IFAC World Congress
August. 1981

33.4

MINIMUM VARIANCE CONTROL HARNESSED FOR NON-MINIMUM-PHASE PLANTS

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Abstract. Minimum variance control is applied to non-minimum phase plants augmented with adaptive compensators. The objective of the compensators is to achieve, asymptotically, a minimum phase property for the augmented plant. With this property, the minimum variance controller gives a stabilizing control signal.

Results are developed for discrete-time, linear, stochastic plants for which the series and feedforward parallel moving average compensators have coefficients updated based on plant parameter estimates.

The schemes perform well on simulations and were further supported by a global convergence theory.

For tracking problems, preprocessing the desired trajectory may give improved performance. The design of the preprocessor can be achieved using a systematic approach.

Keywords. Adaptive control; self-adjusting systems; stochastic control.

INTRODUCTION

In this paper, we develop a novel approach, yet involving standard techniques, to the adaptive control of general, linear, finite-dimensional, discrete-time plants. Also, a convergence analysis theory is developed to refine and justify the scheme.

In particular, the power of adaptive minimum variance control for minimum phase system is harnessed for non-minimum-phase plants. Minimum variance control is applied to the plant augmented with adaptive series and parallel compensators which adaptively assign the augmented plant "transfer function zeros". State variable techniques can be employed in tracking schemes to optimally preprocess the desired trajectory if required.

The nearest competitor for the scheme of this paper is the adaptive pole assignment scheme of Wellstead, Prager and Zanker (1979). For the deterministic case, using a least squares parameter estimation scheme and easily constructed "sufficiently rich" control signals, the plant parameters are learnt in a finite number of steps (as low as n steps where n is the dimension of the state vector). With knowledge of the true plant parameters, the plant poles can be assigned arbitrarily by means of feedback compensators. A local convergence theory for the case of a suboptimal projection parameter-update algorithm is given in Goodwin and Sin (1979), where also the least squares case is considered without resort to "sufficiently rich" control signals.

One property of the controllers in Wellstead, Prager and Zanker (1979), Goodwin and Sin (1979) is that the zeros of the plant are not influenced by the controller. Also, it is not clear what happens in the case of plant noise, or how best to preprocess the input to achieve a suitable compromise between tracking performance and control signal level. These properties of the adaptive pole assignment approach may well be viewed as disadvantages in some applications.

In the approach of this paper where adaptive zero assignment is used, the zeros of the augmented plant are moved to any desired location, and the case of plant noise is dealt with in the stochastic theory. Also, the potential is there to apply the Riccati theory to improve tracking performance. The convergence analysis builds on the ideas in Goodwin and Sin (1979), Kumar and Moore (to appear), Goodwin, Ramadge and Caines (1979).

COMPENSATORS AND MINIMUM VARIANCE CONTROL

Minimum variance control schemes for known plants are simple to implement, and effective for minimum phase plants. For non-minimum phase plants, the control signals become unbounded. In this section, we harness the power of minimum variance control to control a known non-minimum-phase plant, augmented with compensators, so that the augmented plant is minimum-phase. The scheme is of interest in its own right, but is here presented as a stepping stone to the adaptive versions of the next section.

Plant. Consider a linear, time invariant, discrete-time finite dimensional and stochastic signal generating system. To keep the notation simple, we work with scalar input-output models, also known as autoregressive, moving average, exogenous variable (ARMAX) models

$$z_k = -\sum_{i=1}^{n} a^{(i)} z_{k-i} + \sum_{i=0}^{m} b^{(i)} u_{k-N-i} + \sum_{i=1}^{p} c^{(i)} w_{k-i} + w_k \qquad (1)$$

where z_k are the measurements, u_k the external known inputs (controls) with an N unit delay associated, and w_k is zero mean white bounded variance unknown noise[1]. The parameter $a^{(i)}$, $b^{(i)}$, $c^{(i)}$ are assumed known in this section.

