# ON THE SELF-TUNING REGULATOR AND A PRIORI ELS CONVERGENCE.

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## ABSTRACT

It is only now, after one decade of adaptive control theory development following on from the self-tuning-regulator (STR) algorithms of Astrom and Wittenmark, that the original insights imbedded in its design are justified by a rigorous global convergence theory, as outlined in the paper.

The original STR is of current interest from a theoretical point of view because it is the first example of a direct adaptive stochatic control scheme for which the plant noise, when present, is sufficient to achieve the persistence of excitation required for asymptotic optimality. This aspect is featured in the theory of the paper.

The theory of the paper also generalizes open-loop extended least squares (recursive maximum likelihood) global convergence theory to the case where state estimates involve a priori noise estimates are employed rather than a posteriori ones as in existing theories.

## 1. INTRODUCTION

The challenge that the self-tuning regulator (STR) [1] presents to the theorist is to match the simplicity and robustness of the algorithm with a theory as to why it works as well as it does.

Early simulations hinted at global convergence under a plant minimum phase assumption for ARMAX (autoregressive moving average exogenous input) plants. However, analysis in [2,3] and simulations in [4] show that, as in open-loop extended least squares schemes [2,5,6], for "highly" correlated plant noise, there may not be asymptotic optimality. Thus a now familiar strict positive real condition on the noise model is introduced or side-stepped using the techniques of [7].

The first convergence analysis techniques applied to the STR assumed a **priori** closed-loop stability and then gave asymptotic results [1,2,3]. Then global convergence results where achieved for stochastic approximation based versions [8]. What then of the least squares based adaptive control?

To date, extended least squares techniques in adaptive estimation and control have global convergence theories as in [8-13] only when the state estimates include the a **posteriori** estimates, and the parameter updates include noise model parameter estimates. The theories apply only to indirect adaptive control schemes. It is known [14,15] that the original STR can be viewed as an extended least squares scheme employing a **priori** noise estimates in such a way that the plant noise model parameter estimates are not updated or used in the controller. The STR is a truly direct adaptive controller not requiring explicit estimates of the plant noise model parameters.

The intuition behind the original STR design and some subsequent "direct" adaptive control algorithms is that only the minimum variance controller parameters need to be identified, and these can be consistently estimated on line in a self-tuning scheme by virtue of plant noise alone.

The addition of persistently exciting input or reference signals, at the expense of asymptotic optimality, is not required to achieve this. This is in contrast to the case for the direct adaptive control schemes of [9,10] with persistence of excitation results as in [11,13].

In this paper, the open-loop and closed-loop global convergence theories of [9,13] are generalized to cope with a priori noise estimates in the state estimates, and thereby to cope with the original STR modified by a (weighting coefficient) selection factor. Moreover, for the closed-loop STR, it is shown that plant noise, assumed to exist, is sufficient to give persistence of excitation and thereby asymptotic optimality.

## 2. THE STR AND A PRIORI NOISE ESTIMATION

In this section, a rationale is given for the orignal STR based on a priori noise estimation.

The Plant: Consider the ARMAX plant in standard polynomial delay operator notation as:

$$Ay_k = Bu_k + Cw_k \qquad (2.1)$$

with $w_k$ the plant zero mean "white" noise, $y_k$ the plant output, and $u_k$ the plant exogenous input. For the theory to follow, $w_k$ is assumed to satisfy

$$E[w_k|F_{k-1}] = 0, \; E[w^2{}_k|F_{k-1}] \leq \sigma_w \text{ and } \limsup_{k \to \infty} \frac{1}{k} \sum_{1}^{k} w_i{}^2 < \infty \text{ a.s.}$$

where $F_k$ is the $\sigma$-algebra generated by $w_0$, $w_1, \ldots w_k$. The plant is assumed to be strictly minimum phase so that $B^{-1}$ and $C^{-1}$ are exponentially asymptotically stable operators. An alternative expression for (2.1) is:

$$y_k = b_1 u_{k-1} + \theta' x_k + w_k \qquad (2.2)$$

where, without loss of generality taking $\ell = n$,

$$\theta' = [(c_1 - a_1)\cdot\cdot(c_n - a_n)b_2\cdot\cdot b_m c_1\cdot\cdot c_n] \qquad (2.3)$$

$$x_k' = [y_{k-1}\cdot\cdot y_{k-n}\; u_{k-2}\cdot\cdot u_{k-m}(w_{k-1}-y_{k-1})\cdot\cdot(w_{k-n}-y_{k-n})] \qquad (2.4)$$

An additional requirement of our theory development is that the polynomials B and (A-C) be relatively prime.

