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Discrete time LQG controls with control dependent noise

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Abstract

This paper presents some studies on partially observed linear quadratic Gaussian (LQG) models where the stochastic disturbances depend on both the states and the controls, and the measurements are bilinear in the noise and the states/controls. While the Separation Theorem of standard LQG design does not apply, suboptimal linear state estimate feedback controllers are derived based on certain linearizations. The controllers are useful for nonlinear stochastic systems where the linearized models include terms bilinear in the noise and states/controls and are significantly more accurate than if the bilinear terms are set to zero. The controllers are calculated by solving a generalized discrete time Riccati equation, which in turn has properties relating to well posedness of the associated LQG problem.

1. Introduction

The classical linear quadratic Gaussian (LQG) control theory for stochastic linear systems assumes that the stochastic disturbances are additive and not control or state dependent [1–4]. Relaxing this assumption to allow state and control dependence in the noise terms leads to a broad class of stochastic models, which have applications for real-world control. For example, in a stock market the investments (controls) made by so-called “large investors” are going to affect fluctuations (disturbances) of the market. Working with models involving a bilinear noise dependence allows an improved approximation of the underlying nonlinear stochastic system.

Recently, linear quadratic regulator (LQR) theory has been generalized for a class of linear/bilinear stochastic systems in continuous time [3, 6]. The associated optimal state feedback control laws are linear, being calculated by solving a so-called *stochastic* Riccati equation which specializes to the familiar conventional Riccati equation when the disturbances are independent of the states and controls. The stochastic Riccati equations are by no means as well understood as in the standard case, at least in the continuous time setting. There remains open questions concerning existence and uniqueness of the solutions of these equations. There is also an intriguing property that the control weighting matrix R in a standard quadratic integral cost term need not be positive definite, even in the continuous time case.

What is the situation then for the partially observed case? To what extent does the standard LQG methodology [1] with its Separation Theorem apply? Can we achieve useful *linear* state estimate feedback laws?

In this paper the above questions are addressed for the discrete time case and some initial results are presented. The expectation is that since the

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models are bilinear in the state and the noise, as well as in the control and the noise, some of the virtues of the standard linear Gaussian theory will be lost. Certainly, even if the noise signals are Gaussian, the states and control signals will in general be non-Gaussian. Consequently, optimal (information) state estimators will be infinite dimensional, in general; see for example [5]. Even so, since a conditional *linear* minimum square error (LMSE) covariance state estimator is known for the models of interest, and is finite dimensional, it makes sense from an implementation point of view to work with such a state estimator and the resulting linear state estimate feedback law, even if such a law is suboptimal.

The conditional LMSE filter has the structure of a Kalman filter, see [1], but with a Kalman gain which is state estimate and control dependent. Likewise, the quadratic state cost when expressed in terms of state estimates instead of true states is nonlinear. Appropriate linearizations of the filter equations and cost terms, neglecting higher order terms but allowing terms bilinear in the noise and controls/state estimates in the filter, allows application of a discrete-time analogy of the recently studied LQR theory in [3]. This leads to an 'optimal' linear state estimate feedback law under assumptions of negligible higher order terms. In practise, this law has some degree of sub-optimality because the neglected higher-order terms may be significant. However, the neglected terms do not include terms bilinear in the innovations (prediction errors) and the state estimates/controls, so there is a chance for improved performance over the standard LQG approach which neglects these terms as well as higher-order terms.

The paper is organized as follows. In Section 2 an optimal feedback controller is derived for a completely observed discrete time, linear quadratic regulators (LQR) with state- and control-dependent noise. As in the standard case, solving a discrete time Riccati equation is a key step in calculating the optimal controller. In fact, the associated Riccati equation is a generalization of the standard discrete time Riccati equation. The existence properties of this equation and its relationship to the well posedness of the control problem is discussed in Section 3. Section 4 is concerned with an approximate Kalman filter for the partially observed LQG model. Finally, suboptimal linear state estimate feedback laws are obtained in Section 5 by combining the results in Sections 2 and 4.

2. Discrete time LQR results

In this section, we derive parallel results to those of [3], but in discrete time rather than continuous time. These will be useful in a later section. The results in this section are also of interest on their own right, as discrete time algorithms are useful in practice.

