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Periodic Structure Controller Design.[1]

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Abstract

This paper first demonstrates via averaging theory an approach whereby any stable linear system can be approximated by a simple periodic-structure system.

Next is proposed the control of continuous-time, linear, time-invariant plants via a periodic structure control scheme. It is established that for continuous-time minimal plants it is possible to design periodic-structure stabilizing first-order controllers which asymptotically approach the performance of an n^{th} order stabilizing time-invariant controller, such as an optimal (LQG) controller, in the limit as the switching rate increases. The proposed controllers suffer only a small loss of performance compared with the n^{th} order controller, are attractive from a computational point of view, and may be implemented in either discrete or continuous time.

Finally, the question of achieving a low order robust variable-structure controller for a high order uncertain plant is addressed.

Simulation results are shown which demonstrate the efficacy of the periodic structures proposed.

1 Introduction

Averaging theory [1], which allows the study of an "intractable" system by working with an approximation, termed the "averaged" system has found application to the areas of adaptive and robust control. It has provided useful insight and theoretical support for various existing designs, and facilitated the emergence of novel designs. Being such a powerful tool, it seems reasonable for us to explore a dual field of application for averaging theory where an easy-to-analyse system is approximated by an "intractable" system. For example, consider the case of the approximation of a given linear system, the averaged system, by means of a periodic dynamical system. Such "intractable" approximations could be attractive for implementation and perhaps for enhanced robustness in the case of a controller. To lead into such developments, we now review certain results for periodic systems and more general switched systems.

Certainly, it is well known that controllers with periodic gains can lead to more robust performance than time invariant ones, even for time-invariant plants. Gain and phase margin improvement are possible, see for instance [2] and its references for discrete-time results. However, these results do not always generate acceptable inter-sampling performance [3].

For continuous-compensators, averaging theory is used in [4] to explain the superior performance of vibrational controllers by demonstrating that they have zero-placement as well as pole relocation capabilities. Also [4] extends the discrete-time gain margin improvement results to continuous time. In the area of decentralised control, it has been shown that periodically switched-gain feedback laws in one feedback channel lead to decentralised controllability and observability in 2-channel systems with fixed modes, whereas time-invariant feedback laws do not, [5],[6]. There is thus a sound theoretical and practical basis for working with periodic controllers, even for time-invariant plants.

A more general class of time-varying/nonlinear controllers are termed state dependent switched controllers. Such consist of a set of continuous-time or discrete-time controllers , one of which is active at any time. The switched controllers may be time invariant or time varying, or even nonlinear. The switching may be periodic, thus including the class of periodic structure controllers, or may be driven by a rule which assesses which controller will work best according to some measure, given the data. Switched controllers are useful for the control of a system whose dynamics contain jump discontinuities in the dynamics, such as a robot arm which picks up a heavy object. To this end Khargonekar and Poolla [7] state, "in robust multivariable control problems nonlinear time-varying controllers yield advantages ... if there is parametric or structured uncertainty". Again we see that there is strong motivation for switched systems to achieve robustness. How then are such systems analysed ?

In the case of discrete-time periodic systems, methods exist to find an equivalence between discrete-time periodic systems, and higher order time-invariant systems [8],[9],[10]. Standard design results are then applied to the time-invariant representations, and re-interpreted to achieve results for periodic systems. For example, [11] finds the regulator gains for discrete-time periodic systems by solving algebraic Riccati equations for the associated time-invariant system. For continuous-time plants, these conversion methods are less well understood. Thus [9] reports that the methods of [8] can be extended to continuous-time systems with the drawback that the input and output spaces are in general infinite dimensional Hilbert spaces. Thus there are few explicit results dealing with the convergence of continuous-time switched systems. In fact, short of an infinite series (the Peano-Baker series), there is no analytic solution to the calculation of the transition matrix of a time-varying system unless the update matrix is semi-proper [12]. However approximations exist for the cases where the switching speed is high or low with respect to the dynamics of the system. Thus the stability of high frequency periodically switched systems has been tackled in [13],[14] where the author uses an approximation result on the exponentials of certain matrices to study the dependence of the stability on the frequency of switching. The author gives an explicit formula which facilitates checking the stability of such systems in the limiting case of high frequency switching. However, these results only apply to systems with piece-wise constant periodic gains, and there is no development of explicit error bounds or regions of convergence. There is therefore a need for stability results which are not restricted to piecewise constant periodic gains, and for which explicit error bounds and regions of convergence can be calculated.

To help focus ideas in meeting the needs for further work in this area, let us keep in mind an application which seems ideally suited to such developments. One possible application of state dependent switched systems is to achieve resonance suppression. Such a task has been tackled [15] where square-law nonlinearities in the control law are used to encourage energy flow from the resonances to the controller, and from one frequency band to the entire frequency band. In [16] there is implemented an adaptive resonance suppression controller/plant system with an unstable resonance, possibly due to a badly modelled plant, or changing plant dynamics. Could there be an advantage to a switched controller design approach?

In this paper we investigate by means of averaging theory two novel avenues of application of the switched and periodic-structure systems. The first is to use easy-to-implement periodic-structure systems to achieve low order approximations to a continuous-time controller.

Thus, in this paper, we seek approximations of a linear time-invariant (LTI) system via certain periodic-structure systems. We do not seek improvements in the gain or phase margin. In this way we recover the performance of the LTI system without any performance enhancement, but with considerable computational advantages, and with tight bounds on the inter-sampled behaviour.

