move of one dot in the Ferrer's diagram
- Find Gray listing for:
  1. All partitions
  2. Partitions into parts of size at most k (Denoted $P(n,k)$)
  3. Partitions into distinct parts
  4. Partitions into odd parts
  5. ...

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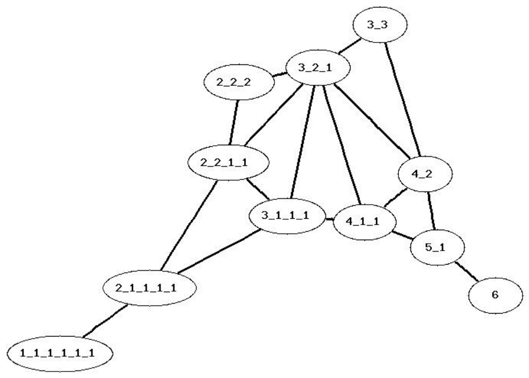
- Bijection between these does not preserve Gray Codes!
- We concentrate on the Gray Graphs for $P(n, n)$.

Gray Graph for $P(6, 6)$

Finding a Hamiltonian path "by eye" is not hard in this case.



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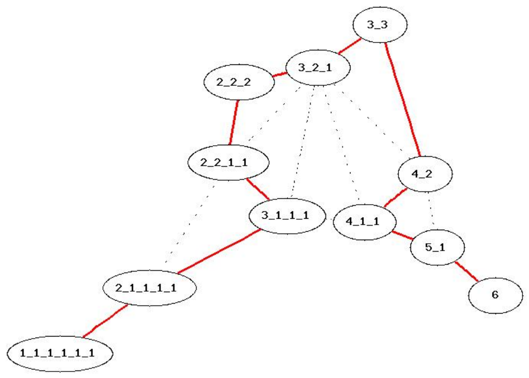

Gray Graph for $P(n, n)$

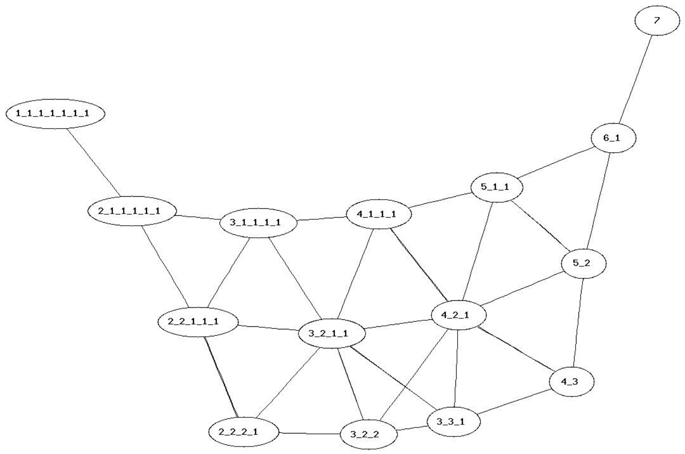

Figure: $n = 7$

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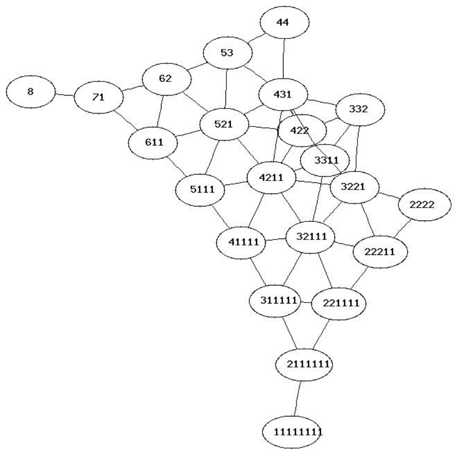

Figure: $n = 8$

Gray Graph for $P(n, n)$

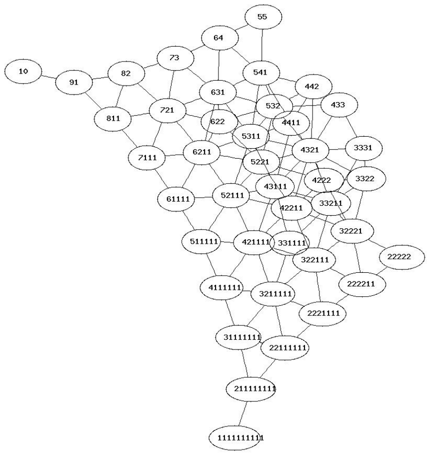

Figure: $n = 10$

Gray Graph for $P(n, n)$

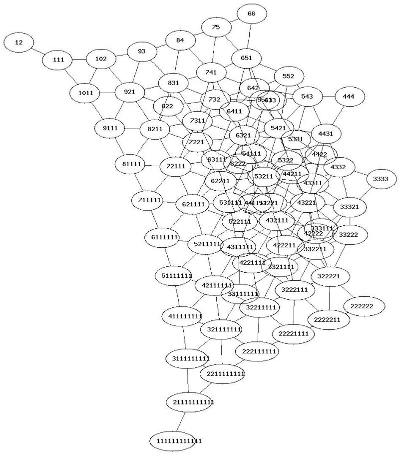

Figure: $n = 12$

Enumerating Edges

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n	1	2	3	4	5	6	7	8	9	...
$\#Edges$	0	1	2	5	9	17	28	47	73	...
$\#HP$	1	1	1	1	1	1	52	652	298,896	...

Where *HP* stands for *Hamiltonian Paths.*

Questions