An 'a priori' Extended Least Squares Scheme: Consider state estimates:

$$\hat{x}_k' = [y_{k-1}\cdots y_{k-n}\; u_{k-2}\cdots u_{k-m}(\hat{w}_{k-1|k-2}-y_{k-1})\cdots(\hat{w}_{k-n|k-n-1}-y_{k-n})] \qquad (2.5)$$

and parameter estimates:

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \hat{\gamma}_k \hat{B}_k \hat{x}_k (y_k - b_1 u_{k-1} - \hat{\theta}'_{k-1} \hat{x}_k). \tag{2.6a}$$

$$\hat{B}_k = \hat{B}_{k-1} - \frac{\hat{B}_{k-1} \hat{x}_k \hat{x}_k' \hat{B}_{k-1}}{\hat{\gamma}_k^{-1} + \hat{x}_k' \hat{B}_{k-1} \hat{x}_k} \tag{2.6b}$$

for $k \geq r$ where $r$ is the first integer such that, in obvious notation.

$$\hat{B}_r^{-1} = \sum_1^r \hat{\gamma}_i \hat{x}_i \hat{x}_i' + \text{diag } \{0_{n+m}, I_n\} \tag{2.6c}$$

is not singular. The a priori noise estimates are given from

$$\hat{w}_{k|k-1} = y_k - b_1 u_{k-1} - \hat{\theta}_{k-1} \hat{x}_k \tag{2.7}$$

and the control $u_k$ is chosen so that

$$\hat{y}_{k+1|k} \triangleq b_1 u_k + \hat{\theta}'_{k+1} \hat{x}_{k+1} = 0 \tag{2.8}$$

Observe that the control selection (2.8) forces the a priori noise estimates $\hat{w}_{k|k-1}$ to be the plant outputs $y_k$, so that the state estimates (2.5) can be written as:

$$\hat{x}_k' = [y_{k-1} \cdots y_{k-n} \; u_{k-2} \cdots u_{k-m} \; 0 \cdots 0] \tag{2.5}$$

The algorithms (2.6), (2.8) now simplifies as the standard (but weighted by $\hat{\gamma}_k$ in (2.6)) STR algorithm, deleting zero blocks of vectors and matrices. Thus denoting:

$$\hat{x}_k' = [\bar{x}_k' \; 0 \cdots 0], \; \hat{\theta}_k' = [\hat{\bar{\theta}}_k' \; 0 \cdots] \tag{2.9a}$$

$$\hat{\bar{B}}_k^{-1} = \text{diag. } \{\bar{P}_k^{-1}, I\}, \; \bar{P}_k^{-1} = \sum_1^k \bar{x}_i \bar{x}_i' \tag{2.9b}$$

$$\hat{\theta}_k' \hat{x}_{k+1} = \hat{\bar{\theta}}_k' \hat{\bar{x}}_{k+1} \tag{2.9c}$$

The standard STR algorithm is expressed in terms of $\bar{x}_k$, $\bar{P}_k$ and $\hat{\bar{\theta}}_k$.

The term diag$\{0_{n+m}, I_n\}$ in (2.6c) ensures that $B_k$ exists, although its necesity is a hint to the persistence of excitation issues to arise later.

Weighting Coefficient Selection: In practice, the $\hat{\gamma}_k$ selection is taken as a constant in the presence of stability. (Note that $\hat{\gamma}_k\hat{B}_k$ is invariant of $\hat{\gamma}_k$ when $\hat{\gamma}_k=\gamma\neq 0$, so that without loss of generality take $\gamma=1$.) However, in the presence of instability it is known from simulations that improved performance can be achieved by giving less weighting to the recent data, or equivalently having a decreasing $\hat{\gamma}_k$. Our theory calls for a monotonically non-increasing $\hat{\gamma}_k$ (at least asymptotically). The $\hat{\gamma}_k$ selection defined below appears cumbersome at first glance, but allows what we believe is the "simplest" convergence analysis. When there is "stablity" and "persistence of excitation" with bounded noise inputs, $\hat{\gamma}_k$ is constant, otherwise $\hat{\gamma}_k$ decays. The following selection has some features in common with that for a posteriori based extended least squares in [9,10, and more recently 16], but has new features which were not envisaged at the time of development of the a posteriori schemes. Let us define, for some bounds and arbitrary small $\varepsilon>0$,

$\gamma_k = 1$ if $tr[\bar{P}_k]$ and $\hat{x}_k'\hat{B}_{k-1}\hat{x}_k$ decay to zero "faster" than $s_k$, and the condition number of $\bar{P}_k^{-1}$ is bounded above ($k\varepsilon S_1$).

