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INTERLEAVED PULSE TRAIN SPECTRUM ESTIMATION

Robert J. Orsi *, *John B. Moore* * *and Robert E. Mahony* †

* Dept. of Systems Engineering, RSISE, Australian National University, Canberra ACT 0200, Australia.
† CRC for Robust and Adaptive Systems, Australian National University, Canberra ACT 0200, Australia.

ABSTRACT

In this paper we consider signals consisting of a finite though unknown number of periodic time-interleaved pulse trains. For such signals, we present a novel approach for determining both the number of pulse trains present and the frequency of each pulse train. Our approach requires only the time of arrival data of each pulse. It is robust to noisy time of arrival data and missing pulses, and above all is very computationally efficient. If N is the number of pulses being processed, the computation required is of the order of $N \log N$.

1. INTRODUCTION

A periodic pulse train consists of a sequence of periodically spaced pulses. Often a single channel receiver will receive periodic pulse trains from a number of sources simultaneously. The superposition of all the received pulse trains is known as an *interleaved pulse train*. The process of determining the number of pulse trains present in this signal and associating each received pulse with a source is termed *pulse train deinterleaving*. This process relies on the assumption that the different pulse train sources have different characteristics such as period of pulse emission. One application of pulse train deinterleaving is in radar detection [1]. Potential applications include computer communications and neural systems.

Typical approaches to pulse train deinterleaving are sequential search [2] and histogramming [2, 3]. A practical disadvantage of these algorithms is the computational effort they require. If N is the number of pulses being processed, computations are of the order of N^2 [4].

A recent novel approach to pulse train deinterleaving is given in [5] where the problem is first formulated as a stochastic discrete-time dynamic linear model. Like sequential search and histogramming, this method is quite computationally expensive.

In this paper, rather than trying to deinterleave a received interleaved pulse train, we focus solely on determining the number of pulse trains present and the frequency of each pulse train. We term this information the *interleaved pulse train spectrum*. We present a novel approach for estimating the spectrum of a signal consisting of a finite though unknown number of periodic time-interleaved pulse trains. Only the time of arrival data of each pulse is used. No knowledge of any other pulse train characteristics such pulse energy are required nor is prior knowledge of the transmitter characteristics required. An advantage of our scheme is that, if N is the number of pulses being processed, computations are of the order $N \log N$. Note that once the interleaved pulse train spectrum is known it is a relatively easy task to deinterleave the received signal using standard methods such as those used after histogramming [2].

The paper is structured as follows. A problem formulation is first presented followed by an overview of the proposed scheme. The remainder of the paper then discusses aspects of the approach in greater depth. Firstly an analysis of a special class of non-generic pulse train sequences that satisfy various simplifying assumptions is undertaken. Using insight gained from this non-generic case, some simulation results for the generic case are presented and discussed. The paper ends with some further discussion and concluding remarks.

2. PROBLEM FORMULATION AND APPROACH

Consider M periodic pulse train sources. Let T_i, f_i and ϕ_i denote respectively the period, frequency and phase of the ith source. The received interleaved signal consists of the superposition of the M pulse trains produced by these sources. Let $t_0, t_1, \ldots, t_N$ denote the times of arrival of $N+1$ consecutive pulses in this signal nominally setting $t_0 := 0$. The problem is as follows:

Problem: *Given $t_0, \ldots, t_N$, determine both the number of pulse trains present and the frequency of each pulse train.*

The first step in the proposed scheme is to calculate

$$x(n) := e^{j\frac{2\pi}{t_{max}}t_n} \qquad \text{for } n = 0, \ldots, N-1, \tag{1}$$

where $t_{max} := t_N$. The signal $x(n)$ can be thought of as taking the interval $[t_0, t_{N-1}]$, containing the first N pulse times of arrival, normalising its length to approximately 2π and then wrapping this normalised interval around the unit circle. Note that as mentioned before $t_0 = 0$.

The next step is to take the N-length discrete Fourier transform (DFT) of (1). The magnitude of the transformed signal contains the information necessary to both determine how many pulse trains are present and to make a good estimate of their frequencies. (The phase response seems to

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contain little information.) Redundant information within the spectrum can be used to improve confidence of results.

3. A NON-GENERIC SPECIAL CASE

In this section the proposed scheme is analysed for a class of non-generic pulse trains that satisfy various simplifying assumptions. The analysis of the scheme under these assumptions provides insight into behaviour in the generic case.

