Gradient Flow Techniques for Pose Estimation of Quadratic Surfaces

John B. Moore
Dept. of Systems Engineering
Australian National University
Canberra, ACT

Moonhong Baeg
Dept. of Electric and
Electronic Engineering
University of Tokyo

Uwe Helmke
Department of Mathematics
University of Regensburg
Regensburg, Germany

1. Introduction

A key problem in robotics is the estimation of the location and orientation of objects from surface measurement data. This is termed pose estimation. A fundamental task is the pose estimation of known quadratic surfaces from, possibly noisy, data. A solution for this task facilitates pose estimation for more complex objects. Current algorithms [3] – [6] frequently converge to local minima of the performance index and are unsuited for on-line applications because of the intensive computer effort required.

In this paper, following the general approach outlined in [1], [2], we study gradient flows on the Euclidean group towards a solution of the pose estimation problem of quadratic surfaces. Discretizations of the flow leads to recursive numerical methods for pose estimation. Recursions using second derivative information can achieve quadratic convergence rates. The goal is to develop a complete phase portrait analysis of the algorithms, which establishes the theoretical foundation for developing fast algorithms for pose estimation. At this stage there is only a theory with implicit, rather than explicit conditions.

To set the stage for our investigation, consider a known quadratic surface in $\mathbb{R}^3$ described by the equation $x'Q_{11}x + 2Q'_{12}x + Q_{22} = 0$, for $x \in \mathbb{R}^3$. It is useful to pass to the equivalent description in homogeneous coordinates $\xi' = [x', 1] \in \mathbb{R}^4$, given as $\xi' Q \xi = 0$ for the symmetric coefficient matrix $Q = (Q_{ij}) \in \mathbb{R}^{4\times4}$. Without loss of generality let us consider that this object is located at the origin and that $Q_{22} < 0$ and $\det Q_{11} \neq 0$. Now a translation by a vector $p \in \mathbb{R}^3$ and a 3×3 rotation matrix $R \in SO(3)$ leads to a new quadratic surface coefficient matrix

$$A(R,p) := T'(R,p)QT(R,p) \quad , \quad T(R,p) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \tag{1.1}$$

satisfying $\xi' A(R,p)\xi = 0$. Here the set $SO(3)$ of rotation matrices is the group of real orthogonal 3×3 matrices R satisfying $R'R = I_3$ and $\det R = 1$. If Q is such that Q_{11} is invertible, then there exists $(R,p) \in SO(3) \times \mathbb{R}^3$ such that $A(R,p)$ is diagonal.

2. Gradient Flows for Pose Estimation

Consider a quadratic object with known coefficient matrix $Q = Q' \in \mathbb{R}^{4\times4}$. In the event of noise free surface data x_i, or $\xi_i' = [x_i'\ 1]$ for $i = 1, 2, \cdots k$, then pose estimation is the estimation of $R \in SO(3)$ and $p \in \mathbb{R}^3$ such that $\xi_i' T'(R,p)QT(R,p)\xi_i = 0$ for all i with $T(R,p)$ given by (1.1). In the noisy data case, an appropriate index is a least squares index

$$\Phi(R,p) = \frac{1}{2k} \sum_{i=1}^{k} \phi_i^2 \ , \ \phi_i = \xi_i' T'(R,p)QT(R,p)\xi_i \tag{2.1}$$

where $R \in SO(3)$ is the rotation matrix and $p \in \mathbb{R}^3$ is the position vector. Using the definitions

$$\tilde{H}(R,p) := \frac{1}{k}\sum_{i=1}^{k}\phi_i(R,p)\xi_i\xi_i' := \begin{bmatrix}\tilde{H}_{11} & \tilde{H}_{12}\\ \tilde{H}_{12}' & \tilde{H}_{22}\end{bmatrix}\quad,\quad |\tilde{H}_{22}| = 1 \tag{2.2}$$

then the index can be reformulated as

$$\Phi(R,p) = \frac{1}{2}\, tr\ \left(T'(R,p)QT(R,p)\tilde{H}(R,p)\right)\ . \tag{2.3}$$

Now the minimization task is

$$\min_{R\in SO(3),p\in\mathbb{R}^3} \Phi(R,p)\ . \tag{2.4}$$

To achieve a minimizing gradient flow, exploiting techniques in [1], [2], first note that the Euclidean group $SO(3)\times\mathbb{R}^3$ is a smooth manifold with tangent space

$$T_{(R,p)}\left(SO(3)\times\mathbb{R}^3\right) = \left\{(R\Omega\ ,\zeta)\mid \Omega' = -\Omega, \zeta\in\mathbb{R}^3\right\}. \tag{2.5}$$