$\ln(\sum_{1}^{k}\hat{x}_i'\hat{x}_i)^{-1-\varepsilon}$ if $k\notin S_1$ but condition number of $\bar{P}_k^{-1}$ is bounded ($k\varepsilon S_2$).

$s_k^{-\frac{1}{2}}(\ln s_k)^{-\frac{1}{2}(1+\varepsilon)}$ otherwise ($k\varepsilon S_3$), where $s_k = \sum_{1}^{k}\hat{x}_i'\hat{B}_{i-1}\hat{x}_i$

$$\gamma_k^* = \min[\gamma_k, \gamma^*_{k-1}]$$

$\hat{\gamma}_k = \gamma_k^*$ if $\gamma_k\hat{x}_k'\hat{B}_{k-1}\hat{x}_k < \varepsilon/2$ or $k<j$ the first time instant that the inequality is satisfied.

$\min[\hat{\gamma}_k^*, \varepsilon(2\hat{x}_k'\hat{B}_{k-1}'\hat{x}_k)^{-1}]$ otherwise ($k\varepsilon S_4$). (2.10)

## 3. A PRIORI EXTENDED LEAST SQUARES AND STR CONVERGENCE RESULTS

The challenge addressed in this section is to generalize the extended least squares based global convergence analysis techniques of [9-13] to cope with state estimates involving a **priori** noise estimates rather than a **posteriori** noise estimates as in the original theory. The work of the previous section then allows interpretation to give STR global convergence results.

Ironically, the first global convergence results for the open-loop estimation case [5a] were for the case of a **priori** noise estimates in the state estimates, but then it was observed that the theory simplified and could be more complete by working with the a **posteriori** noise estimates [5b,6].

For the closed-loop case, in generalizing the earlier stochastic approximation results to a least squares version, an important observation of [9] is that the convergence theory is inherently more straightforward for the case of a **posteriori** noise estimates in the state estimate vector than if a **priori** estimates are used. Now dealing with the closed-loop STR case we must of necessity cope with the more difficult case involving a **priori** noise estimates.

To achieve our objective, we generalize the first lemma and theorem of a sequence of results in [9,12], and then observe that the remaining results carry through mutatis mutandis.

Lemma 3.1: With the step size selection rule (2.10) associated with the a **priori** Extended Least Squares algorithm, including STR schemes of the previous section, then for arbitrary $\varepsilon>0$ and bounds in (2.10),

$$\sum_{1}^{\infty} \delta_k \hat{\gamma}_k^2 \hat{x}_k' \hat{B}_k \hat{x}_k < \infty \quad \text{a.s.} \tag{3.1a}$$

$$\hat{\gamma}_k \hat{x}_k' \hat{B}_k \hat{x}_k < \varepsilon \text{ for } k>j \text{ [defined in (2.10)]} \tag{3.1b}$$

$$\liminf_{k\to\infty} \delta_k \hat{\gamma}_k \left(\sum_1^k \hat{x}_i{}'\hat{x}_i\right) > 0 \text{ a.s.} \tag{3.1c}$$

where $\delta_n = (\lim \sum_1^k \hat{x}_i'\hat{x}_i)^{-1-\varepsilon}$ if $k\varepsilon S$ and $\delta_k = 1$ otherwise.

Proof: See Appendix.

Remark: The results (3.1a), (3.1c) are not surprising in the light of earlier theory for weighting coefficient selection for a **posteriori** extended least squares schemes [9,10,16]. However, result (3.1b) is stronger than for earlier schemes and is crucial to the convergence results to follow for a **priori** based extended least squares.