It is assumed that the pulse trains considered in this section satisfy the follow properties:

(P1) The period of each pulse train is rational, that is, $T_i \in \mathbb{Q}$, $i = 1, \ldots, M$.

(P2) The phase of each pulse train is zero, that is, $\phi_i = 0$, $i = 1, \ldots, M$.

The above properties imply that if the received signal contains a sufficiently large number of pulses, it will be periodic. It is assumed that a sufficiently large number of pulses have been received such that this is the case. The overall signal period will be denoted by T.

In addition to properties (P1) and (P2), the following assumption is also made:

(P3) The received signal consists of exactly an integer number of overall signal periods.

Let r_i denote the number of pulses from pulse train i appearing in one period of the received signal. Then

$$T = T_1 r_1 = \ldots = T_M r_M \tag{2}$$

and the total number of pulses in one period of the received signal is

$$N_T := \sum_{i=1}^{M} r_i.$$

Remark 1 It is assumed that the pulse time of arrival data, $t_0, \ldots, t_N$, is noise free and that there are no missing pulses. □

The prior assumptions imply that $N/N_T \in \mathbb{Z}$, and that

$$t_{max} = \frac{N}{N_T} T. \tag{3}$$

Theorem 2 *Consider a signal consisting of M interleaved pulse trains satisfying properties (P1), (P2) and (P3). Let $x(n)$ be defined as in (1) and let $X(k)$, $k = 0, \ldots, N-1$, denote its discrete Fourier transform. Then, defining $k' = k - 1$,*

$$X(k') = \begin{cases} \frac{N}{N_T} \sum_{l=0}^{N_T-1} e^{j\frac{2\pi}{N}\left(\frac{N_T}{T} t_l - k' l\right)}, \\ \qquad \text{if} \quad k' = \frac{pN}{N_T},\ p = 0, \ldots, N_T - 1, \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Furthermore, for $p = r_j$, $\frac{k'}{t_{max}}$ equals f_j, the frequency of the jth pulse train. □

PROOF. By definition

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) e^{-jk\left(\frac{2\pi}{N}\right)n} \\ &= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{t_{max}} t_n} e^{-jk\left(\frac{2\pi}{N}\right)n}. \end{aligned}$$

Replacing n by $mN_T + l$ and noting that (P2) implies that $t_n = t_{mN_T+l} = mT + t_l$,

$$\begin{aligned} &X(k) \\ &= \sum_{m=0}^{\frac{N}{N_T}-1} \sum_{l=0}^{N_T-1} e^{j\frac{2\pi}{t_{max}}(mT+t_l)} e^{-jk\left(\frac{2\pi}{N}\right)(mN_T+l)} \\ &= \sum_{m=0}^{\frac{N}{N_T}-1} \sum_{l=0}^{N_T-1} e^{j2\pi\frac{N_T}{NT}(mT+t_l)} e^{-jk\left(\frac{2\pi}{N}\right)(mN_T+l)} \\ &\qquad \text{by (3)} \\ &= \sum_{m=0}^{\frac{N}{N_T}-1} \sum_{l=0}^{N_T-1} e^{j\frac{2\pi}{N}\left((1-k)N_T m + \frac{N_T}{T} t_l - kl\right)} \\ &= \sum_{l=0}^{N_T-1} \left[\sum_{m=0}^{\frac{N}{N_T}-1} \left(e^{j\frac{2\pi}{N}(1-k)N_T}\right)^m \right] e^{j\frac{2\pi}{N}\left(\frac{N_T}{T} t_l - kl\right)}. \end{aligned}$$

Consider the summation in square brackets in the line above. Letting

$$z := e^{j\frac{2\pi}{N}(1-k)N_T},$$

$$\begin{aligned} \sum_{m=0}^{\frac{N}{N_T}-1} \left(e^{j\frac{2\pi}{N}(1-k)N_T}\right)^m &= \sum_{m=0}^{\frac{N}{N_T}-1} z^m \\ &= \begin{cases} \frac{z^{\frac{N}{N_T}}-1}{z-1}, & \text{if } z \neq 1, \\ \frac{N}{N_T}, & \text{if } z = 1. \end{cases} \end{aligned}$$

Note that $z^{\frac{N}{N_T}} = 1$ and hence that

$$\sum_{m=0}^{\frac{N}{N_T}-1} \left(e^{j\frac{2\pi}{N}(1-k)N_T}\right)^m = \begin{cases} 0, & \text{if } z \neq 1, \\ \frac{N}{N_T}, & \text{if } z = 1. \end{cases}$$

Also note that

$$\begin{aligned} z = 1 \quad &\Leftrightarrow \quad \frac{1-k}{N} N_T = -p \\ &\qquad \text{where} \quad p \in \mathbb{Z} \quad \text{and} \quad k \in \{0, \ldots, N-1\} \\ &\Leftrightarrow \quad k - 1 = \frac{pN}{N_T}, \qquad p = 0, \ldots, N_T - 1. \end{aligned}$$

Note that k above is indeed always an integer as $N/N_T \in \mathbb{Z}$.

