

Reprinted from the PROCEEDINGS OF THE IEEE

VOL. 55, NO. 10, OCTOBER, 1967

pp. 1749-1750

COPYRIGHT © 1967—THE INSTITUTE OF ELECTRICAL AND ELECTRONICS ENGINEERS, INC.

PRINTED IN THE U.S.A.

Dual Form of a Positive Real Lemma

Abstract—Systems theory descriptions alternative to those already known are presented for transfer function matrices which are positive real. They are derived using the fact that the transpose of a positive real matrix is itself positive real.

In Anderson [1] a system theory criterion for rational positive real matrices was given. [A matrix $Z(s)$ of rational functions of a complex variable s is positive real if 1) the poles of $Z(s)$ lie in $\text{Re } s < 0$, or are simple on $\text{Re } s = 0$,

and at any pole on $\text{Re } s = 0$, the associated residue matrix is non-negative definite hermitian; 2) elements of $Z(s)$ are real for real s ; 3) $Z(j\omega) + Z^*(-j\omega)$ is non-negative definite hermitian for all ω , other than those values for which $j\omega$ is a pole of an element of Z .] The criterion is as follows.

Positive Real Lemma

Let $Z(s)$ be an $n \times n$ matrix of rational functions of s , with $Z(\infty) < \infty$, and with (F, G, H, J) a minimal realization for $Z(s)$ in the sense that

Manuscript received July 7, 1967.

$$Z(s) = J + H'(sI - F)^{-1}G \quad (1)$$

with F square and of minimal dimension. Then $Z(s)$ is positive real if and only if there exist matrices P , L , and W_0 , with P positive definite symmetric, such that

$$PF + F'P = -LL'; \quad PG = H - LW_0; \quad W_0'W_0 = J + J'. \quad (2)$$

There are computational difficulties in finding P , L , and W_0 , which in essence define a matrix $W(s)$ such that

$$Z(s) + Z'(-s) = W'(-s)W(s) \quad (3)$$

via the formula

$$W(s) = W_0 + L'(sI - F)^{-1}G. \quad (4)$$

If (3) can be solved for $W(s)$ see, e.g., Youla [2], Davis [3], and Brockett [4], then P , L , and W_0 can be found. Another way of finding P is discussed in Anderson [5], which reduces, in the case where $J + J'$ is nonsingular, to solving the quadratic equation

$$P[F - G(J + J')^{-1}H'] + [F' - H(J + J')^{-1}G']P + PG(J + J')^{-1}G'P + H(J + J')^{-1}H' = 0. \quad (5)$$

In Anderson [5], a technique for solving this equation is discussed. Once a P has been found, L follows from the second of equations (2) (note that W_0 is nonsingular since $J + J'$ is); W_0 also follows from (2), at least to within multiplication by an arbitrary orthogonal matrix. Then $W(s)$ in (4) may be constructed.

It is often of interest to solve (3), i.e., to find $W(s)$ given $Z(s)$, with an additional constraint: not only should $W(s)$ be analytic in the right half-plane $\text{Re } s > 0$, but also $W^{-1}(s)$ should be analytic there. [Note that the analyticity of $W(s)$ in the right half-plane follows from the fact that $W(s)$ and $Z(s)$ have the same poles, viz., the eigenvalues of F , and $Z(s)$ is analytic in $\text{Re } s > 0$, being positive real.]

It is possible to ensure the constraint on W^{-1} in the case where there exists a $W^{-1}(s)$ analytic in $\text{Re } s \geq 0$ as distinct from $\text{Re } s > 0$. One way is to solve (5) with the additional constraint,

$$\text{Re } \lambda_i[F + G(J + J')^{-1}(PG - H)] < 0. \quad (6)$$

It is shown in Anderson [5] that there is a unique P satisfying (5) and (6) and that it generates a $W(s)$ having the desired property.

A second method is to use the following lemma, a trivial variant on a result in Anderson [6].