The theoretical justification for this approach is developed from stability results for certain continuous-time time-varying systems (which may or may not be piece-wise continuous), as well as the calculation of specific error bounds as functions of the speed of time variations (ϵ), and a lower bound on the time-variation to ensure exponential stability of the periodic system. This is achieved via an averaging theoretic approach together with linear theory mathematically reminiscent of that employed by [17]. The results are then extended to ensure low sensitivity with respect to input disturbances. Examples illustrate effective controller simplification and resonance suppression.

Another application of switched systems we explore is to achieve effective resonance suppression in the presence of lightly damped resonances for uncertain systems, and in particular the simultaneous control of a number of LTI subsystems, each representing a resonance, via a switching between controllers, each designed to suppress one of the resonances. Viewing the plant as a partial fraction representation, or equivalently as a set of parallel resonances, the specific control objective we seek is to suppress the most excited resonance as indicated by

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some resonance detector.

The paper is outlined as follows. Section 2, provides preliminary definitions and notation, and gives the results linking certain periodic systems and time-independent systems. Section 3 uses these results to prove our main theorems on the approximation of a system (or controller) via fast switching between a set of lower order systems. In Section 4 we present simulation examples of certain periodic systems and of controller order reduction and compare the results with the theoretical bounds set in Sections 2.3. Section 5 contains simulation results of a switching controller resonance suppression problem. Conclusions are drawn in Section 6.

2 An averaging analysis of periodic systems

2.1 Introduction

This section presents the theoretical basis for the applications of periodic systems developed later. First we introduce necessary notation and quote a fundamental theorem from averaging theory. Then further results are derived for linear systems. In particular, for certain periodic systems and their time-averaged LTI systems with easily calculable state trajectories, we provide explicit bounds for the norm of the difference between the state trajectories of two systems. These bounds are expressed as ϵQ where ϵ is the switching speed, and Q is a constant derived for the specific systems.

2.2 Definitions:

1. $\delta_1(\epsilon) = O(\delta_2(\epsilon))$ for $\epsilon \to 0$ if there exists a constant k such that $|\delta_1(\epsilon)| \le k|\delta_2(\epsilon)|$ for continuous functions, $\delta_1, \delta_2 : \epsilon \to 0$. Also $\delta_1(\epsilon) = o(\delta_2(\epsilon))$ for $\epsilon \to 0$ if $\lim_{\epsilon \to 0} \frac{\delta_1(\epsilon)}{\delta_2(\epsilon)} = 0$. Refer to [1] for additional information.

2. Consider the vector field $f(t,x)$ with $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, Lipschitz-continuous in x on $D \subset \mathbb{R}^n, t \in [0,T]$; f continuous in t and x on $\mathbb{R}^+ \times D$. If the average
$$\overline{f}(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t,x)dt$$
exists, f is called a KBM-vectorfield.

3. Let r be a scalar, x a vector, and A a matrix. Then $|r|$ is the absolute value of r, $||x||$ is the L_2 norm of x, and $||A||$ is the induced L_2 norm of A.

4. Two time-varying functions $f(t), g(t)$ have *exclusive support* iff $f(t)g(t) = 0 \;\; \forall t$.

5. We employ a shorthand notation for systems:
$$P: \begin{array}{rcl} \dot{x} &=& A(t)x + B(t)u; \;\; x(0) = x_0 \\ y &=& C(t)x + D(t)u \end{array} \quad : \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & D(t) \end{array}\right]_{x_0}$$

6. $f(t,x)$ is *periodic* in t with period T iff $f(t,x) = f(t+T, x), \;\; \forall x, t$.

7. If $f(t,x)$ is periodic in t with period T, then the average $\overline{f}(x)$ is defined as
$$\overline{f}(x) \equiv \frac{1}{T}\int_0^T f(\tau, x)d\tau \tag{2.1}$$

8. Let $A(t)$ be a periodically time-varying matrix. We denote its period by T_A, its average $\overline{A}$, and define $D_A := \sup_t ||A(t) - \overline{A}||$, $S_A := \sup_t ||A(t)|| \ge ||\overline{A}||$, $B_A(t) = \int_0^t A(\tau) - \overline{A}d\tau$. Then $||e^{\overline{A}t}|| \le C_A e^{-\lambda_A t}, \forall t$, for some $C_A \ge 1$, for some λ_A which may be chosen via Proposition 1, $||B_A|| \le \frac{1}{2}T_A D_A$.

9. Denote the piecewise constant function
$$\Delta_i(n,t,T) \equiv \begin{cases} n, & (i-1)T/n \le t - mT < iT/n, \;\; m \in \mathbb{Z} \\ 0, & \text{otherwise.} \end{cases}$$

10. Given a square matrix A of size n, we construct a time varying matrix $A^{\#}(t,\epsilon)$ with period ϵ, and average A such that the rows of $A^{\#}(t,\epsilon)$ have independent support by
$$A^{\#}(t,\epsilon) = \begin{bmatrix} \Delta_1(n,t,\epsilon) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta_n(n,t,\epsilon) \end{bmatrix} A \tag{2.2}$$

11. Given a system $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, then
$$P^{\#}(t,\epsilon) = \left[\begin{array}{cc} A^{\#}(t,\epsilon) & B^{\#}(t,\epsilon) \\ \hline C & D \end{array}\right].$$

12. Denote the Lie brackett $[A, B] \equiv AB - BA$.

2.3 Proposition:

To facilitate calculation, we include the following proposition.