Lemma 3.2: Consider the plant (2.1) (not necessarily stable orminimum phase) and the a **priori** (weighted) extended least squares cheme with $\hat{\gamma}_k$ selection (2.10). Consider also that $[C^{-1}(z)-\frac{1}{2}]$ is strictly positive real. Then (see proof for details):

(i) $\limsup_{k\to\infty} \delta_k \tilde{\theta}_k' \hat{B}_k^{-1} \tilde{\theta}_k < \infty$ a.s., $\tilde{\theta}_k = \theta - \hat{\theta}_k$

(ii) $\sum_1^\infty ||\hat{B}_{k-1}||^{-1} \; ||\hat{\theta}_k - \hat{\theta}_{k-j}||^2 < \infty$ for all finite j

(iii) $\sum_1^\infty \delta_k \hat{\gamma}_k (\hat{w}_{k|k-1} - w_k)^2 < \infty$, $\sum \delta_k \hat{\gamma}_k ||\tilde{x}_k||^2 < \infty$ a.s. $\tilde{x}_k = x_k - \hat{x}_k$

(iv) for the STR case [control given from (2.8)]

$$\sum_1^\infty \delta_k \hat{\gamma}_k (y_k - w_k)^2 < \infty \text{ a.s.}$$

Proof: See Appendix.

Lemma 3.3: (Linear boundedness) Under the conditions of Lemma (3.2) and the additional constraint that for some $\bar{k},\bar{\kappa}$ and all $k \geq \bar{k}$,

$$\frac{1}{k} \sum_{i=1}^{k} ||\hat{x}_i||^2 \leq \frac{\bar{\kappa}}{k} \sum_{i=1}^{k} (\hat{w}_{i|i-1} - w_i)^2 + \kappa \qquad (3.2)$$

then

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{1}^{k} ||\hat{x}_i||^2 < \infty \quad \text{a.s.} \qquad (3.3)$$

$$\liminf_{k \to \infty} \delta_k \hat{\gamma}_k k > 0 \qquad (3.4)$$

$$\sum_{1}^{\infty} k^{-1} (w_{k/k}-w_k)^2 < 0 \qquad (3.5)$$

$$\sum_{1}^{\infty} k^{-1} (y_k-w_k)^2 < \infty , \quad \sum_{1}^{\infty} k^{-1} (||\tilde{x}_k||^2 < \infty \quad \text{a.s.} \qquad (3.6)$$

Proof: The result (3.3) follows that of [15] as applied in [9,10]. That (3.4) follows from (3.3) and the $\hat{\gamma}_k$ selection property (3.1c) is immediate. The strengthening of the results of Theorem (3.1) then follows from (3.4).

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Lemma 3.4: Consider the open-loop estimation case in which for all $k \geq \bar{k}$ (as in Lemma 3.2), the following stability is assumed:

$$\frac{1}{k} \sum_{1}^{k} ||x_i||^2 < \infty$$

Consider the closed-loop STR control case in which the plant is minimum phase with, for some $K,\bar{K}$ and all $k>\bar{k}$,

$$\frac{\bar{K}}{k}\sum_{1}^{k} u_k^2 \le \frac{1}{k}\sum_{1}^{k} ||x_k||^2 \le \frac{K}{k}\sum_{1}^{k} y_k^2 + K \qquad (3.8)$$

Then for the above cases, the constraint (3.2) is satisfied and the stablity result (3.3) of Lemma 3.2 follows.

Proof: Follows in a straightforward manner the relevant part of the proof of Theorem 3.2 of [9].

Remarks:

1. The result (3.3) in the STR case gives a Cesaro boundedness result for the plant inputs and outputs.
2. So far in the theory, there is achieved asymptotically optimal one-step ahead prediction and minimum variance control without there being consistent parameter estimation. The results of [17] point to lack of robustness unless there is persistently exciting input signals for related algorithms. The following results we believe are crucial to understand the practical success of the original STR.

Theorem 3.1: (STR Convergence with Sufficiently Rich Plant Noise) !neQmhao the STR algorithm of this paper applied to the plant (2.1), assumed strictly minimum phase, with no pole/zero cancellations and satisfying $[C^{-1}(z)-\frac{1}{2}]$ strictly positive real. In addition, assume that the noise $w_k$ is sufficiently rich so that

$$\liminf_{k\to\infty} \frac{1}{k}\sum_{1}^{k} w_i^2 > 0 \qquad \text{a.s.} \qquad (3.9)$$

Then (3.3) - (3.7) hold with, using the definition (2.9),

$$\lim_{k\to\infty} \delta_{\bar{k}}^{-1}\bar{P}_k = 0, \ \lim_{k\to\infty} \bar{\theta}_k = \hat{\theta} \qquad (3.10)$$

Moreover, with the condition number bound of (2.10) suitably large, there is a $\bar{k}$ such that for all $k \geq \bar{k}$, then $k \in S_1 \cup S_2$. Furthermore, with $w_k$ bounded in norm,

$$\lim_{k\to\infty} \delta_k^{-1} \hat{x}_k' \hat{B}_{k-1} \hat{x}_k = 0 \qquad \text{a.s.} \tag{3.11}$$

for $\bar{k}$ suitably large $\hat{\gamma}_k = \gamma_k^*$, $k \in S_1$, for all $k \geq \bar{k}$ and

$$\liminf_{k\to\infty} \hat{\gamma}_k > 0 \qquad \text{a.s.} \tag{3.12}$$

Proof: See Appendix.

