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INDEFINITE STOCHASTIC LINEAR QUADRATIC CONTROL AND GENERALIZED DIFFERENTIAL RICCATI EQUATION*

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Abstract. A stochastic linear quadratic (LQ) control problem is indefinite when the cost weighting matrices for the state and the control are allowed to be indefinite. Indefinite stochastic LQ theory has been extensively developed and has found interesting applications in finance. However, there remains an outstanding open problem, which is to identify an appropriate Riccati-type equation whose solvability is *equivalent* to the solvability of the indefinite stochastic LQ problem. This paper solves this open problem for LQ control in a finite time horizon. A new type of differential Riccati equation, called the generalized (differential) Riccati equation, is introduced, which involves algebraic equality/inequality constraints and a matrix pseudoinverse. It is then shown that the solvability of the generalized Riccati equation is not only sufficient, but also *necessary*, for the well-posedness of the indefinite LQ problem and the existence of optimal feedback/open-loop controls. Moreover, all of the optimal controls can be identified via the solution to the Riccati equation. An example is presented to illustrate the theory developed.

Key words. stochastic LQ control, indefinite costs, generalized Riccati equation, matrix pseudoinverse, matrix minimum principle, dynamic programming

AMS subject classifications. 93E20, 49K45

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1. Introduction.

Consider the following stochastic linear quadratic (LQ) optimal control problem in a finite time horizon $[0, T]$:

(1.1)

$$\begin{array}{ll} \text{Minimize} & J = E\left\{ \int_0^T [x(t)'Q(t)x(t) + u(t)'R(t)u(t)]dt + x(T)'Hx(T) \right\}, \\ \text{subject to} & \left\{ \begin{array}{l} dx(t) = [A(t)x(t) + B(t)u(t)]dt + [C(t)x(t) + D(t)u(t)]dW(t), \\ x(0) = x_0 \in \mathbf{R}^n. \end{array} \right. \end{array}$$

Here $W(t)$ is a standard one-dimensional Brownian motion, and the control $u(\cdot)$ takes value in some Euclidean space.

In optimal LQ control theory, the Riccati equation approach has been used systematically to provide an optimal feedback control (see [14, 20, 4] for the deterministic case, and [23, 6, 11] for the stochastic case). In the literature it is typically assumed that the cost function has a positive definite weighting matrix, R, for the control term, and a positive semidefinite weighting matrix, Q, for the state term. In this case, the solvability of the Riccati equation is both necessary and sufficient for the solvability of the underlying LQ problem.

However, it was found in [7] for the first time that a stochastic LQ problem with *indefinite* Q and R may still be well-posed. This phenomenon has to do with the

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deep nature of the uncertainty involved; see [7] for a detailed discussion and many examples. Follow-up research on indefinite stochastic LQ control in a finite time horizon has been carried out in [8, 16, 9] to incorporate more complicated features such as random coefficients and integral constraints. The infinite-time-horizon case, in which the stability becomes a crucial issue, was treated in [2, 24] via techniques in linear matrix inequality and semidefinite programming [21]. On the other hand, applications of indefinite LQ control to portfolio selection problems and a contingent claim problem can be found in [25, 17] and [15], respectively. We would also like to mention a recent paper [12] in which the stochastic H^∞ problem is dealt with via a Riccati equation that has a structure similar to the one in [7].

In the first paper [7] on indefinite stochastic LQ control, it is shown that if the following type of Riccati equation, called the stochastic Riccati equation (t is suppressed),

$$\begin{cases} \dot{P} + PA + A'P + C'PC - (PB + C'PD)(R + D'PD)^{-1}(B'P + D'PC) + Q = 0, \\ P(T) = H, \\ R + D'PD > 0, \quad \text{a.e. } t \in [0, T], \end{cases} \tag{1.2}$$

has a solution $P(\cdot)$, then the original (indefinite) LQ problem is well-posed and an optimal feedback control can be constructed explicitly via $P(\cdot)$. (Note that the third positive definiteness constraint in (1.2) is *part* of that equation and must be satisfied by any solution.) In other words, the solvability of the stochastic Riccati equation (1.2) is *sufficient*, but not *necessary* in general, for the well-posedness as well as the solvability of the LQ problem. A natural question then is what can we say about the indefinite LQ problem if (1.2) does *not* have a solution at all? Note that the positive definiteness constraint of $R + D'PD$ in (1.2) is very restrictive, which likely leads to the nonexistence of its solutions. As a consequence, it may happen that the original indefinite LQ problem is well-posed and there exist optimal controls, while (1.2) still has no solution, in which case (1.2) becomes useless. This is quite different from the deterministic counterpart (as mentioned earlier), which in turn suggests that (1.2) may not be *the* right Riccati equation for indefinite LQ control.

Let us look at an example to illustrate the above discussion.

Example 1.1. Consider LQ problem (1.1) with $T = 1$, $A(t) = B(t) = D(t) = 1$, $C(t) = -1$, $Q(t) = -1$, $R(t) = -\frac{2e^{3(1-t)}+1}{3}$, and $H = 1$. Note that Q and R are both *negative* here. Equation (1.2) in this case specializes to

$$\begin{cases} \dot{P}(t) + 3P(t) - 1 = 0, \\ P(1) = 1, \\ R(t) + P(t) > 0, \quad \text{a.e. } t \in [0, T]. \end{cases} \tag{1.3}$$

The only solution that satisfies the first two constraints of the above equation is $P(t) = \frac{2e^{3(1-t)}+1}{3}$. Hence $R(t) + P(t) \equiv 0$, violating the third constraint. This shows that (1.3) has no solution and the result in [7] fails to apply to this case. However, the original LQ problem is well-posed. To see this, let $P(t) = \frac{2e^{3(1-t)}+1}{3}$, and apply Itô's formula to $P(t)x(t)^2$. We then obtain

$$d[P(t)x(t)^2] = [x(t)^2 + P(t)u(t)^2]dt + \{\cdots\}dW(t).$$

Integrating from 0 to 1, and taking expectation, we have $J \equiv P(0)x_0^2$. This implies that the cost function takes a *constant* value $P(0)x_0^2$ regardless of the control being applied. In particular, the LQ problem is well-posed and does have optimal controls.

The above example suggests that the stochastic Riccati equation (1.3) introduced in [7] may not be able to handle certain indefinite stochastic LQ problems. Finding a more appropriate Riccati-type equation, in the sense that its solvability should be *equivalent* to that of the underlying LQ problem, remains an outstanding open problem. The objective of this paper is to tackle this open problem, thereby enabling us to deal with general indefinite stochastic LQ problems, including pathological situations such as the one in Example 1.1. The key to achieving this goal is the introduction of a new type of differential Riccati equation—called a *generalized Riccati equation*—where the positive definiteness constraint of $R + D'PD$ is relaxed. This equation involves a matrix pseudoinverse and an additional algebraic constraint due to the possible singularity of the term $R+D'PD$. This new Riccati equation turns out to be the right one for studying indefinite LQ problems, as the solvability of this equation is not only sufficient, but also *necessary*, for the well-posedness of the LQ problem as well as the attainability of its optimal controls. Moreover, we are able to derive *all* optimal controls via the solution of the generalized Riccati equation.