Proposition 1 *[17] For any n by n matrix F with $Re[\lambda_i(F)] < 0 \;\; \forall i$, there exist constants $\lambda_F > 0$, $C_F > 1$ such that $||e^{Ft}|| \le C_F e^{-\lambda_F t}, \quad \forall t \ge 0$. Furthermore, if $||F|| \le M$, $Re[\lambda_i(F)] < \alpha$, and $0 < \mu < 2M$ then one choice of C, λ may be generated via the inequality*
$$||e^{Ft}|| \le (2M/\mu)^{n-1} e^{(\alpha+\mu)t}, \; for \; t \ge 0 \tag{2.3}$$

2.4 Fundamental Averaging Result.

Theorem 2.1 (Eckhaus/Sanchez-Palencia[18],[1].) *Given*
$$\dot{x} = \epsilon f(t,x), \;\; x(0) = x_0 \tag{2.4}$$
with $x_0, x \in D \subset \mathbb{R}^n$. Suppose f is a KBM-vectorfield producing the averaged equation
$$\dot{y} = \epsilon f^0(y), \;\; y(0) = x_0$$
where f^0 is continuously differentiable with respect to y in D. Let $y = 0$ be an asymptotically stable critical point in the linear approximation with domain of attraction $D^0 \subset D$. Then for all $x_0 \in D^0$
$$x(t) - y(t) = O(\sup_{z \in D}\sup_{t \in [0, \frac{L}{\epsilon})} \epsilon || \int_0^t [f(\tau, z) - f^0(z)]d\tau||), \;\; 0 \le t < \infty$$
for any L independent of ϵ. Furthermore, if $f(t,x)$ is periodic in t, then
$$x(t) - y(t) = O(\epsilon), \;\; 0 \le t < \infty \tag{2.5}$$

The main results of this section can be verified by making use of the above averaging theorem. However, here we rederive these results in the context of linear systems. This provides explicit error bounds and regions of convergence in terms of ϵ.

2.5 Explicit Averaging Results for Linear Systems

We now introduce lemmas on the trajectories of linear-time-varying systems when compared to the trajectories of their time-averaged linear-time-invariant analogues. These results form the basis for the applications of the following sections.

Lemma 2.1 *Consider the following systems*
$$\begin{array}{llll} G_1: & \dot{x} = & \overline{A}x; & x(0) = x_0 \\ G_2: & \dot{y} = & A(t\epsilon^{-1})y; & y(0) = x_0 \end{array} \tag{2.6}$$
where $Re[\lambda_i(\overline{A})] < 0$, $A(t)$ is periodic with period T_A, average $\overline{A}$, and Definition (8) holds. Define
$$\epsilon_0 = \frac{\lambda_A}{2T_A D_A C_A[||\overline{A}|| + S_A]} \tag{2.7}$$
then
$$\begin{aligned} ||y(t) - x(t)|| \le & \; \epsilon T_A D_A C_A ||x_0||[1 + (||\overline{A}|| + S_A)C_A t]e^{-\lambda_A t} \\ & + (\epsilon T_A D_A C_A)^2 ||x_0||(||\overline{A}|| + S_A)[t + (||\overline{A}|| + S_A)C_A \frac{t^2}{2}]e^{-\lambda_A t/2} \\ & , \forall \epsilon < \epsilon_0 \end{aligned} \tag{2.8}$$
Moreover, defining $\phi_y(.,.)$ as the state transition matrix of $y(.)$, then $\forall \epsilon < \epsilon_0$,
$$\begin{aligned} ||\phi_y(t, t_0)|| \le & \; C_A e^{-\lambda_A/4(t-t_0)}\{1 + \epsilon T_A D_A(1 + \frac{4C_A(||\overline{A}|| + S_A)}{e\lambda_A}) \\ & + (\epsilon T_A D_A)^2 C_A(||\overline{A}|| + S_A)(\frac{4}{e\lambda_A} + 32\frac{(||\overline{A}|| + S_A)C_A}{e^2\lambda_A^2})\} \\ \stackrel{\Delta}{=} & \; C_y e^{-\lambda_y(t-t_0)} \end{aligned} \tag{2.9}$$

Remarks

1. There is a trade-off between the constant multiples and the decay rate in the error bounds. This proof is intended to show exponential convergence of the trajectories $x(t), y(t)$ given sufficiently fast time variation within $A(t)$.

2. Lemma (2.1) can easily be generalised to the case $x_0 \neq y_0$. Here we keep equal initial conditions since this is the case necessary for our applications.

Lemma (2.1) demonstrates that with the switching rate above some limit, any periodic system, with linear average which is exponentially asymptotically stable, is itself exponentially asymptotically stable. In the next lemmas, we extend these results to systems with input disturbances $\omega(\cdot)$. Thus we consider jointly, and separately the implications of assumptions (2.10),(2.11) and (2.12)

Assumptions on Noise : Disturbances $w(\cdot)$ obey the relation

$$\left\|\int_0^S w(t)dt\right\| \leq LS^\alpha, \text{ for some } L, 0 < \alpha < 1, \forall S > 0 \quad (2.10)$$

$$\|w(t)\| \leq W; \quad \forall t \quad (2.11)$$

$$\|\dot{w}(t)\| \leq \eta; \quad \forall t \quad (2.12)$$

Lemma 2.2 *Consider the system:*

$$G_3 : \quad \dot{z} = A(t\epsilon^{-1})z + \omega(t); \quad z(0) = z_0 \quad (2.13)$$

with the conditions of Lemma (2.1) and the Definitions (δ) and Bound (2.7) holding. Then $\omega(.)$ obeying Assumption (2.10) implies

$$\|z(t)\| \leq 2t^\alpha C_y L + C_y e^{-\lambda_y t}\|z_0\|, \quad \forall \epsilon < \epsilon_0 \quad (2.14)$$

and $\omega(.)$ obeying Assumption (2.11) implies

$$\|z(t)\| \leq C_y W/\lambda_y + C_y(\|z_0\| - W/\lambda_y)e^{-\lambda_y t} \quad (2.15)$$

where C_y and λ_y are constants defined via (2.9) such that

$$\|\phi_y(\tau, r)\| \leq C_y e^{-\lambda_y(\tau - r)}, \quad \forall \epsilon < \epsilon_0 \quad (2.16)$$