Remarks:

1. We stress that the above results are truly global convergence results and do not include any requirement for projection into a stability-domain as in earlier theories for the original STR algorithm. Notice that with the property (3.12), the STR scheme is asymptotically the original STR scheme with equal weightings ($\hat{\gamma}_k \equiv 1$).
2. The persistence of excitation due to measurement noise alone, is specific to the least squares style regression vector used. If one uses an a posteriori based extended least squares estimation procedure, there is no guarantee of adaptive control with consistent parameter estimation unless a persistently exciting or "continuously disturbed" reference trajectory $y_k^*$ is used [12,13,19]. Enriching $y_k^*$ to achieve persistency will adversely affect asymptotic optimality [19].
3. For the case when the STR algorithm is organized so as to simultaneously estimate $b_1$, then the above analysis shows that plant noise alone may not be sufficient for persistence of excitation of the states $\bar{x}_k' = [y_{k-1} \cdots y_{k-n} \; u_{k-1} \cdots u_{k-m}]$, so that $\bar{P}_k \to 0$ as $k \to \infty$, and will not be sufficient for $\bar{P}_k$ to go to zero linearly. This means that to guarantee estimation of $b_1$ in such an STR scheme, variations to the controller or the addition

of persistence of excitation signals could be required. Degree one of lack of persistence manifests itself in the results of [20] where the stochastic approximation variant of the STR [8] is shown to produce parameter estimates converging to the current value except for a single degree of freedom (a random scaling).

## 4. CONCLUSIONS

The paper has presented global convergence results for an estimation/control scheme based on a priori noise estimates and thereby for the original STR of [1]. The results are significant in that they are the first known global convergence results for a truly direct least squares based stochastic adaptive control scheme, although clearly there is a need to modify the usual least square algorithm through weighting coefficient selection. The resulting algorithm behaves asymptotically like least squares but can contain some transient elements similar to stochastic approximation. This appears to be the price necessary to establish global convergence.

Of particular interest is that the convergence results confirm that the plant noise, assumed to exist, is sufficient to give persistence of excitation of the plant states and state estimates. Consquently, there is guaranteed parameter estimate convergence to the optimal controller parameters. That is, there is guaranteed asymptotic optimality which is in contrast to the situation of indirect adaptive control where the introduction of persistently exciting input (reference) signals to guarantee parameter convergence precludes achievement of asymptotic optimality.

The fact that the direct adaptive control algorithms such as the STR are simpler for implementation than indirect schemes and no additional persistence of excitation input disturbances are required for their asymptotic optimality, suggests that further global convergence studies be developed for more sophisticated direct adaptive stochastic schemes such as direct adaptive pole assignment and LQG (linear quadratic gaussian) schemes. We believe that the techniques of this paper could provide a crucial key for obtaining global convergence results for such schemes.

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## APPENDIX

Lemma A1: For any $a_t$, $r_t = \int_0^t a_\tau' a_\tau d\tau$, $\varepsilon>0$

$$\limsup_{t\to\infty} \int_{t_1}^{t} a_\tau' a_\tau \, r_\tau^{-1} (\ln r_\tau)^{-1-\varepsilon} d\tau < \infty \qquad \text{(A1)}$$

Also in discrete time,

$$\limsup_{k\to\infty} \sum_{1}^{k} a_i' a_i \, r_i^{-1} (\ln r_i)^{-1-\varepsilon} < \infty \qquad \text{(A2)}$$

Proof: The integral in (A1) can be reformulated as

$$\int_{t_1}^{t} \frac{1}{r_\tau (\ln r_\tau)^{1+\varepsilon}} \frac{dr}{d\tau} d\tau = \int_{t_1}^{t} \frac{1}{(\ln r_\tau)^{1+\varepsilon}} \frac{d \ln r_\tau}{d\tau} d\tau$$

$$= - \int_{t_1}^{t} \frac{(d \ln r)^{-\varepsilon}}{d\tau} d\tau$$

$$= [(\ln r_{t_1})^{-\varepsilon} - (\ln r_t)^{-\varepsilon}]/\varepsilon$$

$$< \infty$$

and (A1) is established. Also the summation in (A2) can be bounded by an integral such as in (A1).