Alternative notation is with operators, A, B, C being polynomials in the delay operator q^{-1}. Then

$$Az_k = Bu_{k-N} + Cw_k \quad \text{or} \quad z_k = Bq^{-N}A^{-1}u_k + CA^{-1}w_k \qquad (2)$$

where A, B, C are respectively of order n, m, p with coefficients given from the elements $a^{(i)} b^{(i)} c^{(i)}$ of vectors a, b, c and without loss of generality $a^{(0)} = c^{(0)} = 1$.

Also define $\bar{a}$ from $a' = [1 \ \bar{a}']$ and likewise $\bar{c}$. Now with the parameters and state estimates

$$\theta = [\bar{a}' \quad b \quad \bar{c}']$$

$$x_k = [-z_{k-1} \ldots -z_{k-n} u_{k-n} \ldots u_{k-N-m} w_{k-1} \ldots w_{k-p}] \qquad (3)$$

then the plant has the state space description

$$z_k = \theta' x_k + w_k \qquad (4)$$

[1] More precisely, with Σ_k denoting the σ-algebra generated by $w_0 w_1 \ldots w_k$, then $E[w_k | \Sigma_{k-1}] = 0$, $E[\| w_k \|^2 | \Sigma_{k-1}] = \sigma_w^2 \le \bar{\sigma} < \infty$. The bounds $\limsup_{k \to \infty} \frac{1}{k} \sum \| w_i \|^2 < \infty$, $E[\| w_k \|^2 | \Sigma_{k-1}] = \sigma_w^2 \le \bar{\sigma} < \infty$ are assumed for some of the convergence theory.

Compensators. With a view to achieving assigned zeros in a plant augmented with compensators, consider the above plant with a moving average pre- or post-compensator having an operator $F(q^{-1})$ together with a parallel feedforward moving average compensator having an operator $[E(q^{-1})q^{-1}]$. Thus we introduce the definition

$$E(q^{-1}) = e^{(0)} + e^{(1)}q^{-1} + \ldots e^{(r-1)}q^{-r+1},$$
$$e' = [e^{(0)} e^{(1)} \ldots e^{(r-1)}], \qquad (5)$$

and likewise for $F(q^{-1})$ and f'. Here $r = \max[n, m+N-1]$ is assumed finite. The selection of E and F is considered following a study of the augmented plant equations.

Time varying (adaptive) compensators are fundamental to the algorithms of the next sections. For these the notation E_k, F_k is employed where now e_k, f_k are time varying.

Augmented Plant. The augmented plant with input $\bar{u}_k$ and output $\bar{z}_k$ has operator equations for time invariant compensators as

$$A\bar{z}_k = \bar{H}\bar{u}_k + Gw_k, \quad \bar{H} = Eq^{-1}A + FBq^{-N} \qquad (6)$$

where for the post-compensator case $G = FC$ and for the pre-compensator case $G = C$. We associate vectors $\bar{h}$ and $\bar{g}$ with $\bar{H}$ and G using the earlier convention.

It proves convenient to work with the relationship between the operators as algebraic equations obtained by equating powers of q^{-i} for each i giving

$$[\mathbf{A} \quad \bar{\mathbf{B}}] \begin{bmatrix} e \\ f \end{bmatrix} = [\mathbf{E} \quad \bar{\mathbf{F}}] \begin{bmatrix} a \\ b \end{bmatrix} = \bar{h} \qquad (7a)$$

$$g = \mathbf{F}c \ \text{(post comp)}, \quad g = c \ \text{(pre comp)} \qquad (7b)$$

where the matrices have the structures in terms of vectors a, b, c, f as

$$\mathbf{A} = \begin{bmatrix} [a] & 0 & 0 \\ & [a] & 0 \\ & & \ddots \\ 0 & 0 & [a] \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} [b] & 0 & 0 \\ & [b] & 0 \\ & & \ddots \\ 0 & & [b] \end{bmatrix},$$

$$\bar{\mathbf{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{B} \end{bmatrix} \updownarrow N-1 \qquad (8)$$

and likewise for $\mathbf{E}$, $\mathbf{F}$, $\bar{\mathbf{F}}$. The matrix dimensions can be inferred from the relationships in which they hold and may vary. For example in (7a), $\mathbf{A}$ is $2r \times r$, $\mathbf{B}$ is $(2r-N+1) \times r$, $\mathbf{E}$ is $2r \times (n+1)$ $\mathbf{F} = (2r-N+1) \times (m+1)$ and in (7b) $\mathbf{F}$ is $(r+p) \times (p+1)$.