Replacing $k-1$ with k', the DFT of $x(n)$ can now be seen to be the expression given in the theorem statement. Furthermore by (3),

$$\frac{k'}{t_{max}} = \frac{pN/N_T}{NT/N_T} = \frac{p}{T},$$

and for $p = r_j$, (2) implies that

$$\frac{k'}{t_{max}} = \frac{r_j}{T_j r_j} = \frac{1}{T_j} = f_j.$$

■

Theorem 2 shows that the N-length DFT of $x(n)$ is non-zero at at most N_T points. Furthermore, M of these possibly non-zero points correspond to the M pulse train frequencies. Additionally, if f is a pulse train frequency, the theorem predicts the existence of harmonics at $2f, 3f, \ldots$.

3.1. The Non-Generic Case: A Simulation Example

The proposed methodology was applied to a signal satisfying properties (P1), (P2) and (P3). The signal consisted of $M = 3$ interleaved pulse trains with respective frequencies $f_1 = 0.25$Hz, $f_2 = 0.75$Hz and $f_3 = 0.8$Hz. The magnitude of the signal produced by the DFT is shown in Figure 1.

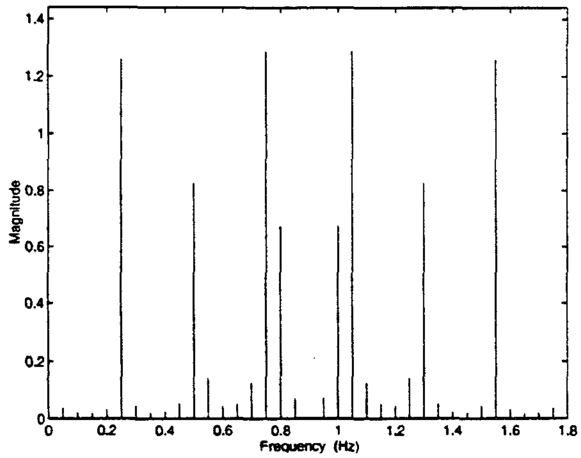

Figure 1: A non-generic magnitude plot.

As predicted the magnitude plot contains only a small number of non-zero values uniformly spaced in frequency. Notice that three of the largest values correspond to pulse train frequencies. Notice also that there are other large values that do not correspond to pulse trains.

As the simulation shows, the M largest values produced do not necessarily correspond to the M pulse trains. Simulations indicate, however, both for non-generic and generic cases, that the largest value, or one of the equal largest values if such is the case, corresponds to a pulse train. Having identified a pulse train frequency, standard methods [2] can be used to deinterleave the corresponding pulse train. By deinterleave we mean that all pulses in the received interleaved signal that are members of the identified pulse train can be removed. This produces a new interleaved signal with one less pulse train present than the original. Our scheme can then be applied to this new signal and another pulse train can be identified and deinterleaved. This process can be repeated until all pulse trains are identified.

Though not apparent from Figure 1, the DFT magnitude at 0Hz is very large, being approximately equal to N. This term is an artifact of the processing method and is ignored in the spectrum analysis.

4. THE GENERIC CASE

In this section a generic case simulation is presented and discussed.

The simulated signal consists of ten interleaved pulse trains. Each pulse train has an arbitrary frequency and a random phase. $N = 2^{12} = 4096$. The output produced by applying our approach is shown in Figure 2. Though a little hard to discern from the plot, the ten largest magnitudes in the spectrum correspond to the ten pulse trains. (As in the non-generic case, the spectrum contains a large term at 0Hz which is ignored.)

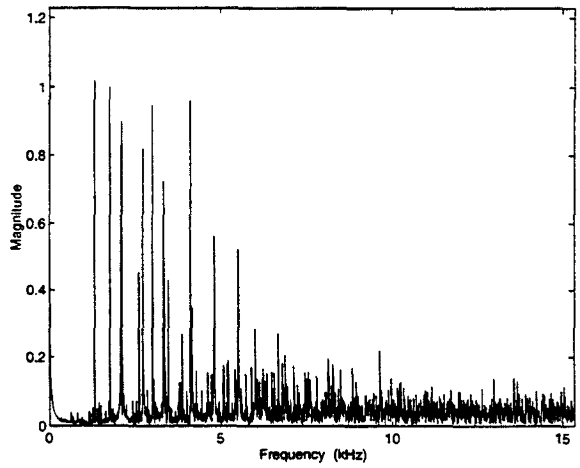

Figure 2: A generic magnitude plot.