It is worth mentioning that even for deterministic singular LQ problems (see [22, 13, 18, 10], among others), which are a special case of the problem treated in this paper, our formulation and results are still new; for details see section 3.

The remainder of this paper is organized as follows. Section 2 formulates the indefinite stochastic LQ problem and gives some preliminaries. The generalized Riccati equation (GRE) is also introduced. Section 3 shows that the solvability of the GRE is sufficient for the well-posedness of the LQ problem and the existence of an optimal control. Moreover, all the optimal controls are identified via the solution of the GRE. Sections 4 and 5 prove that the solvability of the GRE is also necessary for the existence of optimal linear feedback controls and optimal open-loop controls, respectively. An example is presented in section 6 to illustrate the results obtained. Finally, section 7 gives some concluding remarks.

2. Problem formulation and preliminaries.

2.1. Notation. We make use of the following notation in this paper:

$\mathbf{N}$	:	the set of positive integers.
$\mathbf{R}$	:	the set of real numbers.
$\mathbf{R}^n$	:	n-dimensional Euclidean space.
M'	:	the transpose of a matrix M.
$M^\dagger$	:	the Moore–Penrose pseudoinverse of a matrix M.
$\mathbf{Tr}(M)$	:	the sum of diagonal elements of a square matrix M.
$\|x\|$	:	$= \sqrt{\sum x_i^2}$ for a vector $x = (x_1, \dots, x_n)'$.
$\mathbf{R}^{n\times m}$	:	the space of all $n \times m$ matrices.
$\mathcal{S}^n$	:	the space of all $n \times n$ symmetric matrices.
$\mathcal{S}^n_+$	:	the subspace of all positive semidefinite matrices of $\mathcal{S}^n$.
$\hat{\mathcal{S}}^n_+$	:	the subspace of all positive definite matrices of $\mathcal{S}^n$.

Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_t)$, where $t \in [0, T]$, and a Hilbert space X with the norm $\|\cdot\|_X$, define the Hilbert space

$$L^2_{\mathcal{F}}(0,T;X) = \Big\{\phi(\cdot) \mid \quad \phi(\cdot) \text{ is an } \mathcal{F}_t\text{-adapted, } X\text{-valued measurable process on } [0,T], \\ \text{and } E\int_0^T \| \phi(t,\omega) \|_X^2 \, dt < +\infty \Big\},$$

with the norm

$$\| \phi(\cdot) \|_{\mathcal{F},2} = \left(E \int_0^T \| \phi(t,\omega) \|_X^2 \, dt \right)^{\frac{1}{2}}.$$

2.2. Problem formulation. Let $(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_t)$ be a given filtered probability space with a standard one-dimensional Brownian motion $W(t)$ on $[0,T]$ (with $W(0) = 0$). Consider the following linear Itô stochastic differential equation:

$$\text{(2.1)} \qquad \begin{cases} dx(t) = [A(t)x(t) + B(t)u(t)]dt + [C(t)x(t) + D(t)u(t)]dW(t), \\ x(s) = y, \end{cases}$$

where $(s,y) \in [0,T) \times \mathbf{R}^n$ are the initial time and initial state, respectively, and $u(\cdot)$, the admissible control, is an element in $U_{ad} \equiv L^2_{\mathcal{F}}(0,T;\mathbf{R}^{n_u})$. In order to simplify exposition we assume that the Brownian motion is one-dimensional. There is no essential difficulty with the multidimensional case.

For each (s,y) and $u(\cdot) \in U_{ad}$, the associated cost is

$$\text{(2.2)} \quad J(s,y;u(\cdot)) = E\left\{ \int_s^T [x(t)'Q(t)x(t) + u(t)'R(t)u(t)]dt + x(T)'Hx(T) \right\}.$$

The solution $x(\cdot)$ of (2.1) is called the response of the control $u(\cdot) \in U_{ad}$, and $(x(\cdot), u(\cdot))$ is called an *admissible pair*. The objective of the optimal control problem is to minimize the cost function $J(s,y;u(\cdot))$, for a given $(s,y) \in [0,T) \times \mathbf{R}^n$, over all $u(\cdot) \in U_{ad}$. The value function is defined as

$$\text{(2.3)} \qquad V(s,y) = \inf_{u(\cdot) \in U_{ad}} J(s,y;u(\cdot)).$$

DEFINITION 2.1. *The optimization problem* (2.1)–(2.3) *is called* well-posed *if*

$$V(s,y) > -\infty \quad \forall (s,y) \in [0,T) \times \mathbf{R}^n.$$

An admissible pair $(x^*(\cdot), u^*(\cdot))$ *is called* optimal *(with respect to the initial condition* (s,y)*) if* $u^*(\cdot)$ *achieves the infimum of* $J(s,y;u(\cdot))$.

The following basic assumption will be in force throughout this paper.

Assumption **(A)**. The data appearing in the LQ problem (2.1)–(2.3) satisfy

$$\begin{array}{l} A, C \in L^\infty(0,T;\mathbf{R}^{n\times n}), \\ B, D \in L^\infty(0,T;\mathbf{R}^{n\times n_u}), \\ Q \in L^\infty(0,T;\mathcal{S}^n), \\ R \in L^\infty(0,T;\mathcal{S}^{n_u}), \\ H \in \mathcal{S}^n. \end{array}$$

We emphasize again that we are dealing with an *indefinite* LQ problem, namely that Q, R, and H are all possibly indefinite.

2.3. Generalized (differential) Riccati equation. We start by recalling properties of a pseudo matrix inverse [19].

PROPOSITION 2.2. *Let a matrix* $M \in \mathbf{R}^{m\times n}$ *be given. Then there exists a unique matrix* $M^\dagger \in \mathbf{R}^{n\times m}$ *such that*

$$\text{(2.4)} \qquad \begin{cases} MM^\dagger M = M, \quad M^\dagger M M^\dagger = M^\dagger, \\ (MM^\dagger)' = MM^\dagger, \quad (M^\dagger M)' = M^\dagger M. \end{cases}$$

The matrix $M^\dagger$ above is called the *Moore–Penrose pseudoinverse* of M.

Now, we introduce a new type of Riccati equation associated with the LQ problem (2.1)–(2.3).

DEFINITION 2.3. *The constrained differential equation (with the time argument t suppressed)*

(2.5)
$$\begin{cases} \dot{P} + PA + A'P + C'PC - (PB + C'PD)(R + D'PD)^\dagger(B'P + D'PC) + Q = 0, \\ P(T) = H, \\ (R + D'PD)(R + D'PD)^\dagger(B'P + D'PC) - (B'P + D'PC) = 0, \\ R + D'PD \geq 0, \quad \text{a.e. } t \in [0, T], \end{cases}$$

is called a generalized (differential) Riccati equation *(GRE).*

If the term $(R + D'PD)$ is further required to be nonsingular, then the GRE reduces to the stochastic Riccati equation (1.2) that was introduced in [7].