Zero Assignment. The augmented plant zeros are given as the zeros of $\bar{H}(q^{-1})$. Thus to achieve specified zeros, then $E(q^{-1})$ and $F(q^{-1})$ must be chosen to satisfy (2.6), or equivalently the coefficients chosen to satisfy $[A \;\; \bar{B}][e' \;\; f']' = \bar{h}$ [see (7)].

It is known (Theorem 7, 3.20 of Wolovich (1974)) that a unique solution exists if and only if $A(q^{-1})$ and $B(q^{-1})$ are relative prime (no common zeros), or equivalently $[A \;\; \bar{B}]$ is nonsingular. In other words the plant must be a minimal representation in that there are no pole/zero cancellations. A property of this approach to achieve specified zeros in an augmented plant, is that the coefficients in E and F become large when a plant pole is close to a plant zero.

For the application of the control ideas of this paper, the compensators are chosen so that the augmented plant is minimum phase and without loss of generality the zeros of $\bar{H}(q^{-1})$ can be taken at the origin. One possibility is $\bar{h}' = [1\;0\;0\;\ldots\;0]$ and the augmented plant equations (6) simplify as

$$\bar{z}_{k+1} = -\sum_{i=1}^{n} a^{(i)}\bar{z}_{k+1-1} + \sum_{i=0}^{\bar{p}} g^{(i)} w_{k+1-i} + \bar{u}_k \tag{9}$$

where $\bar{p} = p$ for the precompensator case and $\bar{p} = p+r-1$ for the post compensator case.

Noise Estimation. An estimate of the noise w_k, denoted $\hat{w}_k$, can be obtained by processing the plant inputs and outputs as follows. From (2) $w_k = C^{-1}[Az_k - Bu_{k-N}]$. Since initial conditions for this equation are unknown, a noise estimate is calculated via

$$\hat{w}_k = C^{-1}[Az_k - Bu_{k-N}] \tag{10}$$

initialized by $\hat{w}_0 = \hat{w}_{-1} = \ldots \hat{w}_{-p+1} = 0$. Clearly with C^{-1} an exponentially asymptotically stable operator, a standard assumption, then $\hat{w}_k \to w_k$ exponentially as $k \to \infty$.

Minimum Variance Control. In minimum variance control of the augmented plant, the control $(\bar{u}_k)$ is chosen to minimize the variance of the tracking error between the augmented plant output $\bar{z}_k$ and some desired trajectory $\bar{z}^*_k$ for this.

What then is a suitable value for $\bar{z}^*_k$ when only a specified desired plant output z^*_k is given? This question is answered below using an optimization procedure which gives a bounded $\bar{z}^*_k$ from a bounded z^*_k. Of course, for the regulation problem where $z^*_k = 0$, then $\bar{z}^*_k$ is also chosen as $\bar{z}^*_k = 0$.

In minimum variance control, the optimal control $\bar{u}_k$ turns out to be the control which sets the one-step-ahead output prediction $\hat{\bar{z}}_{k+1|k}$ as the desired trajectory $\bar{z}^*_{k+1}$. Thus the control, now denoted $\bar{u}_k$, is selected to satisfy

$$\bar{z}^*_{k+1} = \hat{\bar{z}}_{k+1|k} \equiv -\sum_{i=1}^{n} a^{(i)}\bar{z}_{k+1-i} + \sum_{i=1}^{\bar{p}} g^{(i)}\hat{w}_{k+1-i} + \bar{u}_k \tag{11}$$

A key observation is that the above minimum variance controller of the augmented plant with inputs $\{\bar{z}_k\}$, $\{\bar{z}^*_k\}$, $\{\hat{w}_k\}$ and outputs $\{\bar{u}_k\}$ is exponentially asymptotically stable and thus has a bounded input, bounded-output property. Moreover, as previously noted, $\hat{w}_k \to w_k$ exponentially as $k \to \infty$.

