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OPTIMAL REGULATOR WITH BOUNDS ON THE DERIVATIVE OF THE INPUT*

The problem of minimising $\int_{t_0}^{\infty}(x'Qx + u'Ru)dt$ for a completely controllable linear system, subject to the constraint $|\dot{u}| \leqslant 1$, is shown to be equivalent to a singular optimal-control problem; the known singular theory is then used to obtain the optimal control.

The standard regulator problem of optimal control is to minimise the performance index $V[x(t_0), u, t_0] = \int_{t_0}^{\infty}(x'Qx + u'Ru)dt$ for the linear dynamical system $\dot{x} = Fx + Gu$. F and G may possibly be time varying, and the control u may be either a scalar (single-input system) or a vector (multiple-input system).

If no constraints are imposed on u, the problem is simply the standard regulator problem for which a solution has been known for some years.[1] The regulator problem with the additional constraint $|u| \leqslant 1$ (the 'problem of Letov') has recently been solved; see, for example, Reference 2. However, to the authors' knowledge, no consideration has been given to the regulator problem with the restriction $|\dot{u}| \leqslant 1$, which would limit rapid changes in the control. A little thought will show that there are, in fact, practical applications which require this restriction. This, then, is the problem considered in this letter.

The usual assumptions will be made concerning the system, namely that the pair $[F, G]$ is completely controllable for all $t \geqslant t_0$, and Q and R are nonnegative definite and positive definite, respectively. For clarity of notation, u will be considered to be a scalar, but the results given here are equally applicable to multiple-input systems.

To make the problem tractable, we introduce the new variables

$$\hat{u} = \dot{u} \text{ and } \hat{x} = \begin{bmatrix} x \\ u \end{bmatrix}$$

[where the initial condition $u(t_0)$ will be specified later]. This gives the new system shown in Fig. 1. The state equation of the new system is

$$\dot{\hat{x}} = \hat{F}\hat{x} + \hat{G}\hat{u} \text{ where } \hat{F} = \begin{bmatrix} F & G \\ 0 & 0 \end{bmatrix} \qquad \hat{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the performance index becomes

$$V = \int_{t_0}^{\infty} \hat{x}'\hat{Q}\hat{x}dt \text{ where } \hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$$

The control-variable constraint is now $|\hat{u}| \leqslant 1$.

The new problem is now the 'singular' optimal-control problem, solved for the time-invariant single-input case in

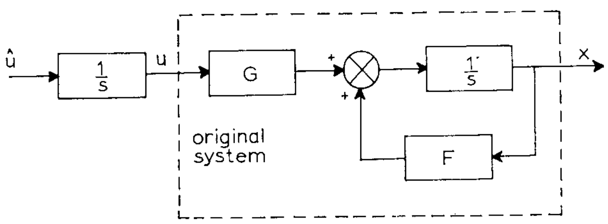

Fig. 1 *Modified system*

References 3 and 4 and for the time-varying multiple-input case in Reference 5. The optimal $\hat{u}$ turns out to be 'bang–bang' almost everywhere, but of the form $\hat{u} = K'\hat{x}$ on the 'singular strip' $K_1'x = 0$, $|K'\hat{x}| \leqslant 1$, where K and K_1 will be given below. Moreover, the optimal bang–bang control law near the origin (excluding the singular strip) is given by

$$\hat{u} = \operatorname{sgn}(K_1'\hat{x})$$

For large $\hat{x}$, this law is no longer optimal, but it is probably the best practical approximation to the true optimal bang–bang law (which is, in general, a rather complicated function of $\hat{x}$).

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In practice, the dual-mode control proposed above—bang–bang in some regions, linear in others—would probably be inconvenient. However, Reference 6 has shown that, if a bang–bang law such as that proposed is used throughout, all trajectories reaching the singular strip will chatter or slide along it in such a way that the effective control law is the linear one above. A practical realisation of the optimal control is then simply $\hat{u} = \operatorname{sgn}(K_1'\hat{x})$, which is strictly optimal for small x, and near optimal for large x.