Another interesting special case is that in which $(R + D'PD) \equiv 0$; the GRE reduces to the following linear differential matrix system:

$$\begin{cases} \dot{P} + PA + A'P + C'PC + Q = 0, \\ P(T) = H, \\ B'P + D'PC = 0, \\ R + D'PD = 0, \quad \text{a.e. } t \in [0, T]. \end{cases} \tag{2.6}$$

2.4. Some useful lemmas.

LEMMA 2.4. *Let $M(\cdot)$ be a given continuously differentiable (in t) matrix function taking values in $\mathcal{S}^n$. Then for any admissible pair $(x(\cdot), u(\cdot))$ of the system* (2.1)*, we have*

$$\begin{aligned} &E[x(t)'Mx(T)] - y'M(s)y - E\int_s^T [x'(\dot{M} + A'M + MA + C'MC)x](t)dt \\ &\quad -E\int_s^T [2u'(B'M + D'MC)x + u'D'MDu](t)dt = 0. \end{aligned} \tag{2.7}$$

Proof. Using Itô's formula, we have (t is suppressed)

$$\begin{aligned} d(x'Mx) &- [(Ax + Bu)'Mx + x'\dot{M}x + x'M(Ax + Bu) - (Cx + Du)'M(Cx + Du)]dt \\ &- [x'M(Cx + Du) + (Cx + Du)'Mx]dW(t) = 0. \end{aligned}$$

Taking expectations and integrations we obtain (2.7). ◻

LEMMA 2.5. *Let a symmetric matrix S be given. Then*

(i) $S^\dagger = S^{\dagger\prime}$.

(ii) $S \geq 0$ *if and only if* $S^\dagger \geq 0$.

(iii) $SS^\dagger = S^\dagger S$.

Proof. Since S is symmetric, it has a singular value decomposition of the following form:

$$S = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V',$$

where Σ is a nonsingular diagonal matrix and V a matrix such that $VV' = V'V = I$. Now, $S^\dagger$ is given by

$$S^\dagger = V \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} V'.$$

Using the above expression of $S^\dagger$, it is easy to show that items (i)–(iii) hold. ◻

LEMMA 2.6 (Extended Schur's lemma [3]). *Let matrices $M = M'$, N, and $R = R'$ be given with appropriate sizes. Then the following conditions are equivalent:*

(i) $M - NR^\dagger N' \geq 0$, $R \geq 0$, *and* $N(I - RR^\dagger) = 0$.

(ii) $\begin{bmatrix} M & N \\ N' & R \end{bmatrix} \geq 0$.

(iii) $\begin{bmatrix} R & N' \\ N & M \end{bmatrix} \geq 0$.

The following lemma plays a key technical role in this paper.

LEMMA 2.7. *Let matrices L, M, and N be given with appropriate sizes. Then the matrix equation*

$$LXM = N \tag{2.8}$$

has a solution X if and only if

$$LL^\dagger NM^\dagger M = N. \tag{2.9}$$

Moreover, any solution to (2.8) *is represented by*

$$X = L^\dagger NM^\dagger + S - L^\dagger LSMM^\dagger, \tag{2.10}$$

where S is a matrix with an appropriate size.

Proof. If X satisfies the equation $LXM = N$, then we have

$$N = LXM = LL^\dagger LXMM^\dagger M = LL^\dagger NM^\dagger M.$$

Conversely, if (2.9) is satisfied, then $L^\dagger NM^\dagger$ is a solution of $LXM = N$. This proves the first part of the lemma. Now, let Y be any matrix with appropriate size and define $\tilde{X} = Y - L^\dagger LYMM^\dagger$. Then $\tilde{X}$ satisfies the homogeneous equation $L\tilde{X}M = 0$. Hence $L^\dagger NM^\dagger + \tilde{X}$ must satisfy (2.8). On the other hand, let X be a solution to (2.8). Then by (2.9), one has $LSM = 0$, where $S = X - L^\dagger NM^\dagger$. Hence

$$X = L^\dagger NM^\dagger + S - L^\dagger LSMM^\dagger.$$

This completes the proof. ☐

3. Sufficiency of the GRE. In this section, we will show that the solvability of the GRE (2.5) is sufficient for the well-posedness of the LQ problem and the existence of an optimal linear state feedback control. Moreover, any optimal control can be obtained via the solution to the GRE.

THEOREM 3.1. *If the GRE* (2.5) *admits a solution $P(\cdot)$, then the stochastic LQ problem* (2.1)–(2.3) *is well-posed. Moreover, the set of all the optimal controls with respect to the initial $(s, y) \in [0, T) \times \mathbf{R}^n$ is determined by the following* (*parameterized by* (Y, z)):

$$\begin{aligned} u_{(Y,z)}(t) = &-\Big\{[R(t) + D(t)'P(t)D(t)]^\dagger[B(t)'P(t) + D(t)'P(t)C(t)] + Y(t) \\ &- [R(t) + D(t)'P(t)D(t)]^\dagger[R(t) + D(t)'P(t)D(t)]Y(t)\Big\}x(t) \\ &+ z(t) - [R(t) + D(t)'P(t)D(t)]^\dagger[R(t) + D(t)'P(t)D(t)]z(t), \end{aligned} \tag{3.1}$$

where $Y(\cdot) \in L^2_{\mathcal{F}}(s, T; \mathbf{R}^{n_u \times n})$ and $z(\cdot) \in L^2_{\mathcal{F}}(s, T; \mathbf{R}^{n_u})$. Furthermore, the value function is uniquely determined by $P(\cdot)$:

$$V(s, y) \equiv \inf_{u(\cdot) \in U_{ad}} J(s, y; u(\cdot)) = y'P(s)y. \tag{3.2}$$

Proof. Let $P(\cdot)$ be a solution of GRE (2.5). Applying Lemma 2.4, we can express the cost function as follows:

$$\begin{aligned} J(s,y;u(\cdot)) = y'P(s)y + E\int_s^T \Big[& x'(\dot{P} + PA + A'P + C'PC + Q)x \\ & + 2u'(B'P + D'PC)x + u'(D'PD + R)u\Big](t)dt. \end{aligned} \tag{3.3}$$

Now, let $Y(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbf{R}^{n_u\times n})$ and $z(\cdot) \in L^2_{\mathcal{F}}(s,T;\mathbf{R}^{n_u})$ be given. Set

$$\begin{aligned} L_1(t) &= Y(t) - [R(t) + D'(t)P(t)D(t)]^\dagger[R(t) + D'(t)P(t)D(t)]Y(t), \\ L_2(t) &= z(t) - [R(t) + D'(t)P(t)D(t)]^\dagger[R(t) + D'(t)P(t)D(t)]z(t). \end{aligned}$$