Control Scheme Properties: From (9) and (11)

$$\bar{z}_{k+1} = \bar{z}^*_{k+1} + \sum_{i=1}^{\bar{p}} g^{(i)}(w_{k+1-i} - \hat{w}_{i+1-i}) + g^{(0)} w_{k+1} \tag{12}$$

With the boundedness assumptions

$$\limsup_{m\to\infty} \frac{1}{m}\sum_{0}^{m} \| \bar{z}^*_k \|^2 < \infty, \quad \limsup_{m\to\infty} \frac{1}{m}\sum_{0}^{m} \| w_k \|^2 < \infty \tag{13}$$

then since $\hat{w}_k \to w_k$ exponentially as $k \to \infty$, the sequence $\{\bar{z}_k\}$ is likewise bounded. Since the augmented plant is minimum phase, a bounded output $\{\bar{z}_k\}$ implies a bounded input $\{\bar{u}_k\}$. As a consequence of the bound on $\{\bar{u}_k\}$, the original plant input $\{u_k\}$, is also bounded. Moreover, for the pre-compensator case $z_k = \bar{z}_k - q^{-1}E\bar{u}_k$ and consequently $\{z_k\}$ is bounded. For the post-compensator case with $\bar{u}_k = u_k$,

$$\bar{z}_k = z_k + q^{-1}E\bar{u}_k = (Fq^{-N}BA^{-1} + q^{-1}E)u_k + FCA^{-1}w_k$$

from which

$$\bar{z}_k = \bar{H}A^{-1}u_k + FCA^{-1}w_k = \bar{H}q^{N}B^{-1}(z_k - CA^{-1}w_k) + FCA^{-1}w_k$$

and

$$\begin{aligned} z_k &= \bar{H}^{-1}q^{-N}B(\bar{z}_k - FCA^{-1}w_k) + CA^{-1}w_k \\ &= \bar{H}^{-1}[q^{-N}B\bar{z}_k - q^{-1}ECw_k] \end{aligned} \tag{14}$$

With $\{\bar{z}_k\}$ and $\{w_k\}$ bounded in the sense above, then from (14), $\{z_k\}$ is likewise bounded.

The relationship corresponding to (14) for the pre-compensator case is readily established as

$$z_k = \bar{H}^{-1}[q^{-N}BF\bar{z}_k - q^{-1}ECw_k] \tag{15}$$

The above results are summarized as a theorem.

Theorem 1: Consider the original plant (1) augmented with compensators to achieve an augmented plant (6) with zeros at the origin. Then minimum variance control applied to the augmented plant with the control law given from the implicit solution of (11), achieves a bounded-input, bounded-output controller. Also with the desired trajectory $\bar{z}_k^*$ and noise w_k bounded as in (13), the plant signals $\{u_k, z_k\}$ and augmented plant signals $\{\bar{u}_k, \bar{z}_k\}$ are bounded likewise. The augmented plant output $\bar{z}_k$ approaches $\bar{z}_k^* + g_0 w_k$ exponentially fast for the stochastic case and $\bar{z}_k = \bar{z}_k^*$ in r steps in the noise free case. Moreover, z_k tracks $\bar{z}_k$ as in (14) (15).

Remark 1. If the noise and tracking error bounds are strengthened to an upper norm bound, then the boundedness results for $u_k, z_k, \bar{u}_k, \bar{z}_k$ can be likewise strengthened.