Lemma 2.3 *Consider the systems*

$$\begin{aligned} G_3 : \quad \dot{z} &= A(t\epsilon^{-1})z + B(t\epsilon^{-1})\omega(t); \quad & z(0) = s_0 \\ G_4 : \quad \dot{s} &= \overline{A}s + \overline{B}\omega(t); & s(0) = s_0 \end{aligned}$$

With the conditions of Lemma (2.1), Assumption (2.11), and Definition (δ) holding for matrices A, B, then

$$\begin{aligned} \|z(t) - s(t)\| \quad &\times \left\{1 + \frac{\epsilon(\|A\| + S_A)D_A T_A C_A}{\lambda_A - \epsilon(\|A\| + S_A)D_A T_A C_A}\right\} \\ &\leq \frac{WC_A}{\lambda_A}\left\{2D_B + \frac{\epsilon D_A T_A}{\lambda_A}[C_A(\|A\| + S_A) + 2\lambda]\right\}, \\ &\forall \quad W > \lambda_A\|s_0\|, \ \epsilon < \frac{\lambda_A}{D_A T_A C_A(\|\overline{A}\| + S_A)} \end{aligned} \quad (2.17)$$

and given Assumption (2.12),

$$\begin{aligned} \|z(t) - s(t)\| \quad &\leq \frac{\epsilon}{\lambda_A}\left\{\begin{array}{l} D_B T_B[W\lambda_A + C_A(W + \eta)] + \\ \frac{WC_A T_A D_A}{\lambda_A}[C_A(\|A\| + S_A) + 2\lambda] \end{array}\right\} \\ &\times \left\{1 + \frac{\epsilon(\|A\| + S_A)D_A T_A C_A}{\lambda_A - \epsilon(\|A\| + S_A)D_A T_A C_A}\right\}, \\ &\forall \quad W > \lambda_A\|s_0\|, \ \epsilon < \frac{\lambda_A}{D_A T_A C_A(\|\overline{A}\| + S_A)} \end{aligned} \quad (2.18)$$

Remarks:

1. Lemma (2.3) is a powerful sensitivity result. It shows for bounded disturbances, and sufficiently fast time variation, that the averaged and the periodic systems will have trajectories whose difference is tightly bounded. It is a consequence of the low pass nature of an integrator, and the high frequency of the perturbations introduced by $A(t\epsilon^{-1})$.

2. Assumptions guaranteeing the independence of the time variation of the noise, and the matrix $B(t\epsilon^{-1})$ lead to a lower expectation value of the error $\|z(t) - s(t)\|$. The aim of this paper is to provide hard bounds on the errors, and as such, stochastic calculations are beyond its scope, except to note that the actual errors will in most cases be much smaller than the bounds calculated.

3. When $\|w(t)\|$ is bounded, then the condition that $\dot{\omega}(t)$ is bounded will be automatically satisfied in the case of low-pass filters being applied to the inputs of a feedback loop.

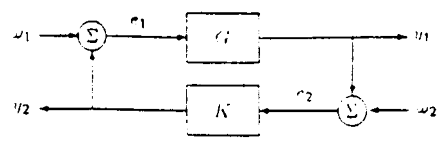


Figure 3.1: The feedback system $\{G, K\}$.

3 Controller Simplification

In this section we apply the results of the previous section to prove that any linear, continuous time stabilizing controller (in a feedback loop with any LTI plant which it stabilizes) can be approximated by certain variable structure controllers. These controllers may be designed to have the advantages of simple structure, or of being equivalent to first order systems, and thus much computationally cheaper than the full order controllers they are replacing. Moreover, we will show that as the speed of variation increases, the state trajectories of the feedback systems associated with these controllers, and the indices associated with their corresponding LQG cost functions, asymptotically approach those of the full order controller.

To this end we reformulate the results of Lemma (2.3) into our main results on the approximation of controllers:

Theorem 3.1 *Consider a plant, and full state feedback controller pair:*

$$\begin{aligned} G : \quad \dot{x} &= Ax + Bu + \omega_1(t); \quad x(0) = x_0 \\ K(t\epsilon^{-1}) : \quad u &= K(t\epsilon^{-1})x \end{aligned}$$

where $K(t)$ is periodic with period T and average K^0, and $(A + BK^0)$ is exponentially stable, and $\omega_1(t)$ is a noise term obeying Assumptions (2.10),(2.11). Consider also the "averaged" system

$$\begin{aligned} G : \quad \dot{x}^0 &= Ax^0 + Bu + \omega_1(t); \quad x^0(0) = x_0 \\ K^0 : \quad u &= K^0 x^0 \end{aligned}$$

Then the state x, of the system $\{G, K(t\epsilon^{-1})\}$ and x^0 of the system $\{G, K^0\}$ are related by

$$\|x(t) - x^0(t)\| \leq \epsilon D, \quad \forall \epsilon \leq \epsilon_0 \quad (3.1)$$

where ϵ_0 and D can be calculated from Lemma (2.3).

This result can be extended to controllers employing state estimation.