To compute K_1, we use the following procedure (described more fully in References 4 and 5): Form the matrices

$$G_1 = \hat{F}\hat{G} - \dot{\hat{G}} \qquad S_1 = \hat{Q}\hat{G} \qquad R_1 = \hat{G}'\hat{Q}\hat{G}$$

and let $\Pi(t, t_1)$ be the solution of the equation

$$-\dot{\Pi} = \Pi(\hat{F} - G_1R_1^{-1}S_1') + (\hat{F}' - S_1R_1^{-1}G_1')\Pi$$
$$- \Pi G_1R_1^{-1}G_1'\Pi - S_1R_1^{-1}S_1' + \dot{\hat{Q}} \text{ with } \Pi(t_1, t_1) = 0$$

Now put

$$P(t) = \lim_{t_1 \to \infty} \Pi(t, t_1)$$

The assumptions made concerning Q and R and the complete controllability of $[F, G]$ ensure that this limit will, in fact, exist. Then

$$K_1 = -(PG_1 + S_1)R_1^{-1}$$

Also, the matrix K (if it is required) is given by

$$K = \dot{K}_{11} + \hat{F}'K_1$$

In the present case, we can partition P as

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix}$$

where the partitioning scheme is the same as that of $\hat{F}$ or $\hat{Q}$. Substituting all partitioned matrices into the above Riccati differential equation, it is found that

$$-\dot{P}_1 = P_1F + F'P_1 - P_1GR^{-1}G'P_1 + Q \qquad P_2 = 0$$
$$P_3 = 0$$

Also

$$K_1 = -(PG_1 + S_1)R_1^{-1} = \begin{bmatrix} -P_1GR^{-1} \\ -1 \end{bmatrix}$$

The equation of the singular strip is $K_1'\hat{x} = R^{-1}G'P_1x + u = 0$, and, if we set $u(t_0) = -R^{-1}G'P_1x(t_0)$, and $x(t_0)$ is sufficiently small for $|\hat{u}| < 1$ to hold, the system trajectory will always stay on the singular strip, and u will always be given by $u = -R^{-1}G'P_1x$.

This, as it turns out, is precisely the optimal control for no constraints on $\dot{u}$. This is an expected result, since the singular

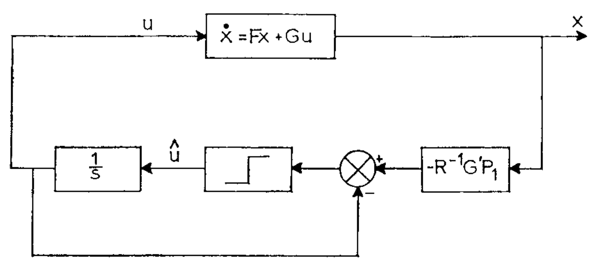

Fig. 2 *Feedback configuration*

strip is merely that region where the control lies within its constraints. The significance of this result is that, if we are free to specify $u(t_0)$, we should choose it so that the initial point in (x, u) space is on the singular strip; for a given $x(t_0)$, any other u will certainly be 'less optimal' than a result which was derived without assuming any constraints on u.

For large $x(t_0)$, it is, in general, impossible to choose a $u(t_0)$ so that the initial point is on the singular strip. The optimal (or, strictly speaking, the suboptimal best practical approximation to the optimal) u is then given by a bang–bang law; i.e.

$$\hat{u} = \operatorname{sgn} K_1'\hat{x}$$

or $$\hat{u} = -\operatorname{sgn}(R^{-1}G'P_1x + u)$$

This gives the feedback configuration of Fig. 2. If we are not free to choose $u(t_0)$, this completes the solution.

Sufficient conditions for the stability of the closed-loop system may be found in References 3–5. In general, it is difficult to guarantee global asymptotic stability, but the system may always be made asymptotically stable in a neighbourhood of the singular strip by appropriate choice of Q.

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