We define a Riemannian metric on $SO(3)\times\mathbb{R}^3$ by

$$\langle\ (R\Omega_1,\zeta_1)\,,(R\Omega_2,\zeta_2)\ \rangle := \ tr\left((R\Omega_1)'\,(R\Omega_2) + 2\zeta_1'\zeta_2\right) \tag{2.6}$$

Straightforward computations then lead to the following explicit description of the gradient flow of $\Phi(R,p)$.

Theorem 1

1) *The gradient flow of the cost function $\Phi: SO(3)\times\mathbb{R}^3\to\mathbb{R}$ with respect to the Riemannian metric (2.6) is*

$$\begin{aligned}\begin{bmatrix}\dot{R}\\ \dot{p}\end{bmatrix} &= -\begin{bmatrix}\nabla_R\Phi\,(R,p)\\ \nabla_p\Phi\,(R,p)\end{bmatrix}\\ &= -\begin{bmatrix} R\left(R'\tilde{L}(R,p) - \tilde{L}'(R,p)R\right)\\ Q_{11}\left(R\tilde{H}_{12}(R,p) + p\tilde{H}_{22}(R,p)\right) + Q_{12}\tilde{H}_{22}(R,p)\end{bmatrix}\end{aligned} \tag{2.7}$$

where

$$\tilde{L}(R,p) = Q_{11}\left(R\tilde{H}_{11}(R,p) + p\tilde{H}_{12}'(R,p)\right) + Q_{12}\tilde{H}_{12}'(R,p)\ . \tag{2.8}$$

2) *The equilibrium points of Φ are characterized as:*

$$R'\tilde{L}(R,p) = \tilde{L}'(R,p)R, \tag{2.9}$$

$$Q_{11}R\tilde{H}_{12}(R,p) + (Q_{11}p + Q_{12})\tilde{H}_{22}(R,p) = 0. \tag{2.10}$$

□

An important feature of the gradient flow (2.7) is that the solutions $(R(t),p(t))$ exist for all time $t\geq 0$ and converge to the equilibrium points. This is made more precise in the following result.

Proposition 2

Assume that the data points $\xi_i\in\mathbb{R}^4$, $i = 1,\ldots,N$, are in general position with $N\geq 10$. Then

(a) *The cost function $\Phi: SO(3)\times\mathbb{R}^3\to\mathbb{R}$ has compact sublevel sets.*

(b) *The solutions $(R(t),p(t))\in SO(3)\times\mathbb{R}^3$ of (2.7) exist for all $t\geq 0$.*

(c) *Every solution $(R(t),p(t))$ of (2.7) converges to an equilibrium point (R_∞,p_∞) characterized by (2.9), (2.10).*

(d) *There exist only a finite number of equilibria $(R_\infty,p_\infty)\in SO(3)\times\mathbb{R}^3$ of (2.7).* □

Remark

Simple manipulations give

$$p = -R\tilde{H}_{12}(R,p)\tilde{H}_{22}^{-1}(R,p) - Q_{11}^{-1}Q_{12} \tag{2.11}$$

$$[\ R^{'}Q_{11}R\ ,\ \tilde{\mathcal{H}}_{11}(R,p)\ ] = 0 \quad , \ \tilde{\mathcal{H}}_{11}(R,p) = \tilde{H}_{11}(R,p) - \tilde{H}_{12}(R,p)\tilde{H}_{22}^{-1}(R,p)\tilde{H}_{12}^{'}(R,p) \tag{2.12}$$

for the equilibria of (2.7). Here $[A,B] = AB - BA$ denotes the Lie bracket. Also, at all equilibria

$$\tilde{L} = Q_{11}R\tilde{\mathcal{H}}_{11}(R,p)\ . \tag{2.13}$$

Moreover, using the diagonalizations

$$\mathcal{H}_{11}(R,p) = V_{\mathcal{H}}(R,p)\Lambda_{\mathcal{H}}(R,p)V_{\mathcal{H}}^{'}(R,p)\ , \quad Q = V_Q\Lambda_Q V_Q^{'} \tag{2.14}$$

with $V_{\mathcal{H}}$, $V_Q \in SO(3)$ and $\Lambda_{\mathcal{H}}$, Λ_Q diagonal in *reverse* order, then the equilibria condition (2.12) holds if and only if

$$R = V_Q V_{\mathcal{H}}^{'}(R,p)\Pi \ \in \ SO(3) \tag{2.15}$$

where Π is an arbitrary permutation matrix.