Remark 2. Setting $\bar{z}_k^* = z_k^*$ gives that z_k tracking $q^{1-N}Bz_k^*$ or $q^{1-N}BFz_k^*$ when $\bar{H} = q^{-1}$. Improved tracking can be achieved as below.

Preprocessing the Specified Trajectory. The relationships (14), (15) suggest the following recursive calculation of $\bar{z}_k^*$ from z_k^*, for the case $\bar{H} = q^{-1}$

$$B\bar{z}_k^* = z_{k+N-1}^* \quad \text{post-compensator case}$$

$$BF\bar{z}_k^* = z_{k+N-1}^* \quad \text{pre-compensator case} \qquad (16)$$

Note that if $z_k^* = 0$ as in the regulator, $\bar{z}_k = 0$ is a solution. With the original plant minimum phase there is also no difficulty in solving for $\bar{z}_k^*$, but the case of interest here is when the plant is nonminimum phase and thus B^{-1} and $(BF)^{-1}$ are unstable operators with the solution $\bar{z}_k^*$ becoming unbounded.

One scheme for avoiding an unbounded $\bar{z}_k^*$ is to reset the states of (2.16) associated with past $\bar{z}_k^*$ periodically, or during times when the introduction of transient error in $(z_k - z_k^*)$ does not matter so much. A refinement to avoid large transients in $(z_k - z_k^*)$ is to switch in a stabilizing feedback gain until the appropriate states are sufficiently small. The stabilizing gain can be selected from Kalman Regulator theory (Anderson and Moore (1969)) as one possibility.

A further refinement is to apply tracking theory (Anderson and Moore (1969)), in an appropriate discrete-time form to keep the terms $\|\bar{z}_k^*\|^2$ and $\|z_{k+N-1}^* - B\bar{z}_k^*\|^2$ or $\|z_{k+N-1}^* - BF\bar{z}_k\|^2$ small. The first term is interpreted as the "control energy" and the second as the "tracking error cost" in applying the tracking theory. Of course, there are two possibilities. In the first, if z_k^*, the desired trajectory, can be modelled as the output of a linear system with zero input, then the tracker is simply a regulator for an augmented system. Otherwise, knowledge of $\bar{z}_k^*$ in the future is required for difference equations solved backwards in time at least for two or three times the relevant time constant. This level of refinement is in fact not justified here since similar tracking algorithms could be better employed on the original plant. However, for the adaptive schemes of the next section, this approach could well be justified. This is not discussed further.

ADAPTIVE TRACKING

In this section, the tracking scheme of the previous section for known stochastic linear systems is made adaptive by performing parameter estimation and using these estimates in lieu of the unknown true parameters for compensation and control. Two variations are considered.

Parameter Estimation. One possibility is to apply a weighted least squares parameter update scheme, as in Anderson and Moore (1969), to estimate θ as $\hat{\theta}_k$ from the plant inputs $\{u_k\}$ and outputs $\{z_k\}$ and model (4) as

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \hat{\gamma}_k \hat{P}_k \hat{x}_k (z_k - \hat{z}_{k|k-1}), \quad \hat{z}_{k|k-1} = \hat{\theta}_{k-1}' \hat{x}_k \qquad (16a)$$

$$\hat{P}_k = \hat{P}_{k-1} - \hat{P}_{k-1}\hat{x}_k\hat{x}_k'\hat{P}_{k-1}(\hat{\gamma}_k^{-1} + \hat{x}_k'\hat{P}_{k-1}\hat{x}_k)^{-1} \qquad (16b)$$

A posteriori estimates of the noise and states are taken as

$$\hat{w}_k = z_k - \hat{\theta}_k' x_k \qquad (17a)$$

$$\hat{x}_k = [-z_{k-1} \cdots -z_{k-n} u_{k-n} \cdots u_{k-N-m} \hat{w}_{k-1} \cdots \hat{w}_{k-p}] \qquad (17b)$$

For the noise free case in which $\hat{x}_k = x_k$, then a suitable selection of $\hat{\gamma}_k$ is $\hat{\gamma}_k = 1$ and the algorithm is an "unweighted least squares" scheme. If P_k is set as $P_k = I$ and γ_k is set as $\gamma_k = \alpha_1(\alpha_2 + x_k'x_k)^{-1}$ for some $0 < \alpha_1 < 2$ and $0 < \alpha_2$, then the algorithm is a simple "gradient projection" scheme.