Applying Proposition 2.2 and Lemma 2.5(iii), we have

$$[R(t) + D'(t)P(t)D(t)]L_i(t) = [R(t) + D'(t)P(t)D(t)]^\dagger L_i(t) = 0, \qquad i = 1,2, \tag{3.4}$$

and

$$[P(t)B(t) + C'(t)P(t)D(t)]L_i(t) = 0, \qquad i = 1,2. \tag{3.5}$$

Then identity (3.3) can be expressed as

$$\begin{aligned} & J(s,y;u(\cdot)) \\ &\quad = y'P(s)y + E\textstyle\int_s^T \Big\{x'[\dot{P} + PA + A'P + C'PC + Q \\ &\qquad - (PB + C'PD)(R + D'PD)^\dagger(B'P + D'PC)]x \\ &\qquad + \big[u + \big((R + D'PD)^\dagger(B'P + D'PC) + L_1\big)x + L_2\big]' \\ &\qquad \times (R + D'PD)\big[u + \big((R + D'PD)^\dagger(B'P + D'PC) + L_1\big)x + L_2\big]\Big\}(t)dt \\ &\quad = y'P(s)y + E\textstyle\int_s^T \Big\{\big[u + \big((R + D'PD)^\dagger(B'P + D'PC) + L_1\big)x + L_2\big]' \\ &\qquad \times (R + D'PD)\big[u + \big((R + D'PD)^\dagger(B'P + D'PC) + L_1\big)x + L_2\big]\Big\}(t)dt. \end{aligned} \tag{3.6}$$

Hence, $J(s,y;u(\cdot))$ is minimized by the control given by (3.1) with the optimal value being $y'P(s)y$.

What remains to show is that *any* optimal control can be represented by (3.1) for some $Y(\cdot)$ and $z(\cdot)$. To this end, let $u(\cdot)$ be an optimal control. Then by (3.6) the integrand in the right-hand side of (3.6) must be zero almost everywhere in t. This implies (t is suppressed)

$$(R + D'PD)^{1/2}\big[u + \big((R + D'PD)^\dagger(B'P + D'PC) + L_1\big)x + L_2\big] = 0,$$

which leads to

$$(R + D'PD)[u + \big((R + D'PD)^\dagger(B'P + D'PC) + L_1\big)x + L_2] = 0,$$

or, equivalently (noting (3.4)),

$$[R(t) + D(t)'P(t)D(t)]u(t) + [B(t)'P(t) + D(t)'P(t)C(t)]x(t) = 0, \quad \text{a.e. } t \in [s,T]. \tag{3.7}$$

To solve the above equation in $u(t)$, we apply Lemma 2.7 with

$$L = R(t) + D(t)'P(t)D(t), \quad M = I, \quad N = -[B(t)'P(t) + D(t)'P(t)C(t)]x(t).$$

Notice that condition (2.9) in the present case is implied by the third constraint in GRE (2.5); hence the general solution (2.10) with $z(t) = S$ and $Y(t) = 0$ yields that $u(t)$ can be represented by (3.1). ◻

COROLLARY 3.2. *The optimal controls are obtained in the following special cases:*

(i) *If* $R(t) + D(t)'P(t)D(t) \equiv 0$, a.e. $t \in [s, T]$, *then any admissible control is optimal.*

(ii) *If* $R(t) + D(t)'P(t)D(t) > 0$, a.e. $t \in [s, T]$, *then there is a unique optimal control that is given by the following linear feedback law:*

$$u(t) = -[R(t) + D(t)'P(t)D(t)]^{-1}[B(t)'P(t) + D(t)'P(t)C(t)]x(t). \tag{3.8}$$

Proof. The proofs here are straightforward from Theorem 3.1. ◻

As an immediate consequence of Theorem 3.1, we have the uniqueness of the solution to GRE (2.5).

COROLLARY 3.3. *If there is a solution to the GRE* (2.5), *then it must be the only solution to* (2.5).

Proof. Let $P_1(\cdot)$ and $P_2(\cdot)$ be two solutions of GRE (2.5). Then Theorem 3.1 implies that

$$y'P_1(s)y = y'P_2(s)y \ \ \forall y \in \mathbf{R}^n \ \ \forall s \in [0, T].$$

Hence $P_1(t) \equiv P_2(t)$. ◻

It is interesting to see the specialization of our results in the deterministic case (i.e., $C(t) = D(t) \equiv 0$). The control weight $R(t)$ is given as satisfying $R(t) \geq 0$, so it is a possibly singular case. The corresponding GRE is

$$\begin{cases} \dot{P}(t) + P(t)A(t) + A(t)'P(t) - P(t)B(t)R(t)^{\dagger}B(t)'P(t) + Q(t) = 0, \\ P(T) = H, \\ R(t)R(t)^{\dagger}B(t)'P(t) - B(t)'P(t) = 0 \ \ \forall t \in [0, T]. \end{cases} \tag{3.9}$$

According to Theorem 3.1, if the above equation admits a solution $P(\cdot)$, then there may be infinitely many optimal controls, and any optimal control has the following form:

(3.10)
$$u_{Y,z}(t) = [-R(t)^{\dagger}B(t)'P(t) + Y(t) - R(t)^{\dagger}R(t)Y(t)]x(t) + z(t) - R(t)^{\dagger}R(t)z(t),$$

where $Y(\cdot) \in L^2(s, T; \mathbf{R}^{n_u \times n})$ and $z(\cdot) \in L^2(s, T; \mathbf{R}^{n_u})$.

4. Necessity of the GRE. In the previous section, we proved that the solvability of GRE (2.5) is *sufficient* for the well-posedness of the stochastic LQ problem (2.1)–(2.3), and that optimal feedback control laws can be constructed based on the solution to the Riccati equation. In particular, if (2.5) admits a solution, then there must be an optimal *linear feedback* control, obtained by taking $Y(t) \equiv 0$ and $z(t) \equiv 0$ in (3.1). In this section we shall show that the solvability of the GRE is also *necessary* for there to exist an optimal *linear feedback* control for the LQ problem.

4.1. A linear feedback control formulation. If a linear feedback control is optimal for the LQ problem (2.1)–(2.3), then it must be optimal also in the class of linear feedback controls of the following form:

$$u(t) = K(t)x(t), \qquad K(t) \in \mathbf{R}^{n_u \times n}. \tag{4.1}$$

The corresponding closed-loop system with the initial $(s, y) = (0, x_0)$ is

$$\begin{cases} dx(t) &= [A(t) + B(t)K(t)]x(t)dt + [C(t) + D(t)K(t)]x(t)dW(t), \\ x(0) &= x_0 \in \mathbf{R}^n. \end{cases} \tag{4.2}$$

Now, if $x(\cdot)$ satisfies (4.2), then by Itô's formula the matrix $X(t) \equiv E[x(t)x(t)']$ satisfies the differential matrix equation

$$\begin{cases} \dot{X}(t) = [A(t) + B(t)K(t)]X(t) + X(t)[A(t) + B(t)K(t)]' \\ \qquad\qquad + [C(t) + D(t)K(t)]X(t)[C(t) + D(t)K(t)]', \\ X(0) = X_0 \equiv E[x_0 x_0'] \in \mathcal{S}_n^+, \end{cases} \tag{4.3}$$

with the associated cost function J expressed equivalently as

$$J(K(\cdot)) \equiv \int_0^T \mathbf{Tr}[(Q(t) + K'(t)R(t)K(t))X(t)]dt + \mathbf{Tr}(HX(T)). \tag{4.4}$$