Theorem 3.2 *Consider the feedback loop $\{G^0, K^0\}$ as in Figure (3.1), with the disturbance noise ω_1, ω_2 obeying the Assumptions (2.10),(2.11), (2.12), and the systems G^0, K^0 being*

$$G : \left[\begin{array}{c|c} A^0 & B^0 \\ \hline C^0 & D^0 \end{array}\right]_{x_0}$$

$$K^0 : \left[\begin{array}{c|c} A^0 + B^0F^0 + H^0C^0 + H^0D^0F^0 & -H^0 \\ \hline F^0 & 0 \end{array}\right]_{\hat{x}_0 = x_0}$$

where the gains H^0, F^0 have been chosen (by LQG design or otherwise) such that $(A^0 + B^0F^0), (A^0 + H^0C^0)$ are exponentially stable.

Consider a second feedback system, $\{G(t\epsilon^{-1}), K(t\epsilon^{-1})\}$ with the systems being:

$$G(t\epsilon^{-1}) : \left[\begin{array}{c|c} A^1(t\epsilon^{-1}) & B^1(t\epsilon^{-1}) \\ \hline C^1(t\epsilon^{-1}) & D^1(t\epsilon^{-1}) \end{array}\right]_{x_0}$$

$$K(t\epsilon^{-1}) : \left[\begin{array}{c|c} A^2(.) + B^2(.)F(.) + H(.)C^2(.) + H(.)D^2(.)F(.) & -H(.) \\ \hline F(.) & 0 \end{array}\right]$$

where the time dependence of $A^2(t\epsilon^{-1})$, etc. have not been shown due to space limitations, and the matrix blocks have been defined so that

$$\xi^0 = \frac{1}{T}\int_0^T \xi(t)dt; \quad \xi(t+T) = \xi(T) \tag{3.2}$$

for ξ being each of the blocks $A^1, A^2, B^1, B^2, \ldots$, as well as the multiples B^1F, HC^1, HD^1F, HD^2F. Define the augmented states of the systems $\{G^0, K^0\}$ and $\{G(t\epsilon^{-1}), K(t\epsilon^{-1})\}$ as $X^0 \equiv [x^{0T} \quad \hat{x}^{0T}]^T$, $X(t) \equiv [x^T \ \hat{x}^T]^T$, then

$$\|X^0 - X\| \leq \epsilon L, \quad 0 \leq t < \infty, \quad \forall \epsilon < \epsilon_0 \tag{3.3}$$

where L, ϵ_0 are constants and can be calculated via (2.18), (2.7)

Remarks

1. This result links the state trajectories of any set of plants/controllers obeying the assumptions of the theorem. The power of its application arises from the fact that the matrix blocks of $G(t\epsilon^{-1}), K(t\epsilon^{-1})$ need not be time varying. Thus, we can use Theorem (3.2) to take a stable (time invariant) $\{G, K\}$ system, and link the state trajectories to those of systems where either or both of G, K may be time varying. Furthermore, we can take a stable time-varying system, and design other time variations using concepts such as independent support of Definition (10) to simplify analysis, and provide results pertaining to both systems.

We specialize Theorem (3.2) to the application of controller order reduction.

Theorem 3.3 *Consider a linear time-invariant feedback system $\{G, K\}$ where the augmented state update block is exponentially stable, and K and the periodic structure controller $K^\#$ have the form:*

$$K: \left[\begin{array}{c|c} E & F \\ \hline G & 0 \end{array}\right], \qquad K^\#: \left[\begin{array}{c|c} E^\#(t\epsilon^{-1}) & F^\#(t\epsilon^{-1}) \\ \hline G & 0 \end{array}\right]$$

where $\#$ is the operation of Definition (10). Then $K^\#$ is computationally equivalent to a first order controller. In a given bounded noise environment, with the noise signals obeying Assumptions (2.10),(2.11), (2.12), the LQG indices of the two systems are related :

$$LQG_{\{G,K\}} - LQG_{\{G,K^\#\}} = O(\epsilon), \quad \forall \epsilon < \epsilon_0 \tag{3.4}$$

where the explicit dependence on ϵ, and the value of ϵ_0 may be calculated via Lemma (2.3).

Remark:

To further illustrate our claim that a controller with only one state changing at a given time is computationally equivalent to a first order controller, consider the dynamical system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e(t) \\ f(t) \end{bmatrix} u(t), \ x_1(0), x_2(0). \tag{3.5}$$

where $a(.), b(.), e(.)$ are periodic with period T, and are zero for time $t \in [(2n+1)T/2, nT]$, $n \in J^+$ and which have independent support with respect to the periodic functions $c(.), d(.), f(.)$ which are zero for time $t \in [nT, (2n+1)T/2]$, $n \in J^+$. Then, it is clear that the system (3.5) is equivalent to a combination of the two systems:

$$\left.\begin{array}{l} \dot{y}_1 = a(t)y_1 + b(t)C_1 + e(t)u(t) \\ \dot{y}_2 = 0 \end{array}\right\} \ t \in [nT, (2n+1)T/2], n \in J^+$$

$$\left.\begin{array}{l} \dot{y}_1 = 0 \\ \dot{y}_2 = c(t)C_2 + d(t)y_2 + f(t)u(t) \end{array}\right\} \ t \in [(2n+1)T/2, nT], n \in J^+ \tag{3.6}$$

At $t = nT$, then C_1 is updated to the value of $y_2(nT)$, and when $t = (2n+1)T/2$, C_2 is updated to the value of $y_1((2n+1)T/2)$.

Since $y_1(t)$ and $y_2(t)$ are not updating simultaneously, a single first order system, with a switching of initial conditions and constants at time $t = nT/2, n \in J^+$, can exactly follow the trajectories of $y_1(t), t_2(t)$, which in turn exactly follow the trajectories of $x_1(t), x_2(t)$.

4 A Case Study

In this section, we present examples of simulations pertaining to the Lemmas of Section 2,and the Theorems of Section 3.