Of course $\tilde{H}_{12}$ and $\mathcal{H}_{11}$, $V_{\mathcal{H}}$ are R,p dependent, so (2.11) (2.15) are implicit, rather than explicit equation for R and p. Since the diagonalizations in (2.16) are not unique, then R and p satisfying (2.11) (2.15) are not unique. Under the equilibrium condition (2.11), the index $\Phi(R,p)$ can be re-organized by simple manipulation as

$$\Phi(R,p) = \ tr\left[R^{'}Q_{11}R\tilde{\mathcal{H}}_{11}(R,p) + \tilde{H}_{22}(R,p)\ (Q_{22} - Q_{12}^{'}Q_{11}^{-1}Q_{12})\right]\ . \tag{2.16}$$

Observe that this index is minimized by an R,p selection satisfying (2.11) (2.15) with $\Pi = I$. The minimal index is

$$\Phi^* = tr\ [\Lambda_Q\Lambda_{\mathcal{H}} + \tilde{H}_{22}(R,p)(Q_{22} - Q_{12}^{'}Q_{11}^{-1}Q_{12})] \tag{2.17}$$

□

The stability properties of the gradient flow can be obtained from the linearization at the equilibrium points. We do omit the details here.

3. Recursive Implementations

Recursive numerical versions of the gradient flow can be derived, paralleling the recursive schemes in [1]. Thus we propose

$$R_{k+1} = R_k e^{\alpha_k[R_k^{'}Q_{11}R_k,\ \tilde{\mathcal{H}}_{11}(R_k,p_k)]} \tag{3.1}$$

$$p_{k+1} = e^{-\tilde{H}_{22}(R_k,p_k)Q_{11}\delta}p_k - (Q_{11}R_k\tilde{H}_{12}(R_k,p_k) + Q_{12}\tilde{H}_{22}(R_k,p_k)\delta \tag{3.2}$$

for suitable small $\delta > 0$ and step-size adaptation

$$\alpha_k = \frac{1}{4\|Q_{11}\|\ \|\tilde{\mathcal{H}}_{11}(R_k,p_k)\|}\ . \tag{3.3}$$

A second approach to achieve recursive results is to seek solutions of the implicit optimal conditions (2.11), (2.12) recursively.

Thus a resonable selection is

$$\begin{aligned} R_{k+1} &= V_Q V_{\tilde{\mathcal{H}}_{11}}(R_k, p_k) \\ p_{k+1} &= R_{k+1} + \tilde{H}_{12}(R_k, p_k)\tilde{H}_{22}^{-1}(R_k, p_k) - Q_{11}^{-1}Q_{12}. \end{aligned} \tag{3.4}$$

No theory is given for these recursions, but simulation studies suggest that it is effective for local convergence. The calculation time of these recursive versions is 50 or 100 times faster than for the gradient flow using Matlab.

4. Conclusions

Optimum pose estimation procedures for quadratic surfaces have been presented. A general steepest descent method for pose estimation is proposed. For generic measurement data the gradient flow converges exponentially fast to a local minimum. Explicit recursive implementation of the gradient flow are proposed.

References

[1] U. Helmke and J.B. Moore, "Optimization and Dynamical Systems", Springer-Verlag, Communication and Control Engineering Series, Nov., 1993.

[2] R. Brockett, "Least squares matching problems", Linear Algebra Appl., 122-124. pp.761-777.

[3] O.D. Faugeras and M. Hebert, "The representation, Recognition, and Locating of 3-D Objects", IEEE Inter. J. of Robotics Research, Vol. 5, No. 3, 1986.

[4] G.L. Bilbro and W.E. Snyder, "Linear Estimation of Object Pose from Local Fits to Segments", IEEE Inter. Conf. on Robotics & Automation, pp. 1747-1752, 1987.

[5] D.S. Kalivas, A.A. Sawchuk, " A Region Matching Motion Estimation Algorithm", CVGIP: Image Understanding, pp. 275-288, 1991.

[6] Y. Han and W.E. Snyder and G.L. Bilbro, "Pose Determination using Tree Annealing", IEEE Inter. Conf. of Robotics & Automation, pp. 427-432, 1990.