For the stochastic case, the weighting coefficients $\hat{\gamma}_k$ are chosen according to a stability measure $\hat{M}_k = \hat{x}_k'\hat{P}_{k-1}\hat{x}_k$ which is available in the calculation. One selection scheme termed a "weighred least squares" scheme selects $\hat{\gamma}_k$ as follows

$$\hat{\gamma}_k = \min\{\gamma_k, \hat{\gamma}_{k-1}\} \qquad (18a)$$

$$\gamma_k = \begin{cases} k^{-\varepsilon} & \text{if } k \in S_1 \triangleq \{k \mid \hat{M}_k \le \alpha_1 k^{-(1-\varepsilon)}\} \\ \sqrt{\alpha_2}(k^{1+\varepsilon}\hat{M}_k)^{-\frac{1}{2}} & \text{if } k \in \{k \mid k \notin S_1, \ \hat{M}_k \le \alpha_2^{(1-\varepsilon)}\} \\ [\max(k, \sum_{i=1}^{k} \hat{M}_i)]^{-1} & \text{otherwise} \end{cases} \qquad (18b)$$

for some $1 \gg \varepsilon > 0$ and $\alpha_1, \alpha_2 > 0$ selected to keep γ_k from decreasing excessively.

Stochastic approximation versions take $P_k + I$ and γ_k as the third selection in (18b) for all k. Other variations also can be constructed.

Compansator Update. Should the plant be in fact minimum phase, then the compensator selection $F=1, E=0$ are known to be suitable since adaptive minimum variance control of minimum phase plants is globally convergent (Kumar and Moore to appear) and has attractive properties. Should the plant be non-minimum-phase, then the control signals will become excessive with $F=1, E=0$. In the case of unknown plants, it makes sense then to initialize the algorithm with $F=1$ and $E=0$ and if $|u_k|$ is below some threshold keep $F=1, E=0$. Otherwise switch to the schemes now described without returning more than a finite number of times to the case $F=1, E=0$. With estimates $\hat{\theta}_k = [\hat{\bar{a}}_k' \hat{b}_k' \hat{\bar{c}}_k']'$, a first trial reasonable selection for e_k, $\hat{f}_k$ is simply $[\hat{e}_k' \hat{f}_k']' = [\hat{A}_k \ \hat{\bar{B}}_K]^{-1}\hat{h}$ where $\hat{A}_k$, $\hat{\bar{B}}_k$ are given from (8) with a,b replaced by $\hat{a}_k$, $\hat{b}_k$. For the convergence theory and indeed for practical considerations we introduce the constraint.

$$|| [\hat{e}_k' \ \hat{f}_k'] || \leq \beta < \infty \tag{19}$$

for β sufficiently large. Also, there is the problem of what to do if $[\hat{A}_k \ \hat{\bar{B}}_k]^{-1}$ does not exist.

One possibility is to introduce a random perturbation $\Delta\hat{\theta}_k' = [\Delta\hat{\bar{a}}_k' \ \Delta\hat{b}_k' \ 0]'$ such that

$$\Delta\hat{\theta}_k'\hat{x}_{k+1} = 0, \quad || \Delta\hat{\theta}_k - \Delta\hat{\theta}_{k-1} || < \beta_1 \, || \hat{\theta}_k - \hat{\theta}_{k-1} || \tag{20}$$

for some β_1, β sufficiently large, and select $\hat{e}_k$, $\hat{f}_k$ as[2]

$$[\hat{e}_k' \ \hat{f}_k']' = \begin{cases} [\hat{A}_k + \Delta\hat{A}_k \ \ \hat{\bar{B}}_k + \Delta\hat{\bar{B}}_k]^{-1}\hat{h} & \text{if its norm is less than } \beta \\ [\hat{e}_{k-1}' \ \hat{f}_{k-1}']' & \text{otherwise} \end{cases} \tag{21}$$