Consider the discrete time stochastic signal model

$$x_{k+1} = (A_k + w_k^A \, \Delta A_k)x_k + (B_k + w_k^B \, \Delta B_k)u_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control, and $w_k^A, w_k^B \in \mathbb{R}$ are noise terms, assumed here to be martingale increments on $\mathcal{G}_{k-1}$, where $\mathcal{G}_{k-1}$ is the σ-algebra generated by past noise terms up to $w_{k-1}^A, w_{k-1}^B, w_{k-1}$. Thus x_k is measurable with respect to $\mathcal{G}_k$ and

$$E[w_{k-1}^A|\mathcal{G}_{k-1}] = E[w_{k-1}^B|\mathcal{G}_{k-1}] = E[w_{k-1}|\mathcal{G}_{k-1}] = 0.$$

The covariances are assumed to be

$$E[(w_{k-1}^A)^2|\mathcal{G}_{k-1}] = E[(w_{k-1}^B)^2|\mathcal{G}_{k-1}] = 1,$$
$$E[w_k w_k'|\mathcal{G}_{k-1}] = Q_k$$

and

$$E[w_{k-1}^A w_{k-1}^B|\mathcal{G}_{k-1}] = \rho_k^{AB},$$
$$E[w_{k-1} w_{k-1}^A|\mathcal{G}_{k-1}] = \rho_k^A,$$
$$E[w_{k-1} w_{k-1}^B|\mathcal{G}_{k-1}] = \rho_k^B.$$

Generalizations of the dependent noise terms $w_k^A \, \Delta A_k$ and $w_k^B \, \Delta B_k$ to the case of non-scalar noise is immediate by working with terms $\sum_{i=1}^N w_k^{A^i} \, \Delta A_k^i$ and $\sum_{i=1}^N w_k^{B^i} \, \Delta B_k^i$.

It should be noted that the time-varying versions of Eq. (1) result from linearizations of nonlinear stochastic models of the form $x_{k+1} = f(x_k, u_k, w_k)$. Linearizations which set the bilinear terms in the noise to zero result in the standard stochastic models.

The performance index of the problem is given by the standard quadratic sum cost

$$J_T = E\left\{ \sum_{k=0}^{T-1} (x_k' Q_k^c x_k + u_k' R_{k+1}^c u_k) + x_T' Q_T^c x_T - x_0' Q_0^c x_0 \right\}. \tag{2}$$

In this model, all the $A_k, \Delta A_k$, etc. are (deterministic) matrices with appropriate dimensions, Q_k^c and Q are non-negative definite matrices, and R_k^c are symmetric matrices (could be indefinite, as in standard discrete time LQR theory).

Let us solve the above stochastic optimal control problem in two different cases. The results derived below will be applied in Section 4 for partially observed models.

Case I: $\rho_k^A = \rho_k^B = 0$.

Let us consider first the case when $w_k \perp w_k^A, w_k^B$, so that $\rho_k^A = \rho_k^B = 0$. In this case, we are going to show that the optimal control takes the form

$$u_k = K_k^c x_k, \tag{3}$$

where

$$K_k^c = -(\Omega_{k+1}^c)^{-1} L_{k+1}^c,$$

$$L_{k+1}^c = B_k' S_{k+1} A_k + \rho_k^{AB}\, \Delta B_k' S_{k+1}\, \Delta A_k,$$

$$\Omega_{k+1}^c = B_k' S_{k+1} B_k + \Delta B_k' S_{k+1}\, \Delta B_k + R_{k+1}^c.$$

Here, S_k is the solution of a backward matrix Riccati equation

$$\begin{aligned} S_k &= A_k' S_{k+1} A_k - L_{k+1}^{c\prime} (\Omega_{k+1}^c)^{-1} L_{k+1}^c \\ &\quad + (Q_k^c + \Delta A_k' S_{k+1} \Delta A_k), \qquad (4) \\ S_T &= Q_T^c. \end{aligned}$$

In fact, assuming the existence of the solution S_k of Eq. (4), the control law (3)–(4) is seen to be optimal by completion of the square arguments as follows. First, note that

$$\sum_{k=0}^{T-1} (x_k' S_k x_k - x_{k+1}' S_{k+1} x_{k+1}) = x_0' S_0 x_0 - x_T' S_T x_T, \tag{5}$$

and

$$\begin{aligned} &(u_k - K_k^c x_k)' \Omega_{k+1}^c (u_k - K_k^c x_k) \\ &\quad = u_k' \Omega_{k+1}^c u_k + 2x_k' L_{k+1}^{c\prime} u_k \\ &\qquad + x_k' L_{k+1}^{c\prime} (\Omega_{k+1}^c)^{-1} L_{k+1}^c x_k \qquad (6) \\ &\quad = u_k' (R_{k+1}^c + B_k' S_{k+1} B_k + \Delta B_k' S_{k+1} \Delta B_k) u_k \\ &\qquad + 2x_k' (A_k' S_{k+1} B_k + \rho_k^{AB} \Delta A_k' S_{k+1} \Delta B_k) u_k \\ &\qquad + x_k' L_{k+1}^{c\prime} (\Omega_{k+1}^c)^{-1} L_{k+1}^c x_k. \end{aligned}$$