Lemma for Determination of P

With the same hypothesis as for the positive real lemma, let $Z(s)$ be positive real, $J + J'$ nonsingular, $Z(s)$ possess no $j\omega$ -axis poles, and $Z(j\omega) + Z'(-j\omega)$ be positive definite. Then the equation

$$-\dot{\Pi} = \Pi[F - G(J + J')^{-1}H'] + [F' - H(J + J')^{-1}G']\Pi + \Pi G(J + J')^{-1}G'\Pi + H(J + J')^{-1}H' \quad (7)$$

with initial condition $\Pi(t_1) = 0$ has a well-defined solution $\Psi(t, t_1)$ for all $t < t_1$. Moreover, $\lim_{t \rightarrow -\infty} \Psi(t, t_1)$ exists and equals P , the unique solution of (5) and (6).

Dual results follow by noting that if $Z(s)$ is positive real, so is $Y(s) = Z'(s)$; and if $Z(s)$ is given by (1), $Y(s)$ is given by

$$Y(s) = J' + G'(sI - F')^{-1}H. \quad (8)$$

Applying the positive real lemma to $Y(s)$ leads to:

Dual Positive Real Lemma

With the same hypothesis as for the positive real lemma, $Z(s)$ is positive real if and only if there exist matrices Q , M , and V_0 such that

$$QF' + FQ = -MM'; \quad QH = G - MV_0; \quad V_0'V_0 = J + J'. \quad (9)$$

Again, M allows determination of a matrix $V(s)$ such that

$$Y(s) + Y'(-s) = V'(-s)V(s) \quad (10)$$

through

$$V(s) = V_0 + M'(sI - F')^{-1}H. \quad (11)$$

Substituting $Z(s)$ into (10) and transposing leads to

$$Z(s) + Z'(-s) = V'(s)V(-s) \quad (12)$$

and thus $V(s)$ in (11) gives a spectral factorization of $Z(s) + Z'(-s)$, where the first term on the right of (12) is analytic in $\text{Re } s > 0$, rather than the second, as in (3).

If $(J + J')$ is nonsingular, Q satisfies

$$Q[F' - H(J + J')^{-1}G'] + [F - G(J + J')^{-1}H']Q + QH(J + J')^{-1}H'Q + G(J + J')^{-1}G' = 0. \quad (13)$$

The particular solution of (12) leading to a $V(s)$ with stable inverse also satisfies the dual condition

$$\text{Re } \lambda_i[F' + H(J + J')^{-1}(QH - G)] < 0 \quad (14)$$

and, with appropriate conditions on $Z(s)$, is the limiting solution of the Riccati equation

$$-\dot{\Pi} = \Pi[F' - H(J + J')^{-1}G'] + [F - G(J + J')^{-1}H']\Pi + \Pi H(J + J')^{-1}H'\Pi + G(J + J')^{-1}G' \quad (15)$$

with zero initial condition.

Finally, let us note that if P , L , and W_0 are matrices satisfying (2), when F , G , H , and J are known, one set of matrices satisfying (9) is provided by solving the invertible equations

$$QP = I; \quad PM = -L; \quad W_0 = V_0. \quad (16)$$

BRIAN D. ANDERSON
JOHN B. MOORE
Dept. of Elec. Engrg.
University of Newcastle
Newcastle, New South Wales
Australia

REFERENCES

- [1] B. D. O. Anderson, "A system theory criterion for positive real matrices," *SIAM J. of Control*, vol. 5, pp. 171-182, May 1967.
- [2] D. C. Youla, "On the factorization of rational matrices," *IRE Trans. Information Theory*, vol. IT-7, pp. 172-189, July 1961.
- [3] M. C. Davis, "Factoring the spectral matrix," *IEEE Trans. Automatic Control*, vol. AC-8, pp. 296-305, October 1963.
- [4] R. W. Brockett, "Spectral factorization of rational matrices," *IEEE Trans. Information Theory*, to be published.
- [5] B. D. O. Anderson, "An algebraic solution to the spectral factorization problem," *IEEE Trans. Automatic Control*, to be published.
- [6] B. D. O. Anderson, "Quadratic minimization, positive real matrices and spectral factorization," submitted for publication.