Observe that with this selection, not only is (19) satisfied, but the prediction $(\hat{\theta}_k + \Delta\hat{\theta}_k)'\hat{x}_{k+1}$ is independent of $\Delta\hat{\theta}_k$. Observe also that we are now constrained to the following sequences of calculations $\hat{x}_k \to \hat{\theta}_k \to \bar{u}_k \to u_k \to \Delta\hat{\theta}_k \to \hat{e}_k, \ \hat{f}_k$.

Should $|\hat{A}_k \ \hat{\bar{B}}_k|$ be zero or small for a number of iterations, one possibility is that $|A \ \bar{B}|$ is zero, or small, and the signal model order estimate is excessive. A reduction of this value may be appropriate.

Simulations suggest to us that a suitable selection of β_1 above is $\beta_1 = 0$ and thus $\Delta\theta_k = 0$, save in the case when the model order is overspecified. The term $\Delta\theta_k$ is included to rigorously establish global convergence of the algorithms.

The augmented plant equations are now expressed in terms of $Z_k' = [z_k z_{k-1} \dots]$, $\bar{U}_k = [u_k u_{k-1} \dots]$ of appropriate dimension as

$$\bar{z}_{k+1} = \begin{cases} \hat{f}_k' Z_{k+1} + \hat{e}_k'\bar{U}_k & \text{post compensator case} \quad (22a) \\ z_{k+1} + \hat{e}_k'\bar{U}_k & \text{pre compensator case} \quad (22b) \end{cases}$$

$$u_k = \begin{cases} \bar{u}_k & \text{post compensator case} \quad (22c) \\ \hat{f}_{k-1}'\bar{U}_k & \text{pre compensator case} \quad (22d) \end{cases}$$

<u>Stable Controller Adaptive Tracking Scheme (I)</u>

One possible controller in the adaptive case is to replace $\bar{a}$, g in (16) by estimates $(\hat{\bar{a}}_k + \Delta\hat{\bar{a}}_k)$, $\hat{g}_k$ and thus select $\bar{u}_k$ to satisfy

$$\bar{z}^*_{k+1} = -(\hat{\bar{a}}_k + \Delta\hat{\bar{a}}_k)\bar{Z}_k + \hat{g}_k\hat{W}_k + \bar{u}_k + \bar{z}_{k+1} \tag{23a}$$

$$\hat{g}_k = \begin{cases} \hat{F}_k\hat{x}_k & \text{post compensator case} \\ \hat{c}_k & \text{pre compensator case} \end{cases} \tag{23b}$$

where $\hat{W}_k' = [\hat{w}_k \ \hat{w}_{k-1} \dots]$ of appropriate dimension. These implicit equations give a controller with input $\bar{z}_k$, $\hat{w}_k$ and output $\bar{u}_k$ which is inherently bounded-input, bounded-output stable as long as $\hat{a}_k$, $\Delta\hat{a}_k$, $\hat{g}_k$ are bounded. The controller is, however, only asymptotically a minimum variance controller.

<u>Minimum Variance Adaptive Tracking System (II)</u>

Here, the one-step-ahead prediction estimates of the augmented plant $\hat{\bar{z}}_{k+1|k}$ are forced by a control signal $\bar{u}_k$ selection to be the desired trajectory $\bar{z}^*_{k+1}$. The prediction estimates $\hat{z}_{k+1|k}$ can be calculated in terms of the plant prediction estimates

$$\hat{z}_{k+1|k} = \hat{\theta}_k'\hat{x}_{k+1}, \quad \hat{x}_k = [-z_{k-1} \dots -z_{k-n} u_{k-N} \dots u_{k-N-m} \hat{w}_{k-1} \dots \hat{w}_{k-p}] \tag{24}$$