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Proof of Lemma 3.1: Proof of property (3.1a). For $k \varepsilon S_1 U S_2$, and following [16],

$$\lambda_{max} \hat{B}_k \leq \lambda_{min}^{-1} \sum_{1}^{k} \hat{\gamma}_i \hat{x}_i \hat{x}_i'$$

$$\leq \hat{\gamma}_k \lambda_{min}^{-1} \sum_1^k \hat{x}_i\hat{x}_i'$$

$$\leq K_1 \gamma_k \gamma_{max}^{-1} \sum_1^k \hat{x}_i\hat{x}_i'$$

This leads to,

$$\sum_{S_1 U S_2} \delta_k \hat{\gamma}_i^2 \hat{x}_i \hat{B}_i \hat{x}_i \leq K_1 \sum_{S_1 U S_2} \delta_i \hat{\gamma}_i^2 \lambda_{max}^{-1} (\sum_1^k \hat{x}_j\hat{x}_j') \hat{x}_i'\hat{x}_i < \infty$$

For $k \in S_3$, then (3.1a) follows from a direct application of (A2), identifying $a_i'a_i$ as $\hat{x}_i'\hat{B}_{i-1}\hat{x}_i$ and noting $\hat{B}_i \leq \hat{B}_{i-1}$.

<u>Proof of Property</u> (3.1b): Follows trivially from the definition of $\hat{\gamma}_k$ for the case $k \in S_4$ should (3.1b) not be satisfied otherwise.

<u>Proof of Property</u> (3.1c): With the definition of $\hat{\gamma}_k^*$ in (2.10), it follows trivially for both $k \in S_1$, $k \in S_2$ and $k \in S_3$ that

$$\liminf_{k\to\infty} \delta_k \hat{\gamma}_k^* (\sum_1^k \hat{x}_i'\hat{x}_i) > 0 \qquad \text{a.s.}$$

For $k \in S_4$, with $\varepsilon(2\hat{x}_k'\hat{B}_{k-1}\hat{x}_k)^{-1} < \hat{\gamma}_k^*$, and $\delta_k = 1$ of necessity, then

$$\delta_k \hat{\gamma}_k \sum_1^k \hat{x}_i'\hat{x}_i > \frac{\varepsilon \sum_1^k \hat{x}_i'\hat{x}_i}{\hat{x}_k'\hat{x}_k \lambda_{max}(\sum_1^k \hat{\gamma}_i'\hat{x}_i\hat{x}_i')^{-1}}$$

$$> \varepsilon \lambda_{min}(\sum_1^k \hat{\gamma}_i\hat{x}_i\hat{x}_i')$$

so that (3.1c) is established.

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<u>Proof of Lemma 3.2</u>: The convergence analysis is for the parameter and state estimation equations, using the notations $\tilde{\theta} = \theta - \hat{\theta}$, $\tilde{x} = x - \hat{x}$,

$$\tilde{\theta}_k = \tilde{\theta}_{k-1} - \hat{\gamma}_k \hat{B}_k \hat{x}_k \hat{w}_{k|k-1}$$

$$\tilde{x}'_k = [0..00..0(w_{k-1} - \hat{w}_{k-1|k-2})..(w_{k-n} - \hat{w}_{k-n|k-n-1})] \qquad \text{(A3)}$$

Introducing definitions

$$p_k = \theta' \tilde{x}_k + \tfrac{1}{2}\tilde{\theta}'_{k-1}\hat{x}_k \quad , \quad q_k = \tilde{\theta}'_{k-1}\hat{x}_k \qquad \text{(A4)}$$

we see that

$$(p_k - \tfrac{1}{2}q_k) = \theta' \tilde{x}_k = (C-1)(w_k - \hat{w}_{k|k-1})$$

$$(p_k + \tfrac{1}{2}q_k) = \theta' \tilde{x}_k + \tilde{\theta}'_{k-1}\hat{x}_k = (\hat{w}_{k|k-1} - w_k)$$

$$(p_k - \tfrac{1}{2}q_k) = -(C-1)(p_k + \tfrac{1}{2}q_k) \qquad \text{(A5)}$$

$$p_k = (C^{-1} - \tfrac{1}{2})q_k$$

The strict positive real condition (2.20) and the fact that $\hat{\gamma}_k$ is monotonically decreasing now tells us, using manipulations as in [8], that for some $\varepsilon>0$ and all $k>0$,

$$2\sum_{0}^{k}[\hat{\gamma}_i p_i q_i - 3\varepsilon\hat{\gamma}_i(q_i^2 + p_i^2)] \geq 0 \qquad \text{(A6)}$$