To summarize, if we consider only the class of linear feedback controls for the original LQ problem with the initial $x(0) = x_0$, then the problem reduces to the following deterministic optimal control problem:

$$\begin{cases} \text{Minimize}_{K(\cdot)} & \displaystyle\int_0^T \mathbf{Tr}[(Q + K'RK)X]dt + \mathbf{Tr}(HX(T)), \\ \text{subject to} & (4.3). \end{cases} \tag{4.5}$$

4.2. Matrix minimum principle. For the reader's convenience, let us state the matrix minimum principle (see [5]) here. We start by defining the gradient matrix. Let $f(\cdot)$ be a function from $\mathbf{R}^{p\times q}$ to $\mathbf{R}$. Then the gradient matrix of f is a $p \times q$ matrix, denoted by $\frac{\partial f(X)}{\partial X}$, with the ijth component $(\frac{\partial f(X)}{\partial X})_{ij} = \frac{\partial f(X)}{\partial x_{ij}}$.

Consider an $n \times n$ matrix differential system

$$\begin{cases} \dot{X} = F(X(t), U(t), t), \\ X(t_0) = X_0, \end{cases} \tag{4.6}$$

where the control $U(\cdot)$ is a measurable mapping from $[t_0, T]$ to a prescribed set $\Omega \subseteq \mathbf{R}^{n_u \times n}$. With T fixed, consider the cost function

$$J(X_0, U(\cdot)) = \int_{t_0}^T L(X(t), U(t), t)dt + G(X(T)), \tag{4.7}$$

where G and L are scalar-valued functions, satisfying the usual smooth conditions. Let $P(t) \in \mathbf{R}^{n\times n}$ be the costate (adjoint) matrix. Then, the Hamiltonian function is defined as

$$H(X(t), P(t), U(t), t) = L(X(t), U(t), t) + \mathbf{Tr}(F(X(t), U(t), t)P(t)'). \tag{4.8}$$

Now, we present the matrix minimum principle in the form stated in [5].

PROPOSITION 4.1. *If $(X^*(\cdot), U^*(\cdot))$ is optimal for (4.6)–(4.7), then there exists a costate matrix $P^*(t)$ satisfying the costate (adjoint) equation*

(4.9)
$$\begin{cases} \dot{P}^*(t) = -\dfrac{\partial}{\partial X^*(t)} L(X^*(t), U^*(t), t) - \dfrac{\partial}{\partial X^*(t)} \mathbf{Tr}(F(X^*(t), U^*(t), t)P^*(t)'), \\ P^*(T) = \dfrac{\partial}{\partial X^*(T)} G(X^*(T)) \end{cases}$$

such that

$$H(X^*(t), P^*(t), U^*(t), t) \le H(X^*(t), P^*(t), U, t) \ \ \forall U \in \Omega, \ \text{ a.e. } t \in [t_0, T]. \tag{4.10}$$

Note that if $U(\cdot)$ is unconstrained, then (4.10) is equivalent to $\frac{\partial H}{\partial U^*(t)} = 0$.

4.3. Necessity of the GRE. We are going to use the matrix minimum principle to show that if there exists an optimal linear feedback control for the original LQ problem (2.1)–(2.3), then GRE (2.5) must have a solution. Furthermore, we will show that any optimal linear feedback control law has the form (3.1) with $z(t) \equiv 0$.

First we give the following necessary condition.

THEOREM 4.2. *Assume that the LQ problem (2.1)–(2.3) is well-posed. For any $s \in [0, T)$, if there exists $P(\cdot)$ such that*

(4.11)
$$\begin{cases} \dot{P} + PA + A'P + C'PC - (PB + C'PD)(R + D'PD)^\dagger(B'P + D'PC) + Q = 0, \\ P(T) = H, \\ (R + D'PD)(R + D'PD)^\dagger(B'P + D'PC) - B'P - D'PC = 0, \ \text{ a.e. } t \in [s, T], \end{cases}$$

then P must satisfy

$$R + D'PD \ge 0, \ \text{ a.e. } t \in [s, T].$$

Proof. Let $\lambda(t)$ be any fixed eigenvalue of the matrix $R(t) + D(t)'P(t)D(t)$, $t \in [s, T]$. We will show that $\mathbf{mes}(\{t \in [s, T] | \lambda(t) < 0\}) = 0$, where $\mathbf{mes}$ denotes the Lebesgue measure. Let $v_\lambda(t)$ be a unit eigenvector (i.e., $v_\lambda(t)'v_\lambda(t) = 1$) associated with the eigenvalue $\lambda(t)$. Define $I_n(\cdot)$ as the indicator function of the set $\{t \in [s, T] | \lambda(t) < -\frac{1}{n}\}$, $n = 1, 2, \dots$. Fix a scalar $\delta \in \mathbf{R}$ and consider the state trajectory $x(\cdot)$ of system (2.1) under the feedback control

(4.12)
$$u(t) = \begin{cases} 0 & \text{if } \lambda(t) = 0, \\ \frac{\delta I_n(t)}{|\lambda(t)|^{1/2}} v_\lambda(t) & \\ - [R(t) + D(t)'P(t)D(t)]^\dagger [B(t)'P(t) + D(t)'P(t)C(t)]x(t) & \text{if } \lambda(t) \neq 0. \end{cases}$$

Clearly, $u(\cdot) \in L^2_{\mathcal{F}}(s, T; \mathbf{R}^{n_u})$. Now,

$$\begin{aligned} J(s, y; u(\cdot)) = y'P(s)y + E\int_s^T &\big[u + (R + D'PD)^\dagger(B'P + D'PC)x\big]'(R + D'PD) \\ &\times \big[u + (R + D'PD)^\dagger(B'P + D'PC)x\big](t)dt. \end{aligned}$$

It follows from $\lambda(t) \neq 0$ that $|\lambda(t)|^{-1} I_n(t)(R(t) + D(t)'P(t)D(t))v_\lambda(t) = -I_n(t)v_\lambda(t)$. Hence

$$
\begin{aligned}
J(s,y;u(\cdot)) &= y'P(s)y - \delta^2 \int_s^T I_n(t)dt \\
&= y'P(s)y - \delta^2 \mathbf{mes}\left(\left\{t \in [s,T] | \lambda(t) < -\frac{1}{n}\right\}\right).
\end{aligned} \tag{4.13}
$$