Example 1: Illustration of Lemma (2.1)

The system studied with reference to Section 2 is adapted from an example [20] of an unstable periodic system, with a stable averaged system. We demonstrate how increasing the frequency of the periodicity will stabilize the system, and also how the bounds between the sates of the fast periodic system, and the averaged system are calculated.

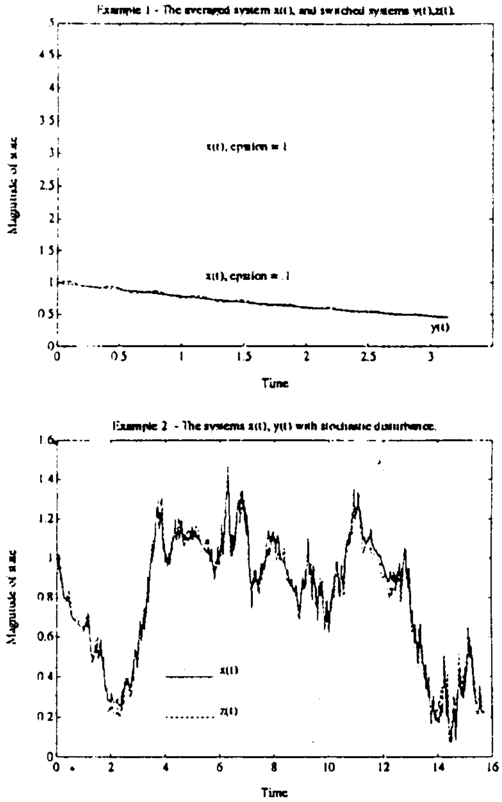


Figure 4.1: Comparison of $x(t), y(t), z(t)$

The periodic system simulated is:

$$\dot{x} = A(t\epsilon^{-1})x; \quad x(0) = [-1 \ 0]' \tag{4.1}$$

$$A(t\epsilon^{-1}) = \begin{bmatrix} -1 + 3/2\cos^2(t\epsilon^{-1}) & 1 - 3/2\cos(t\epsilon^{-1})\sin(t\epsilon^{-1}) \\ -1 - 3/2\sin(t\epsilon^{-1})\cos(t\epsilon^{-1}) & -1 + 3/2\sin^2(t\epsilon^{-1}) \end{bmatrix}$$

The averaged system is:

$$\dot{y}(t) = \overline{A}y; \quad y(0) = [-1 \ 0]', \quad \overline{A} = \begin{bmatrix} -1/4 & 1 \\ -1 & -1/4 \end{bmatrix} \tag{4.2}$$

Thus the averaged system $\overline{A}$ has stable eigenvalues, $\lambda_i(\overline{A}) = -.25 \pm i$, also $\|e^{\overline{A}t}\| = e^{-.25t} = Ce^{-\lambda t}$, and the following constants hold for the systems (4.1),(4.2):

$$\|\overline{A}\| = 1.03, \quad S_A = \sup_t \|A(t)\| = 1.78, \quad \lambda = .25$$

$$D_A = \sup_t \|A(t) - \overline{A}\| = .75, \quad C_A = 1, \quad T_A = 2\pi$$

With $\overline{A}, A(t)$ defined in (4.1),(4.2), then (2.8) and (2.9) become:

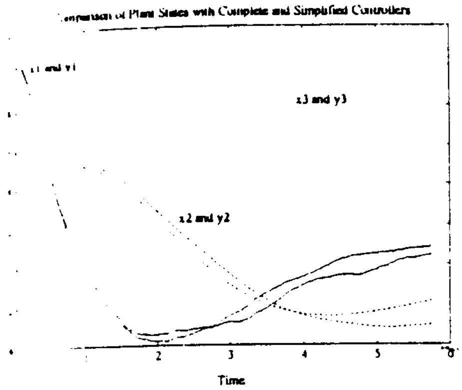
$$\|y(t) - x(t)\| \leq \epsilon\|x_0\|[4.71 + 14t + .81t^2]e^{-.125t}; \quad \forall \epsilon < \frac{1}{34\pi}$$

$$\|\phi_y(t, t_0)\| \leq (1 + 206\epsilon)e^{-.0625(t-t_0)} \ \forall \epsilon < \frac{1}{34\pi}$$

The trajectories for $x(t)$ with $\epsilon = 1$, $x(t)$ with $\epsilon = .1$, and $y(t)$, all with initial condition $x_0 = y_0 = [-10]'$, are shown in Figure 4.1. It is clear that $x(t)$ will grow exponentially for $\epsilon = 1$, and converge to $y(t)$ for $\epsilon = .1$. Thus this simulation, and others not shown here, illustrate that the region of convergence includes the calculated bound of $\epsilon < 1/144$.

Example 2: Illustration of Lemmas (2.2),(2.4)

In this example, equations (4.1), and (4.2) are modified by the addition of a disturbance signal, ω, with a uniform amplitude distribution



...ure 4.2: Comparison of plant trajectories with the full, and first ...der controllers.

...ounded by $|\omega| < 5$.

$$
\begin{array}{rclll}
\dot{z} & = & A(t\epsilon^{-1})z & + \omega(t); & z(0) = x_0 \\
\dot{x} & = & \bar{A}x & + \omega(t); & x(0) = x_0
\end{array}
\tag{4.3}
$$

For the case-study, Lemma 2.2 gives a bound on $z(t)$ $||z(t)|| \leq 81(1+ ...\epsilon)$; and from Lemma (2.4), $||z(t) - x(t)|| \leq 2600\epsilon$, $\forall \epsilon < 0.05$.

Our case study, as depicted in Figure 4.1(b) suggests that the theoretically calculated bounds are extremely conservative.

