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A Note on Feedback Compensators in Optimal Linear Systems

***Abstract*—Two basic methods for designing feedback compensators which may be used to achieve arbitrary closed-loop pole positions are now available. This correspondence shows that when each method is used within the framework of optimal control the same compensator structure results, but the more standard approach using a state estimator and a linear law appears to be the more attractive from the information now available.**

For simplicity, we will consider time-invariant single-input single-output nth-order systems which are completely controllable and completely observable with state equations

$$\dot{x} = Fx + gu, \quad y = h'x. \tag{1}$$

This correspondence gives a comparison of the two basic approaches available for designing feedback compensators (with input y and output u) using optimal control theory.

The most well-known approach is to first calculate a control law $u^* = k'x$ from the solution of an nth-order Riccati equation so as to minimize a performance index

$$V = \int_0^\infty (x'Qx + u^2)\, dt.$$

The matrix Q is nonnegative definite symmetric and satisfies an observability condition which guarantees asymptotic stability of $\dot{x} = (F + gk')x$. Next a state estimator [1],[2] is designed with y and u as input and x_e as output such that x_e approaches x asymptotically. A feedback compensator is then constructed which implements the control law

$$u = k'x_e + u_{\text{ext}} \tag{2}$$

where u_{ext} is used to denote any external input. The state equations of the estimator are

$$\dot{z} = F_e z + g_{1e} u + g_{2e} y \tag{3a}$$

$$x_e = S \begin{bmatrix} z \\ y \end{bmatrix} \tag{3b}$$

where F_e is an arbitrary $(n - 1) \times (n - 1)$ matrix such that $\dot{z} = F_e z$ is asymptotically stable. Once F_e has been chosen, values of g_{1e}, g_{2e}, and S are readily determined using procedures appearing in [1], [2]. The closed-loop autonomous system equations are, from (1)–(3) and properties of g_{1e}, etc.,

$$(d/dt) \begin{bmatrix} x \\ x_e - x \end{bmatrix} = \begin{bmatrix} F + gk' & gk' \\ 0 & F_e \end{bmatrix} \begin{bmatrix} x \\ x_e - x \end{bmatrix}. \tag{4}$$

We now introduce the notation ϕ to denote any arbitrary time-varying gain within the sector $[\frac{1}{2}, \infty)$. An important property from

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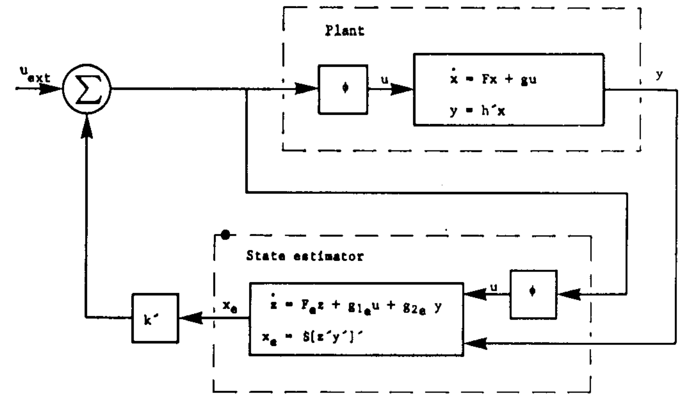

Fig. 1. Feedback compensator involving a state estimator.

the engineering point of view is that the closed-loop system with u replaced by ϕu (Fig. 1) is also asymptotically stable. Equivalently the system (4) with g replaced by $g\phi$ as

$$(d/dt)\begin{bmatrix} x \\ x_e - x \end{bmatrix} = \begin{bmatrix} F + g\phi k' & g\phi k \\ 0 & F_e \end{bmatrix}\begin{bmatrix} x \\ x_e - x \end{bmatrix} \tag{5}$$

is asymptotically stable. To see that this stability property exists we use the fact that $\dot{x} = (F + g\phi k')x$ is asymptotically stable from standard regulator theory [6] and the fact that $(x_e - x)$ is an L_2 function. (These facts also imply that a circle criterion [7] is satisfied for the closed-loop system.)