If $\mathbf{mes}(\{t \in [s,T] | \lambda(t) < -\frac{1}{n}\}) > 0$, then by letting $\delta \to \infty$ in (4.13) we obtain $J(s,y;u(\cdot)) \to -\infty$, which contradicts the well-posedness of the LQ problem. Hence $\mathbf{mes}(\{t \in [s,T] | \lambda(t) < -\frac{1}{n}\}) = 0$. Since

$$
\{t \in [s,T] | \lambda(t) < 0\} = \bigcup_{n \in \mathbf{N}} \left\{t \in [s,T] | \lambda(t) < -\frac{1}{n}\right\},
$$

we conclude that $\mathbf{mes}(\{t \in [s,T] | \lambda(t) < 0\}) = 0$, completing the proof. □

THEOREM 4.3. *If a given linear feedback control $u(t) = K(t)x(t)$ is optimal for the LQ problem* (2.1)–(2.3) *with respect to the initial $(s,y) = (0,x_0)$, then GRE* (2.5) *must have a solution $P(\cdot)$. Moreover, the optimal feedback control $u(t) = K(t)x(t)$ can be represented via* (3.1) *with $z(t) \equiv 0$. In particular, the feedback law $u(t) = K(t)x(t)$ must be optimal with respect to any initial $(s,y) \in [0,T) \times \mathbf{R}^n$.*

Proof. Since the given feedback control $u(t) = K(t)x(t)$ is optimal over the set of all admissible controls, it must in particular be optimal over the class of all linear feedback controls. Therefore, as shown earlier, $K(\cdot)$ must solve the following deterministic optimal control problem:

$$
\left\{
\begin{array}{ll}
\min\limits_{K(\cdot)} & \displaystyle\int_0^T \mathbf{Tr}[(Q + K'RK)X](t)dt + \mathbf{Tr}(HX(T)), \\
\text{subject to} & \left\{ \begin{array}{l} \dot{X}(t) = (A+BK)X + X(A+BK)' + (C+DK)X(C+DK)', \\ X(0) = X_0, \quad X_0 \in \mathcal{S}_n^+. \end{array} \right.
\end{array}
\right. \tag{4.14}
$$

By the minimum principle, Proposition 4.1, we conclude that the Hamiltonian

$$
\mathbf{Tr}\Big((Q + K'RK)X + [(A+BK)X + X(A+BK)' + (C+DK)X(C+DK)']P'\Big)
$$

is pointwise (in t) minimized at $K(t)$ over the space of $\mathbf{R}^{n_u \times n}$. This, together with the costate equation (4.9), leads to the following:

$$
\left\{
\begin{array}{l}
\dot{P} = -Q - K'RK - (C+DK)'P(C+DK) - P(A+BK) - (A+BK)'P, \\
P(T) = H, \\
0 = RKX' + RKX + B'PX' + B'P'X + D'PDKX' + D'PDKX \\
\qquad + D'PCX' + D'P'CX.
\end{array}
\right. \tag{4.15}
$$

Note that in the above calculation we have used the following formulae:

$$
\frac{\partial}{\partial X}\mathbf{Tr}(AX) = A', \quad \frac{\partial}{\partial X}\mathbf{Tr}(AX') = A, \quad \frac{\partial}{\partial X}\mathbf{Tr}(AXBX') = A'XB' + AXB.
$$

Since X and P are symmetric, (4.15) is reduced to

(4.16)
$$\begin{cases} \dot{P} = -Q - K'RK - (C+DK)'P(C+DK) - P(A+BK) - (A+BK)'P, \\ P(T) = H, \\ 0 = (R+D'PD)K + B'P + D'PC. \end{cases}$$

Now, we apply Lemma 2.7 to the equation $(R+D'PD)K + B'P + D'PC = 0$. First of all, we know a priori that it does have a solution K. Thus condition (2.9) must hold, which in the present case specializes to

$$(R+D'PD)(R+D'PD)^{\dagger}(B'P+D'PC) = B'P + D'PC.$$

Moreover, by (2.10), K has the following form:

$$K = -(R+D'PD)^{\dagger}(B'P+D'PC) + Y - (R+D'PD)^{\dagger}(R+D'PD)Y. \tag{4.17}$$

Substituting K into the first equation of (4.16), we can see by a simple calculation that P satisfies

$$\dot{P} + PA + A'P + C'PC - (PB + C'PD)(R+D'PD)^{\dagger}(B'P+D'PC) + Q = 0.$$

Noting that Theorem 4.2 implies that $R + D'PD \geq 0$, we conclude that $P(\cdot)$ solves (2.5). The representation of K is given by (4.17). Finally, the last assertion of the theorem follows from Theorem 3.1. □

5. Open-loop optimal controls. In the previous analysis we have shown that the solvability of the GRE is equivalent to the condition that the LQ problem is solvable by linear feedback controls. In this section we further prove that the solvability of the GRE is also equivalent to the case in which the LQ problem is solvable by continuous *open-loop* controls.

First we need the following lemma.

LEMMA 5.1. *Assume that the LQ problem* (2.1)–(2.3) *is well-posed. Then there exists a symmetric matrix function* $P(\cdot)$ *such that*

$$V(s,y) = y'P(s)y \quad \forall (s,y) \in [0,T] \times \mathbf{R}^n. \tag{5.1}$$

Moreover, assume that $Q(t)$ *and* $R(t)$ *are continuous in* t*, and for any initial* $(s,y) \in [0,T] \times \mathbf{R}^n$ *the LQ problem* (2.1)–(2.3) *has an optimal open-loop control that is continuous in* t*; then the matrix function* $P(\cdot)$ *satisfying* (5.1) *is differentiable on* $[0,T]$.

Proof. First, (5.1) can be shown by a simple adaptation of the well-known result in the deterministic case (see, e.g., [10, 4]). Moreover, since the value function $V(s,y)$ is continuous in s, so is $P(\cdot)$. Next, fix (s,y) and let $(u_*(\cdot), x_*(\cdot))$ be an optimal solution of (2.1)–(2.3) with respect to the initial condition $x(s) = y$ with $u_*(\cdot)$ continuous. Then the dynamic programming optimality principle yields

(5.2)
$$V(s,y) = E\left\{ \int_s^{s+h} [x_*(t)'Q(t)x_*(t) + u_*(t)'R(t)u_*(t)]dt + V(s+h, x_*(s+h)) \right\} \forall h \geq 0.$$

Making use of (5.1)–(5.2), we have

$$\begin{aligned}\tfrac{1}{h}[y'P(s+h)y - y'P(s)y] = {} & \tfrac{1}{h}E[y'P(s+h)y - x_*(s+h)'P(s+h)y] \\ & + \tfrac{1}{h}E[x_*(s+h)'P(s+h)y - x_*(s+h)'P(s+h)x_*(s+h)] \\ & - \tfrac{1}{h}E\int_s^{s+h}[x_*(t)'Q(t)x_*(t) + u_*(t)'R(t)u_*(t)]dt.\end{aligned}$$

Noting that $P(\cdot)$ and $x_*(\cdot)$ are continuous and the integrand above is continuous in t by the assumptions, we can show by a standard argument that the limit of each of the three terms on the right-hand side of the above equation exists as h goes to zero. Therefore $\lim_{h\to 0}\frac{1}{h}[y'P(s+h)y - y'P(s)y]$ exists. Since y is arbitrary, $P(s)$ is differentiable at $s \in [0,T]$. □

The assumption that the optimal control is continuous in t is a rather technical one. From the above proof we can see that only the continuity of the control at the initial time s is actually needed. On the other hand, if we assume that $B(t), C(t), D(t)$, and $R(t)$ are continuous, then by (3.1) the existence of a continuous optimal open-loop control is really *necessary* for the solvability of GRE (2.5).