Example 3: A simulation of controller order reduction via Theorem (3.3)

In this section, we describe results obtained by simulation of the procedure outlined in Theorem (3.3) for the reduction of complexity of a given controller.

Consider the plant (transfer function):

$$
G : \frac{s^2 + 5s + 6}{s^4 - 1.4s^3 + .75s^2 - .14} \tag{4.4}
$$

We design a corresponding LQG regulator for the cost index, and noise covariances (4.5).

$$
I = \int_0^t x'x + u'udt, \quad E([\omega_1 \ \ \omega_2]'[\omega_1 \ \ \omega_2]) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{4.5}
$$

Define this optimal LQG regulator K. Then the periodic-structure, first-order controller $K^{\#}$ can be designed via Definition (11) and Theorem (3.3). The feedback systems $\{G, K\}$ and $\{G, K^{\#}\}$ with a switching speed of $\epsilon = .01$ are simulated with an identical noise signal ω which is uniformly distributed between $-.5$ and $.5$.

As can be seen from Figure (4.2), the states of the two systems track each other well, even though for this example Lemma (2.4) only bounds the tracking error for the case $\epsilon \leq 10^{-9}$. The LQG indices calculated for the two systems are : $LQG_{\{G,K\}} = 6.3$, $LQG_{\{G,K^{\#}\}} = 6.5$.

Remark: The differences between the trajectories of the averaged and variable-structures systems found via simulation are several orders of magnitude smaller than the bounds calculated for the maximum difference. This is because the conservativeness of the bounds is multiplicative, and the theory must at each stage of the calculation bound the maximum possible error.

5 Switching Application for Resonance Suppression

In this section, we explore a further application of state dependent switching systems to the suppression of resonances within a plant. The key idea is to have a time varying system which "switches" its feedback behaviour to that of a controller designed for whichever resonance is most excited at a given time. This switching can be thought of as a type of fast adaptation, where specific knowledge of the plant behaviour is used to enable the controller to "jump" between operating modes.

In gain scheduling, the controller changes its behaviour based on the position of the controller within state space. Here, the scheduling of the controller is decided by an analysis of the output of the feedback loop, and the subsequent decision as to which modes of the plant most need to be controlled at a given time.

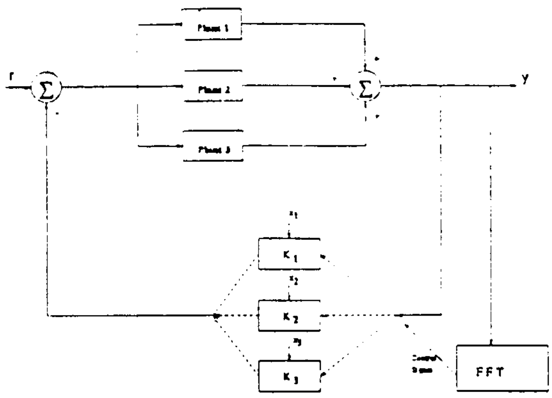


Figure 5.1: Switched control System

Consequently, we employ a partial fraction expansion to view the plant as a set of n resonances as in Figure (5.1). A controller is designed for each resonance, and the switching mechanism we employ switches in the controller for the resonance least well controlled at a given time.

In a previous section, it was shown that it is possible, given fast enough switching, to approximate a given n^{th} order controller by a number of lower order controllers. The drawback with this method is that it requires an extremely fast switching speed, which may not be feasible in certain situations. Also, this method requires that the average of the switched systems equals the original system. However, in the case of a non-nominal plant, or of input disturbances exciting certain modes, performance/robustness enhancement may be possible by an adaptation of the switching mechanism.

Consequently, in this section we are not approximating a higher order controller. Thus for a nominal plant we achieve controller order reduction, but not optimality. In some cases however, we achieve superior robustness properties.

Although we have, as yet, few theoretical results for this algorithm, we present it a useful application of switched systems. In a companion paper [21] we make a more complete study of the algorithm , testing for robustness properties, and optimality in a number of situations.

Description of switching algorithm :

1. Express the plant via a partial fraction description as a number of lower order components (resonances).
2. Calculate a (optimal LQG) controller for each sub-plant.
3. Use a FFT on the Plant output to detect which resonance is excited at any one time, and switch in the controller designed for that resonance.

The parameters to be selected:

- The length of the FFT.
- The speed of switching, and the index used for the switching decision.
- The smoothness of the switching transitions.

The optimal selection of these parameters for a given plant with a given uncertainty and given noise environment is beyond the scope of this paper, but will be further discussed in [21]. Here we demonstrate by simulations that this switching algorithm can be used in a heuristic way to control an uncertain plant.

Simulation Results:

The plant is set to be a group of n resonances (complex conjugate pole pairs). For this example, the poles are all simple, and the calculations are done in discrete time. Consequently, we place a zero at -1, to simulate sampling.

Example 1: An unstable system

In this example we demonstrate how a pair of unstable poles cannot be controlled by certain LQG controllers designed for either one, but can be controlled by a time varying controller which consists of a switching between the two controllers. In effect, this means that a stable control is generated from a switching between two unstable control loops. Of course, there are many cases where even a switched controller cannot stabilize the system. Here we demonstrate the possibility of greater robustness of a controller made of the complete switched

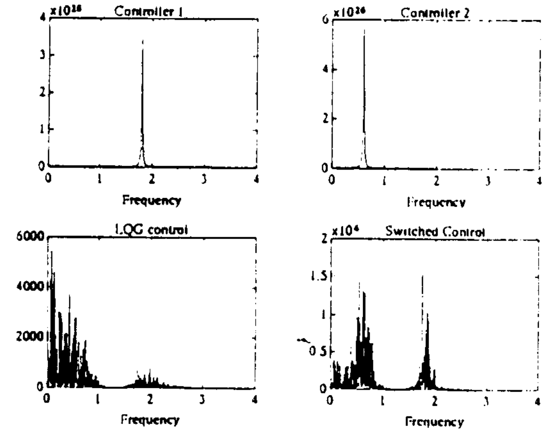


Figure 5.2: Spectral Density of Output for Four controller Configurations

system when compared with a controller comprising of only one of its components.