Consider now a tentative (stochastic) Lyapunov function

$$V_k = \delta_k \tilde{\theta}_k' \hat{B}_k^{-1} \tilde{\theta}_k + 2\sum_{0}^{k}[\hat{\gamma}_i p_i q_k - 3\varepsilon\hat{\gamma}_i(q_i^2 + p_i^2)]$$

$$\geq 0 \qquad \text{(A7)}$$

Simple manipulations now yield

$$E[V_k|F_{k-1}] \leq V_{k-1} + \{E[\Delta_k - 2\hat{\gamma}_k w'_k q_k | F_{k-1}]$$

$$-3\varepsilon\hat{\gamma}_k E[(q_k^2 + p_k^2)|F_{k-1}]\}\delta_k \qquad \text{(A8)}$$

where

$$\Delta_k \triangleq \tilde{\theta}'_k \hat{B}_k^{-1} \tilde{\theta}_k - \tilde{\theta}'_{k-1}\hat{B}_{k-1}^{-1}\tilde{\theta}_{k-1} + 2\hat{\gamma}_k(p_k + w_k)' q_k$$

$$= \tilde{\theta}'_{k-1}(\hat{B}_k^{-1} - \hat{B}_{k-1}^{-1})\tilde{\theta}_{k-1} + \hat{\gamma}_k^2 \hat{x}'_k \hat{B}_k \hat{x}_k \hat{w}^2_{k|k-1} - \hat{\gamma}_k \tilde{\theta}'_{k-1}\hat{x}_k q_k$$

$$= \hat{\gamma}_k^2 \hat{x}_k' \hat{B}_k \hat{x}_k \hat{w}_{k|k-1}^2$$
$$\leq 3\ \hat{\gamma}_k^2 \hat{x}_k' \hat{B}_k \hat{x}_k (p_k^2 + q_k^2 + w_k^2) \qquad \text{(A9)}$$

The second equality follows from a substitution of $\tilde{\theta}_k$ from (A3)(A5), the third from (2.6b)(A4), and the inequality from (A4)(A5).

Now defining

$$\beta_k = 3\delta_k \hat{\gamma}_k^2 \hat{x}_k' \hat{B}_k \hat{x}_k \sigma_w^2$$
$$\alpha_k = 3\delta_k \hat{\gamma}_k (\varepsilon - \hat{\gamma}_k \hat{x}_k' \hat{B}_k \hat{x}_k)(q_k^2 + p_k^2) \qquad \text{(A10)}$$

and noting that $E[\hat{\gamma}_k w_k' q_k | F_{k-1}] = 0$, then (A8) leads to

$$E\{V_k | F_{k-1}] \leq V_{k-1} - E[\alpha_k | F_{k-1}] + \beta_k$$
$$\leq V_{k-1} + \beta_k$$
$$E[\bar{V}_k | F_{k-1}] \leq \bar{V}_{k-1} + \beta_k \quad , \quad \bar{V}_k \triangleq V_k + \sum_1^k \alpha_i \qquad \text{(A11)}$$

We see that the property (3.1a) of the $\hat{\gamma}_k$, $\delta_k$ selections (2.6d) and Lemma 3.1 ensures that $\sum_1^\infty \beta_k < \infty$. Also (3.1c) gives that for some $\bar{k}$ and all k then $\alpha_k > 0$, so that $\sum_1^\infty \alpha_k > 0$. The term $V_k$ is nonnegative under the strict positive real condition on $[C^{-1}(z) - \frac{1}{2}]$ following [9], so that $\bar{V}_k$ is nonnegative. Applying the martingale convergence theorem [18, page 33] to (A11) gives that

$$\sum_0^k \alpha_i \quad , \quad V_k \quad , \quad \delta_k \tilde{\theta}_k' \hat{B}_k^{-1} \tilde{\theta}_k \text{ converge a.s.}$$

$$\sum_1^\infty \delta_k \hat{\gamma}_k (\hat{w}_{k|k-1} - w_k)^2 = \sum_1^\infty \delta_k \hat{\gamma}_k (p_k + \tfrac{1}{2} q_k)^2$$
$$\leq 2\varepsilon^{-1} \sum_1^\infty \alpha_k$$
$$\leq \infty \qquad \text{a.s.} \qquad \text{(A12)}$$