Consider the following convex set of differentiable symmetric matrix functions on $[0,T]$:

$$\mathcal{P} \triangleq \left\{ P(\cdot) \,\middle|\, \left[\begin{array}{c|c} \dot{P} + A'P + PA + C'PC + Q & PB + C'PD \\ \hline B'P + D'PC & R + D'PD \end{array}\right] \geq 0, \right. \\ \left. \text{a.e. } t\in[0,T],\ P(T) \leq H \right\}. \tag{5.3}$$

The following result provides a sufficient condition for the well-posedness of the LQ problem.

THEOREM 5.2. *The LQ problem* (2.1)–(2.3) *is well-posed if the set* $\mathcal{P}$ *is nonempty.*

Proof. Let $P(\cdot) \in \mathcal{P}$. Applying Lemma 2.4, we have, for any admissible (open-loop) control $u(\cdot)$ and any initial $(s,y) \in [0,T) \times \mathbf{R}^n$,

$$\begin{aligned} J(s,y;u(\cdot)) &= y'P(s)y + E[x(T)(H - P(T))x(T)] \\ &\quad + E\int_s^T \begin{pmatrix} x \\ u \end{pmatrix}' \left[\begin{array}{c|c} \dot{P} + A'P + PA + C'PC + Q & PB + C'PD \\ \hline B'P + D'PC & R + D'PD \end{array}\right] \begin{pmatrix} x \\ u \end{pmatrix}(t)dt. \end{aligned} \tag{5.4}$$

Thus $J(s,y;u(\cdot)) \geq y'P(s)y$, implying $V(s,y) > -\infty\ \ \forall (s,y) \in [0,T) \times \mathbf{R}^n$. □

The following is the main result of this section.

THEOREM 5.3. *Assume that* $B(t), C(t), D(t), Q(t)$, *and* $R(t)$ *are continuous in* t. *Then the LQ problem* (2.1)–(2.3) *has an continuous optimal open-loop control for any initial* $(s,y) \in [0,T] \times \mathbf{R}^n$ *if and only if GRE* (2.5) *has a solution* $P(\cdot)$.

Proof. The "if" part follows from Theorem 3.1. Let us now show the "only if" part. First, since the LQ problem is well-posed, Lemma 5.1 yields that there exists a symmetric matrix function $P(\cdot)$ such that

$$V(s,y) = y'P(s)y \ \ \forall (s,y) \in [0,T) \times \mathbf{R}^n.$$

Moreover, by the assumption and Lemma 5.1, $P(\cdot)$ is differentiable. On the other hand, the dynamic programming principle yields

$$\begin{aligned} V(s,y) \leq \quad & E\left\{ \int_s^{s+h} [x(t)'Q(t)x(t) + u(t)'R(t)u(t)]dt + V(s+h, x(s+h)) \right\} \\ & \forall h \geq 0 \ \ \forall u(\cdot) \in U_{ad}. \end{aligned}$$

Applying Itô's formula to $V(t,x(t)) \equiv x(t)'P(t)x(t)$, using the above inequality, and employing Lemma 2.4, we obtain

$$E\int_s^{s+h} \begin{pmatrix} x \\ u \end{pmatrix}' \left[\begin{array}{c|c} \dot P + A'P + PA + C'PC + Q & PB + C'PD \\ \hline B'P + D'PC & R + D'PD \end{array}\right] \begin{pmatrix} x \\ u \end{pmatrix}(t)dt \geq 0.$$

Taking $u(t) \equiv \bar u \in \mathbf{R}^{n_u}$, and then dividing both sides by h and letting $h \to 0$, we obtain

$$\begin{pmatrix} y \\ \bar u \end{pmatrix}' \left[\begin{array}{c|c} \dot P + A'P + PA + C'PC + Q & PB + C'PD \\ \hline B'P + D'PC & R + D'PD \end{array}\right](s) \begin{pmatrix} y \\ \bar u \end{pmatrix} \geq 0, \quad \text{a.e. } s \in [0,T].$$

Since $y \in \mathbf{R}^n$ and $\bar u \in \mathbf{R}^{n_u}$ are arbitrary, we obtain

$$\left[\begin{array}{c|c} \dot P + A'P + PA + C'PC + Q & PB + C'PD \\ \hline B'P + D'PC & R + D'PD \end{array}\right] \geq 0, \quad \text{a.e. } t \in [0,T]. \tag{5.5}$$

Applying Lemma 2.6 to (5.5), we have

(5.6)
$$\begin{cases} \dot P + PA + A'P + C'PC - (PB + C'PD)(R + D'PD)^\dagger(B'P + D'PC) + Q \geq 0, \\ (R + D'PD)(R + D'PD)^\dagger(B'P + D'PC) - B'P - D'PC = 0, \\ R + D'PD \geq 0, \quad \text{a.e. } t \in [0,T]. \end{cases}$$

Now, let $(x_*(\cdot), u_*(\cdot))$ be an optimal open-loop control for (2.1)–(2.3) with respect to the initial condition $x(s) = y$. Applying Lemma 2.4 to $P(\cdot)$, we have

$$\begin{aligned} V(s,y) = y'P(s)y \;&+ E\int_s^T [x_*'\big(\dot P + PA + A'P + C'PC + Q \\ &\qquad - (PB + C'PD)(R + D'PD)^\dagger(B'P + D'PC)\big)x_*](t)dt \\ &+ E\int_s^T [u_* + (R + D'PD)^\dagger(B'P + D'PC)x_*]' \\ &\qquad \times (R + D'PD)[u_* + (R + D'PD)^\dagger(B'P + D'PC)x_*](t)dt. \end{aligned} \tag{5.7}$$

By virtue of the relation $V(s,y) = y'P(s)y$ and (5.6)–(5.7), we obtain

$$\dot P + PA + A'P + C'PC - (PB + C'PD)(R + D'PD)^\dagger(B'P + D'PC) + Q = 0.$$

This completes the proof. □

Theorem 5.1 says that the nonemptiness of the set $\mathcal{P}$ is sufficient for the well-posedness of the original LQ problem. The following result stipulates that the nonemptiness of the set $\mathcal{P}$ is also *necessary* for the attainability of the LQ problem.

THEOREM 5.4. *Under the same assumption of Theorem* 5.3, *the LQ problem* (2.1)–(2.3) *has a continuous optimal open-loop control for any initial* $(s,y) \in [0,T] \times \mathbf{R}^n$ *only if the set* $\mathcal{P}$ *is nonempty.*

Proof. This is seen from (5.5). □

6. An example. In this section we give an example in which the singularity of $R + D'PD$ does occur, but the LQ problem is well-posed and attainable. Moreover, the example shows that a stochastic LQ problem can be well-posed even when *both* Q and R are negative.

