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A Note on Minimal-Order Observers

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***Abstract*—An extension is given to Kalman observer theory which gives necessary and sufficient conditions for reduced-order Kalman observers for the case when it is required to observe a specified linear functional of the states of a system rather than the states themselves. The results are useful for the design of low-order compensators for systems with noisy output measurements.**

The Kalman observer (see, for example, [1]) can be used to achieve at least asymptotically, an arbitrary linear functional of the states of a linear finite-dimensional dynamical system when driven from the system inputs and outputs. This observer has no direct feedthrough from input to output and thus achieves filtering of the system output measurements. Its dimension is the same as that of the original system. The Luenberger observer (see also [1]) is of reduced dimension, but does not filter the system output measurements. It is also less straightforward in derivation and more difficult to design.

Recently [2], a theory of minimal-order observers has been presented. In particular, when it is required to achieve asymptotically a specified linear functional of the system states, readily tested necessary and sufficient conditions for the existence of an observer of given dimension and pole configuration are presented. These conditions are used in an algorithm for designing minimal-order observers. The observers are essentially reduced-order Luenberger observers when a specified, rather than an arbitrary, linear functional of the system states requires estimation. There is no filtering of the system output measurements.

The purpose of this correspondence is to record the corresponding necessary and sufficient conditions for reduced-order Kalman-type observers. These achieve (asymptotically) a specified linear functional of the system states without any direct feedthrough path, thus achieving filtering of noisy measurements. The derivations are simpler than the corresponding ones in [2] and are therefore included for the insight they provide.

For simplicity, we consider the single-input, single-output controllable and observable nth-order system

$$\dot{x} = Fx + gu; \qquad y = h'x, \qquad x(0) = x_0 \tag{1}$$

where F and h are in observable canonical form with

$$\det (sI_n - F) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0. \tag{2}$$

Suppose that we wish to observe $k'x$ using a pth-order completely controllable and observable observer

$$\dot{z} = Az + by + du; \qquad w = l'z, \qquad z(0) = z_0 \tag{3}$$

where we require that $\lim_{t\to\infty} |w - k'x| = 0$ for arbitrary x_0, z_0, and $u(\cdot)$. Notice that the observer has no direct feedthrough component and is thus useful for systems where the measurements y are noisy.

The systems (1) and (3) can be considered as a composite system

$$\begin{bmatrix} \dot{x} \\ \dot{z} - T'\dot{x} \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} x \\ z - T'x \end{bmatrix} + \begin{bmatrix} g \\ d - T'g \end{bmatrix} u \tag{4}$$

$$(w - k'x) = [(l'T' - k') = l'] \begin{bmatrix} x \\ z - T'x \end{bmatrix}$$

where the $(p \times n)$ transformation T' introduced satisfies

$$T'F - AT' = bh'. \tag{5}$$

Examination of (4) and (5) leads to the known result

$$\lim_{t\to\infty} \|z - T'x\| = 0 \Longleftrightarrow d = T'g; \quad \text{Re}\,[\lambda_i(A) < 0], \quad i = 1, 2, \cdots p. \tag{6}$$

In addition, with $[A, l]$ completely observable,

$$\lim_{t\to\infty} \|z - T'x\| = 0 \Longleftrightarrow \lim_{t\to\infty} \|z - T'x\|; \qquad k = Tl$$
$$\Longleftrightarrow d = T'g; \quad k = Tl; \quad \text{Re}\,[\lambda_i(A) < 0], \quad i = 1, 2, \cdots, p. \tag{7}$$

The next step is to eliminate T from the preceding conditions. With t_1 chosen such that $[A, t_1]$ is completely controllable, (6) now has the form

$$b = -\sum_{i=0}^{n} \alpha_i A^i t_1, \qquad \alpha_n = 1 \tag{8}$$

$$T' = [t_1 A t_1 \cdots A^{p-1}t_1 \cdots A^{n-1}t_1]. \tag{9}$$

We now claim that

$$k = Tl \Longleftrightarrow k \subset \boldsymbol{R}[T] \Longleftrightarrow k \subset \boldsymbol{N}[S_p] \tag{10}$$

where

$$S_p = \begin{bmatrix} \beta_0 & \beta_1 & \cdot & \cdot & \cdot & \beta_p & 0 & 0 & \cdot & 0 \\ 0 & \beta_0 & \cdot & \cdot & \cdot & & \beta_p & \cdot & \cdot & \cdot \\ \cdot & & & & & & & & \cdot & \cdot \\ \cdot & & & & & & & & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \beta_0 & \cdot & \cdot & \cdot & \cdot & \beta_p \end{bmatrix} \tag{11}$$

and the β's are defined from

$$\det (sI_p - A) = s^p + \beta_{p-1}s^{p-1} + \cdots + \beta_1 s + \beta_0. \tag{12}$$

This claim is readily established by application of the Cayley Hamilton theorem to the product S_pT to yield $S_pT = 0$. The full rank property of T, guaranteed by $[A, t_1]$ completely controllable, enables standard vector space ideas to lead directly to (10) and (11).

Thus the *necessary and sufficient conditions for a pth-order observer (3) to achieve an asymptotic estimate of $k'x$ where x is the state of the system (1) and (2) is that*

$$k \subset \boldsymbol{N}[S_p] \tag{13}$$

where S_p is given from (11)–(12).

The construction procedure for the estimator (3) is as follows. With A and t_1 chosen such that $[A, t_1]$ is completely controllable and $\text{Re}\,[\lambda_i(A)] < 0$, $i = 1, 2, \cdots, p$ and (11) through (12) are satisfied, then b may be calculated from (8), T from (9), and d and l from $d = T'g$ and $l = (T'T)^{-1}T'k$, respectively.

One general restriction implied by the conditions (13) and not observed in [2] is that the elements k_i of k cannot all be positive or negative since the β_i are all positive for the required stability condition (8) to be satisfied. This rules out the possibility of reduced-order signal estimators where $k = h = [0 \ \ 0 \ \ \cdots \ \ 0 \ \ 1]'$.

The corresponding results for time-varying and multiple-input, multiple-output systems can be obtained along the lines indicated in [2], but their application is considerably more tedious. For the case of single-input, single-output and time-invariant systems, the design of minimal-order observers may still be tedious, although for low-order systems (say, fourth order) the results are readily applied, although it is difficult to say at this stage whether low-order compensator design using this modern control framework has any real advantages over the more empirical classical compensator design techniques.

References

[1] B. D. Anderson and J. B. Moore, *Linear Optimal Control.* Englewood Cliffs, N. J.: Prentice-Hall, 1971.
[2] T. E. Fortmann and D. W. Williamson, "Design of low order observers for linear feedback control laws," in *Proc. 1971 Joint Automatic Control Conf.*; also, *IEEE Trans. Automat. Contr.*, vol. AC-17, June 1972.

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