The plant studied consists of two resonances, each at a radius of 1.01, and at frequencies of 0.6 and 1.8. The LQG controllers are designed for the cost index and noise covariances:

$$I = \frac{1}{T}\int_0^T x'x + u'ud\tau, \quad E(\omega^2) = 1$$

The parameters used for the switched controller are included in the Appendix. The results for the control of the plant by the controllers designed for each resonance (2^{nd} order), the LQG controller designed for the whole plant (4^{th} order), and the (2^{nd} order) switched controller are shown in Table (5.1).

Controller	Index
Controller # 1	10^{20}
Controller # 2	10^{20}
Full order LQG controller	0.017
Switched controller	0.067

Table 5.1: LQG indices for Example 1

Figure 5.2 shows the spectral density of the output for the cases of the controller for the first resonance , the second resonance , the optimal LQG , and the switched controller.

As expected, the two resonance suppression controllers effectively damped energy in the frequency bands they were designed for, but could not control the alternate resonances. The switched controller was able to do this, but there was some energy left in each frequency band from when it switched to control the other frequency band.

6 Conclusion

The idea of using averaging theory for the analysis and design of variable-structure systems has given us new results on the stability and state dynamics of time-varying systems which are not usually analytically tractable. These results are valid for linear, variable-structure systems where the time scale of the switching is fast compared to the dynamics of the plant. We calculate in general, and for a specific example, convergence bounds of the switching speed.

A byproduct of these stability results is the comparison of certain time-varying systems and their timewise averaged, time-independent systems. It is shown that, even in certain "stochastic" environments (with bounded disturbances), their states are bounded, with the difference being $O(\epsilon)$ where ϵ is a measure of the switching speed. This linkage gives us further degrees of freedom in the design of controllers/plants, and these are employed to achieve controller simplification, order reduction, and discretization.

We also develop arguments supporting our view that a logical application of these, and other variable-structure systems is to resonance-suppression, and we demonstrate via simulation some of the power of these techniques.

References

[1] J. A. Sanders, F. Verhults, "Averaging Methods in Nonlinear Dynamical Systems", Springer-Verlag, 1985.

[2] Khargonekar, Poolla, Tannenbaum, "Robust control of linear time-invariant plants using periodic compensation", IEE Trans. Automatic Control, AC-30, No. 11.

[3] Goodwin, Feuer, "Linear Periodic Control": A Frequency Domain Viewpoint", submitted.

[4] Lee, Meerkov, Runolfsson, "Vibrational Feedback Control: Zeros Placement Capabilities", IEEE Trans. Auto. Control, Vol AC-32, No. 7, July 1987.

[5] Anderson, Moore, "Decentralised Control Using Time-Varying Feedback", Control and Dynamical Systems, 1985

[6] Anderson, Moore, "Time-Varying Feedback Laws for Decentralised Control ", IEEE Trans. Auto. Control, AC-26, No. 5, Oct. 1981

[7] Khargonekar, Poolla, "Robust control of Linear Time-Invariant Plants using switching and nonlinear feedback", Proc. 28th CDC, Florida 1989.

[8] Sz.-Nagy, Foias, "Harmonic Analysis of Operators on Hilbert Space", New York, American Elsevier, 1970.

[9] Khargonekar, Poolla, Tannenbaum, "Robust control of Linear Time Invariant Plants Using Periodic Compensation", IEEE Trans. Automatic Control, AC-30, No 11.

[10] Colaneri, "Discrete Time Linear Periodic Systems Stabilization and Zero Error Regulation", Proceedings of IFAC world congress, 1990.

[11] Flamm, "A new shift-invariant representation for periodic linear systems", Systems & Control letters 17 (1991), North-Holland

[12] D. Wiberg, "State Space and Linear Systems", Schaum Outline Series, Mcgraw Hill, 1971

[13] Tokarzewski, "Stability of periodically switched linear systems and the switching frequency", Int. J. of Systems Science, 1987, No 4, 697 - 726.

[14] Tokarzewski, "Stability of linear systems with periodic feedback and sampled-data systems", Int. J. of Systems Science, 1988, No 11, 2339 - 2366.

[15] Hyland, "A Nonlinear Vibration Control Design with a Neural Network Realization", Proc. 29th CDC, Dec. 1990

[16] Telford, Moore, "Adaptive Stabilization and Resonance Suppression", Int. J. Control, 1990, Vol 52, No 3, 725-736.

[17] Coppel W. A., "Dichotomies in Stability Theory", Lecture Notes in Mathematics, 629, Springer-Verlag, 1978.

[18] Sanchez-Palencia, E., "Methode de centrage et comportement des trajectories dans l'espace des phases", Compt. Rend. Acad. Sci., Ser. A, Paris 280, pp. 105-107 (1975)

[19] Desoer, Vidyasagar, "Feedback Systems: Input-Output Properties", Prentice-Hall, 1978

[20] Markus, L. and H. Yamabe, "Global stability criteria for differential systems", Osaka Math J. 12, 1960, 305-317.

[21] Irlicht, Marcels, Moore, "Switched Controller Design", Submitted to IFAC World Congress, 1993.