The control signal $\bar{u}_k$ to achieve minimum variance control is chosen to satisfy

$$\bar{z}^*_{k+1} = \hat{\bar{z}}_{k+1|k} = \begin{cases} \hat{f}_k^{(o)}\hat{z}_{k+1|k} + \hat{\bar{f}}_k' Z_k + \hat{e}_k'\bar{U}_k & \text{post compensator case} \\ \hat{z}_{k+1|k} + \hat{e}_k'\bar{U}_k & \text{pre compensator case} \end{cases} \tag{25}$$

The above implicit equations for $\bar{u}_k$ can be re-organized as explicit equations giving a unique solution of $\bar{u}_k$ if $\hat{e}_k^{(o)} \neq 0$. Notice that $\hat{z}_{k+1|k}$ is not a function of $\bar{u}_k$ when the delay N satisfies $N > 1$. However when $\hat{z}_{k+1|k}$ includes an additive term $\hat{b}_k^{(o)}u_k$, in which case, for uniqueness require $(\hat{e}_k^{(o)} + \hat{b}_k^{(o)}\hat{f}_k^{(o)}) \neq 0$.

The minimum variance tracking scheme controllers appear more complicated than those for

2 A variation which allows $\hat{e}_k \ \hat{f}_k$ to be recursively or only periodically updated so that asymptotically $[\hat{e}_k \ \hat{f}_k]$ approaches that of (21) has implementation simplicity advantages but is not studied in this paper.

the previous stable controller scheme (I), however, the terms involving $\hat{e}_k$, $\hat{f}_k$ are calculated as part of the plant augmentation for both schemes (I) and (II). Thus in fact, the minimum variance schemes are marginally simpler to implement.

The following relationships between the original plant one-step-ahead prediction error and the augmented plant tracking error, or equivalently the augmented plant one-step-ahead prediction error, are immediately from (25). Thus

$$\bar{z}^*_{k+1} = \tilde{\tilde{z}}_{k+1|k} = \begin{cases} \hat{f}^{(o)}_k \tilde{z}_{k+1|k} & \text{(post compensator case)} \\ \tilde{z}_{k+1|k} & \text{(pre compensator case)} \end{cases} \tag{26}$$

For both controllers

$$\tilde{z}_{k+1|k} = \hat{a}_k' Z_{k+1} - \hat{b}_k' U_{k+1-n} - \hat{c}_k \hat{W}_k \tag{27}$$

CONVERGENCE RESULTS

Space limitations preclude us from giving full details but a comprehensive convergence theory is available for these algorithms. In the noise free case, we can establish global convergence using the following "pure least squares" parameter estimation algorithm:

$$P_k^{-1}\hat{\theta}_k = \sum_0^k x_i Z_k', \quad P_k^{-1} = P_{k-1}^{-1} + x_k x_k'; \quad P_0^{-1} = 0 \tag{28}$$

Convergence results are also available in the stochastic case. This theory builds in the work of Kumar and Moore (to appear) on global convergence of adaptive minimum variance control schemes via weighting coefficient selection.

Simulations. Simulations studies have been carried out to give confidence in the robustness of the algorithms. For the regulator case and deterministic plants, the simulations compare favourably with those reported in Goodwin and Sin (1981). For the stochastic cases, the tracking errors have added noise as expected. The added noise term depends on E_k which can be reduced by an appropriate "trial and error" selection of $H(q^{-1})$.

CONCLUSIONS

The paper has shown that the simple and powerful adaptive minimum variance control schemes which are globally convergent for minimum variance plants can be applied by means of adaptive compensators to control nonminimum phase plants.

Given that the minimum variance control scheme is applied to a plant augmented with compensators, the global tracking convergence results demonstrated are as strong as can be expected from such an arrangement.

The schemes can undoubtedly be generalized to give useful results for the multivariable case, and for the case when parameter variations are Markov.

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