Consider the following one-dimensional LQ problem:

$$\text{(6.1)}\qquad \begin{array}{ll} \text{Minimize} & J = E\left\{\int_0^1 [qx(t)^2 + r(t)u(t)^2]dt + hx(1)^2\right\}, \\ \text{subject to} & \left\{\begin{array}{l} dx(t) = [ax(t) + bu(t)]dt + [cx(t) + \delta u(t)]dW(t), \\ x(0) = x_0, \end{array}\right. \end{array}$$

where the coefficients are chosen such that $\delta \neq 0$, $b+\delta c = 0$, $q < 0$, and $2a+c^2+q > 0$. Take $r(t) = -\delta^2 p(t)$, where

$$\text{(6.2)}\qquad p(t) = \frac{e^{(2a+c^2)(1-t)}(2ha + hc^2 + q) - q}{2a + c^2}$$

is the solution to the following equation:

$$\text{(6.3)}\qquad \dot{p}(t) + (2a + c^2)p(t) + q = 0, \qquad p(1) = h.$$

Incidentally, Example 1.1 in section 1 is a special case of this example. It is easy to verify directly that (6.3) is exactly the GRE in the present case. (Note that the singularity arises because $r(t)+\delta^2 p(t) \equiv 0$.) Therefore, by Theorem 3.1 and Corollary 3.1(i), the LQ problem is well-posed, and any admissible control is optimal with an optimal cost $p(s)y^2$.

It is interesting to look at the sign of the solution to the Riccati equation (6.3). First assume that $h < 0$. In this case, since $2a + c^2 > 0$ and $q < 0$, we have from (6.2) that $p(t) \leq h < 0$ $\forall t \in [0, 1]$. Hence, the solution to the Riccati equation could be *negative*, which is quite contrary to the deterministic LQ case. On the other hand, if $h > 0$ is large enough so that $2ha + hc^2 + q > 0$, then $p(t) \geq h > 0$ and $r(t) = -\delta^2 p(t) < 0$. In this case, both q and $r(t)$ are *negative* but the LQ problem is well-posed. Again, this is different from the deterministic situation. The essential reason behind this phenomenon is that the positive terminal cost $hx(1)^2$ *outweighs* the negative running cost.

7. Conclusion. Standard LQ theory, which has proved so useful for control applications in the last decades, has been extended here to signal models with multiplicative noises in both state and control and with quadratic weights that are fundamentally different from those in the literature. Such models better approximate nonlinear stochastic systems and arise naturally in areas of current interest such as in finance. A new Riccati equation is introduced in this paper as an appropriate vehicle for identifying optimal controls and calculating the optimal cost value.

Other properties concerning the existence, uniqueness, and asymptotic behavior of solutions to the GRE associated with an indefinite LQ problem are studied in a companion paper [1], which complements results derived in this paper.

REFERENCES

[1] M. Ait Rami, X. Chen, J. B. Moore, and X. Y. Zhou, *Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls*, IEEE Trans. Automat. Control, 46 (2001), pp. 428–440.

[2] M. Ait Rami and X. Y. Zhou, *Linear matrix inequalities, Riccati equations, and indefinite stochastic linear quadratic control*, IEEE Trans. Automat. Control, 45 (2000), pp. 1131–1143.

[3] A. Albert, *Conditions for positive and nonnegative definiteness in terms of pseudoinverses*, SIAM J. Appl. Math., 17 (1969), pp. 434–440.

[4] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*, Prentice–Hall, Englewood Cliffs, NJ, 1989.

[5] M. Athans, *The matrix minimum principle*, Inform. and Control, 11 (1968), pp. 592–606.

[6] M. Athens, *Special issues on linear–quadratic–Gaussian problem*, IEEE Trans. Automat. Control, 16 (1971), pp. 527–869.

[7] S. Chen, X. Li, and X. Y. Zhou, *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM J. Control Optim., 36 (1998), pp. 1685–1702.

[8] S. Chen and X. Y. Zhou, *Stochastic linear quadratic regulators with indefinite control weight costs*. II, SIAM J. Control Optim., 39 (2000), pp. 1065–1081.

[9] S. Chen and J. Yong, *Stochastic linear quadratic optimal control problems*, Appl. Math. Optim., 43 (2001), pp. 21–45.

[10] D. Clements, B. D. O. Anderson, and P. J. Moylan, *Matrix inequality solution to linear-quadratic singular control problems*, IEEE Trans. Automat. Control, 22 (1977), pp. 55–57.

[11] M. H. A. Davis, *Linear Estimation and Stochastic Control*, Chapman and Hall, London, 1977.

[12] D. Hinrichsen and A. J Pritchard, *Stochastic* H^{∞}, SIAM J. Control Optim., 36 (1998), pp. 1504–1538.

[13] D. H. Jacobson, *Totally singular quadratic minimization problems*, IEEE Trans. Automat. Control, 16 (1971), pp. 651–658.

[14] R. E. Kalman, *Contribution to the theory of optimal control*, Bol. Soc. Mat. Mexican, 5 (1960), pp. 102–119.

[15] M. Kohlmann and X. Y. Zhou, *Relationship between backward stochastic differential equations and stochastic controls: A linear quadratic approach*, SIAM J. Control Optim., 38 (2000), pp. 1392–1407.

[16] A. E. B. Lim and X. Y. Zhou, *Stochastic optimal LQR control with integral quadratic constraints and indefinite control weights*, IEEE Trans. Automat. Control, 44 (1999), pp. 359–369.

[17] A. E. B. Lim and X. Y. Zhou, *Mean–Variance Portfolio Selection with Random Parameters*, preprint.

[18] B. P. Molinari, *Nonnegativity of a quadratic functional*, SIAM J. Control, 13 (1975), pp. 792–806.

[19] R. Penrose, *A generalized inverse of matrices*, Proc. Cambridge Philos. Soc., 52 (1955), pp. 17–19.

[20] V. M. Popov, *Hyperstability and optimality of automatic systems with several control functions*, Rev. Roumaine Sci. Tech. Sér. Electrotech. Energ., 9 (1964), pp. 629–690.

[21] L. Vandenerghe and S. Boyd, *Semidefinite programming*, SIAM Rev., 38 (1996), pp. 49–95.

[22] J. C. Willems, *Least squares stationary control and the algebraic Riccati equation*, IEEE Trans. Automat. Control, 16 (1971), pp. 621–234.

[23] W. M. Wonham, *On the separation theorem of stochastic control*, SIAM J. Control, 6 (1968), pp. 312–326.

[24] D. D. Yao, S. Zhang, and X. Y. Zhou, *Stochastic linear quadratic control via semidefinite programming*, SIAM J. Control Optim., 40 (2001), pp. 801–823.

[25] X. Y. Zhou and D. Li, *Continuous-time mean-variance portfolio selection: A stochastic LQ framework*, Appl. Math. Optim., 42 (2000), pp. 19–33.