The other results of the theorem follow in a straightforward manner as for the corresponding results of [9]. +++

Remark: The crucial construction to generalize the proof technique for the case of a **posteriori** noise estimates in [8,10] to the case of a **priori** estimates here, is the step size $\hat{\gamma}_k$ selection such that (3.1c) holds for sufficiently small $\varepsilon$ required to keep $\alpha_k$ in (A10) positive.

Proof of Theorem 3.1: From the result (i) of Lemma 3.2,

$$\lim_{k\to\infty} \delta_k^{-1}\hat{B}_k = 0 \implies \lim_{k\to\infty} \hat{\theta}_k = \theta \quad \text{a.s.}$$

applying (2.19) this result yields that the desired consistency (3.10) holds for the self-tuning regulator if

$$\lim_{k\to\infty} \delta_k^{-1}\bar{P}_k = 0 \quad , \quad \bar{P}_k^{-1} = \sum_0^k \hat{\gamma}_i \bar{x}_i \bar{x}_i' \tag{A13}$$

Consider that (A13) does not hold, then $k \notin S_1$ subsequent to some time $\bar{k}$, thus $\delta_k$ is asymptotically a constant, and $\liminf_{k\to\infty} \bar{P}_k > 0$.

Now from the Kronecker lemma we see that

$$\lim_{k\to\infty} \sum_1^k i^{-1} \bar{x}_i \bar{x}_i' < \infty \quad ,$$

implies

$$\lim_{k\to\infty} \frac{1}{k} \sum_1^k \bar{x}_i \bar{x}_i' = 0,$$ or equivalently the persistence of excitation condition,

$$\liminf_{k\to\infty} \frac{1}{k} \sum_1^k \bar{x}_i \bar{x}_i' > 0 \tag{A14}$$

implies the result

$$\lim_{k\to\infty} \left( \sum_{1}^{k} i^{-1}\bar{x}_i\bar{x}_i' \right)^{-1} = 0 \qquad \text{(A15)}$$

This in turn implies a contradiction of the result

$$\liminf_{k\to\infty} \bar{P}_k > 0$$

under (3.4). We conclude that (A13) holds, and thereby the desired result (3.10). It remains to establish (A14) to verify (3.10).

Now from (3.6a) we see that $y_k$ has the asymptotic value $w_k$. Moreover, by the minimum phase property of (2.1), $u_k$ is exponentially asymptotically $B^{-1}(A-C)y_k$ which is in turn asymptotically $B^{-1}(A-C)w_k$. Thus $\bar{x}_k$ is asymptotically the state $\bar{x}_k^I$ of the system $B^{-1}(A-C)$ drived by $w_k$. Applying (3.5) tells us that $\bar{x}_k$ in (A14) can be replaced by $\bar{x}_k^I$.

The system $B^{-1}(A-C)$ is time invariant and completely controllable under the coprimeness assumption of the pair B,(A-C) so that the results of [12] apply, giving that persistency of the input $w_k$, as in (3.9), assures persistency of the states $\bar{x}_k^I$, and thereby of $x^I{}_k$ as in (A14). Parallel arguments as in [13,19] lead to the same result.

The persistence of excitaton results (A14) and stability results (3.3),(3.7) together give the result that the condition number of $\sum_1^k \hat{x}_i\hat{x}_i$ is bounded. Thus $k \epsilon S_1 \cup S_2$ (for suitably large bounds) and all $k \geq \bar{k}$ where $\bar{k}$ is sufficiently large.

With closed loop asymptotic time invariance (3.10), and asymptotic stability, then bounded inputs $(w_k)$ give bounded outputs $(\hat{x}_k)$. This together with (A13) implies that (3.11)holds, so that for suitably large $\bar{k}$ and all $k \geq \bar{k}$, then $k \epsilon S_1$ and $k \not\epsilon S_4$. Consequently $\hat{\gamma}_k = \gamma_k^*$ and (3.12) holds for $k \geq \bar{k}$.

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