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TOLERANCE OF NONLINEARITIES IN TIME-VARYING OPTIMAL SYSTEMS

Nominally linear optimal-control regulating systems are examined with a view to assessing the amount of nonlinearity which can be tolerated in the input transducer. Using a suitable Lyapunov function, it is found that a large degree of nonlinearity will not disturb the stability of the system.

In this letter, it is shown that, in a nominally linear, optimal, time-varying control system, a large amount of nonlinearity may be tolerated at the input transducer. The result thus parallels a corresponding result for time-invariant systems.*

Consider a linear system (x the state vector, u the input vector)

$$\dot{x} = Fx + Gu \qquad (1)$$

with F and G in general time varying, together with a performance index

$$J\,(x_0, t_0, u) = \int_{t_0}^{\infty} (u'u + x'Qx)dt \qquad (2)$$

Here $Q = Q'$ is nonnegative definite. It is moreover assumed that the pair $[F, G]$ is uniformly completely controllable (Reference 1), and that the pair $[F, H]$, where H is any matrix such that $H'H$, is uniformly completely observable (Reference 1). (These conditions are placed on the problem to ensure existence of a control law for the infinite time minimisation problem, together with the stability of that law.) As shown in Reference 1, there is a control law

$$u = -K'x \qquad (3)$$

which minimises eqn. 2. The matrix K is defined by

$$K = PG \qquad (4)$$

where P in turn is defined as the solution of

$$-\dot{P} = PF + F'P - PGG'P + Q \qquad (5a)$$

with the initial condition

$$\lim_{T\to\infty} P(T) = 0 \qquad (5b)$$

The matrix P is, of course, time varying; it is also symmetric and satisfies

$$\alpha_1 I \geqslant P(t) \geqslant \alpha_2 I > 0 \qquad (6)$$

for all t and some positive α_1, α_2, provided $Q(.)$ is bounded.

* MOORE, J. B., and ANDERSON, B. D. O.: 'Application of the multidimensional Popov criterion, unpublished

One can conceive of the system input u being generated by some transducer, which in practical situations may be the part of the system most likely to be nonlinear. In any case, suppose that u, instead of being given by eqn. 3, is given by

$$u = -\tfrac{1}{2}K'x - \psi \qquad (7)$$

where ψ is a nonlinear vector function of $K'x$, satisfying

$$\sigma'\psi(\sigma) > 0 \quad \text{with } \sigma = K'x, \quad \sigma \neq 0 \qquad (8)$$

This is a natural generalisation of a typical Popov sector condition, and may well be a result of relations restricting the ith components, σ_i and ψ_i, of the vectors σ and ψ:

$$\psi_i(\sigma) = \psi_i(\sigma_i) \qquad \text{for all } i \qquad (9a)$$

$$\sigma_i\psi_i(\sigma) > 0 \quad \text{when } \sigma_i \neq 0, \text{ for all } i \qquad (9b)$$

With u as in eqn. 7, stability of the closed-loop system can be demonstrated by taking as a Lyapunov function

$$V(x, t) = x'(t)P(t)x(t) \qquad (10)$$

Relations 6 guarantee that V fulfills the necessary requirements to establish stability (Reference 2). Also,

$$\dot{V} = (x'F' - \tfrac{1}{2}x'KG' - \psi'G)Px + x'\dot{P}x + x'P(Fx - \tfrac{1}{2}GK'x - G\psi) \qquad (11)$$

Substituting for $\dot{P}$ using eqn. 5a, and making use of eqn. 4 leads to

$$\dot{V} = -x'Qx - 2\psi'(K'x) \qquad (12)$$

Eqn. 8 guarantees that $\dot{V}$ is nonpositive, and thus stability of the closed-loop system follows. More refined arguments can presumably be used to establish conditions for asymptotic stability, which of course can normally be expected to prevail in view of the demonstration of stability, except in certain limiting situations.

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