It might also be noted, although the proof is not given here, that there is a performance index of the form

$$V_1 = \int_0^\infty ([x_e'(x - x_e)']Q_1[x_e'(x - x_e)']' + u^2)\, dt$$

which is minimized by the control law $u = k'x_e$ rather than $u = k'x$. For the case where Q is positive definite, the matrix Q_1 will be nonnegative definite or positive definite.

The second approach to compensator design presented recently in [3]–[5][1] involves the minimization of a performance index associated with (1), augmented by an arbitrary $(n - 1)$th-order system

$$\dot{x}_a = F_a x_a + g_a u_a, \quad u = k_a' x_a \tag{6}$$

at its input, such that the augmented system is, as the original system, completely controllable. The index has the form

$$V_2 = \int_0^\infty ([x'x_a']Q_2[x'x_a']' + u_a^2)\, dt$$

where Q_2 is nonnegative definite symmetric. The solution of a $(2n - 1)$th-order Riccati equation yields an optimal control law of the form

$$u_a^* = k_2'x + k_a'x_a. \tag{7}$$

Applying a transformation $\tilde{x}_a = x_a + Tx$ to (1), (6), and (7) yields state equations for a compensator of the original system as

$$\dot{\tilde{x}}_a = (F_a + g_a k_a')\tilde{x}_a + Tgu + [TF - (F_a + g_a k_a')T + g_a k_2']x \tag{8a}$$

$$u = h_a'\tilde{x}_a - h_a'Tx + u_{\text{ext}} \tag{8b}$$

where u_{ext} is, as previously, an external input. Ferguson and Rekasius [5] show that a T exists such that (8) may be written as

$$\dot{\tilde{x}}_a = (F_a - g_a k_a')x_a + Tgu + g_2 y \tag{9a}$$

$$u = h_a'\tilde{x}_a + h_2'y + u_{\text{ext}} \tag{9b}$$

for some g_2 and h_2'.

It is immediately apparent that the preceding compensator (9) has the same structure as the compensator involving the state estimator. In fact, it is readily shown that the two compensators are identical iff $k'g = 0$ (more transparently, iff, for some T_0, $T_0F - (F_a + g_a k_a')T_0 - g_a k_2' = 0$, $T_0 g = 0$, $k_a'T_0 = -k'$). For the case when k is an optimal law this condition is never satisfied, and thus the two compensators for this case could not be identical.

Ferguson and Rekasius [5] draw attention to the fact that for system (1) and compensator (9) the circle criterion is satisfied. This means that if in the closed-loop system u_a is replaced by ϕu_a (or equivalently g_a is replaced by $g_a\phi$), then the closed-loop system still remains asymptotically stable. However, it is clear that the control u_a no longer physically exists in the compensator after the transformation T is introduced. We therefore cannot conclude, at least from the fact that the circle criterion is satisfied, that the physical system (1) and compensator (9) can accommodate (in general) time-varying gains ϕ in any part of the loop.

A property common to both forms of compensator is that arbitrary pole positions for the closed-loop systems may be achieved without recourse to optimal control theory but by appropriate selection of the parameters F_e and k' for the first case and the parameters k_2' and k_a' for the second case (see [3], [9]).

In summary then, both compensators investigated have the same structure and both can be used to achieve arbitrary pole positions. When used within the framework of optimal control, for both compensators a circle criterion is satisfied and there is a high-order performance index which is minimized. (It is also possible using the ideas of [8] for both to achieve a prescribed degree of stability.) However, the compensator involving the use of a state estimator has the desirable property [8], not established (at least in general) for the alternative compensator, that time-varying gains in the sector $[\frac{1}{2}, \infty)$ can be accommodated in the plant input transducers without causing instability (see Fig. 1). (This result, of course, implies certain desirable sensitivity properties and tolerance of nonlinearity properties—for example, with saturation nonlinearities the sector condition is satisfied provided that the initial states are not too great.) The alternative compensator, which requires the solution of a $(2n - 1)$th-order Riccati equation, is not as attractive from the calculation cost point of view as the more standard one designed using a state estimator and requiring the solution of only an nth-order Riccati equation. Both compensators are identical only under the restrictive condition $k'g = 0$.

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