If the original signal consists of M interleaved pulse trains, the number of pulses processed, N, will approximately be equal to $t_{max}(f_1+\ldots+f_M)$. The number of pulse trains present is determined by assuming the m largest magnitudes correspond to pulse trains. Starting with $m=1$, m is incremented until $t_{max}(f_1+\ldots+f_m)$ is approximately equal to N. If no such m can be found it means the M largest magnitudes in the spectrum do not correspond to the M pulse trains present. In this case the original interleaved signal can be re-processed in the manner discussed in Section 3.1, that is by identifying one pulse train at a time, deinterleaving this pulse train, and repeating the process.

Actual versus estimated pulse train frequencies for our example are presented in Table 1.

PT No.	Actual Freq. (kHz)	Estimated Freq. (kHz)
1	1.2980	1.2981
2	1.7400	1.7409
3	2.0658	2.0635
4	2.0944	2.0935
5	2.7183	2.7164
6	3.0000	3.0015
7	3.3416	3.3392
8	4.1416	4.1421
9	4.8200	4.8174
10	5.5100	5.5077

Table 1: Actual vs. estimated pulse train frequencies.

As mentioned previously, at least for the non-generic case, if f is a pulse train frequency, Theorem 2 predicts the existence of harmonics at $2f, 3f, \ldots$. Such harmonics also appear in the generic case. Additional processing of the spectrum based on removing the harmonics and adding their magnitudes to the magnitude of the pulse train to which they correspond can reduce much of the noise present. The result of such additional processing for our example is shown in Figure 3.

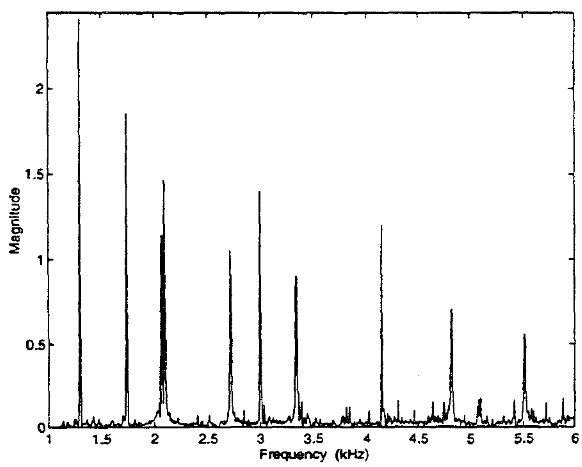

Figure 3: Magnitude plot after additional processing.

5. FURTHER DISCUSSION AND CONCLUDING REMARKS

The proposed scheme is computationally efficient. Other typical deinterleaving methods such as sequential search [2] and histogramming [2, 3] require order N^2 computations [4]. By choosing appropriate data lengths, the proposed scheme can employ the fast Fourier transform. Hence computations for this scheme are of the order of $N \log N$.

The proposed methodology is also quite robust to noise. Though in the example in Section 4 no jitter was added to the pulse times of arrival nor where there any missing pulses, other simulations have shown that the performance of the proposed scheme degrades gracefully as jitter is introduced and increased, and as the percentage of missing pulses increases.

Simulations also indicate that magnitudes corresponding to lower frequency pulse trains tend to be larger than the magnitudes of pulse trains with comparatively higher frequencies. Figures 2 and 3 give some indication of this behaviour. In fact, if the ratio of largest to smallest pulse train frequencies present in an interleaved signal is too large the spectrum magnitudes corresponding to the high frequency pulse trains become submerged in noise. How large this ratio can be is dependent on N and its size increases as N is increased. If the proposed method is having trouble detecting high frequency pulse trains, one option to try to improve detection would be to increase N. Note that increasing N also increases accuracy of frequency estimation.

Commonly used algorithms such as sequential search also suffer significant degradation of performance when the ratio of pulse train frequencies becomes too large. Since these existing methods identify high frequency pulse trains most effectively it is believed the proposed scheme could be used to compliment an existing algorithm for deinterleaving signals with pulse train frequency ratios exceeding these levels.

6. REFERENCES

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