Hence J_T can be re-organized by using Eqs. (4)–(6) and eliminating Q_k^c, R_{k+1}^c to yield

$$\begin{aligned} J_T &= E\left\{ \sum_{k=0}^{T-1} (u_k - K_k^c x_k)' \Omega_{k+1}^c (u_k - K_k^c x_k) \right. \\ &\qquad \left. + x_0' (S_0 - Q_0^c) x_0 \right\} \\ &\quad + E\left\{ \sum_{k=0}^{T-1} [x_{k+1}' S_{k+1} x_{k+1} - x_k' A_k' S_{k+1} A_k x_k \right. \qquad (7) \\ &\qquad - x_k' \Delta A_k' S_{k+1} \Delta A_k x_k - 2x_k' L_{k+1}^{c\prime} u_k \\ &\qquad \left. - u_k' (B_k' S_{k+1} B_k + \Delta B_k' S_{k+1} \Delta B_k) u_k] \right\}. \end{aligned}$$

Substituting for x_{k+1} from Eq. (1), and from L_{k+1}^c, the third term simplifies as $\sum_{k=0}^{T-1} \mathrm{tr}(S_{k+1} Q_k)$. That is,

$$\begin{aligned} J_T &= E\left\{ \sum_{k=0}^{T-1} (u_k - K_k^c x_k)' \Omega_{k+1}^c (u_k - K_k^c x_k) \right\} \\ &\quad + \sum_{k=0}^{T-1} \mathrm{tr}(S_{k+1} Q_k) + x_0' (S_0 - Q_0^c) x_0. \end{aligned} \tag{8}$$

Therefore Eqs. (1) and (2) are well posed if Ω_{k+1}^c is positive definite. In this case, the control law (2)–(4) is the unique optimal control which achieves a minimum cost

$$J_T(\min) = \sum_{k=0}^{T-1} \mathrm{tr}(S_{k+1} Q_k) + x_0' (S_0 - Q_0^c) x_0. \tag{9}$$

However, we have assumed that Ω_{k+1}^c is strictly positive definite for every k. In fact, by defining K_k^c as a solution of the equation $\Omega_{K+1}^c K_k^c = L_{k+1}^c$, and using the concept of pseudo-inverses, the above results can be shown to hold for the case when Ω_{k+1}^c is positive semi-definite. That is, the LQR problem (1)–(2) is well posed if and only if Ω_{k+1}^c is positive semi-definite. In Section 3, we shall address this issue of existence of solutions in more detail. In particular, we shall examine the effect of the bilinear terms on the solution of the Riccati equation (4), and the well posedness of Eqs. (1) and (2).

In continuous time LQR theory, a standard assumption is that the control weighting matrix R is strictly positive definite. This is necessary for the problem to be well posed. Recent results by Chen et al. [3] for the continuous time problem show that R can have negative eigenvalues if the diffusion term in the system equations depends on the control. It is interesting to note therefore that in the discrete time problem, the control weighting matrices R_k can have negative eigenvalues and the problem remain well posed, *even if the bilinear terms ΔA_k and ΔB_k are all zero*! That is, Ω_{k+1}^c can be positive semi-definite, even if some or all the R_k matrices have negative eigenvalues, and ΔA and ΔB are zero. Of course, if ΔA or ΔB are non-zero, then R_k can be 'more' negative-definite and the problem still remain well posed.

Case II: $\rho_k^A \neq 0, \rho_k^B \neq 0$.

In the event that w_k and (w_k^A, w_k^B) are correlated so that $\rho_k^A, \rho_k^B \neq 0$, then the optimal control requires not only the state feedback term as in Eq. (3) but also an external input as

$$u_k = K_k^c x_k + b_k, \tag{10}$$

where b_k is calculated by linear backward recursions as now described.

Without loss of generality, assume $\bar{x}_0 := E[x_0] = 0$, and define

$$\begin{aligned} \alpha_k^A &:= S_{k+1}\Delta A_k \rho_k^A, \qquad \alpha_k^B := S_{k+1}\Delta B_k \rho_k^B, \\ \beta_k^A &:= \Omega_{k+1} K_k, \qquad \beta_k^B := \Omega_{k+1}. \end{aligned} \tag{11}$$

Going through similar calculations as in Eq. (7), one ends up with the following additional cost term involving b_k, ρ_k^A, ρ_k^B:

$$J^{\text{odd}} = \sum_{k=0}^{T-1} ((\alpha_k^A - b_k' \beta_k^A)\bar{x}_k + (\alpha_k^B - b_k' \beta_k^B)\bar{u}_k). \tag{12}$$

Here $\bar{x}_k := E[x_k]$, $\bar{u}_k := E[u_k]$ which satisfy, from taking expectations in Eq. (1), $\bar{x}_{k+1} = A_k \bar{x}_k + B_k \bar{u}_k$. To see that $J^{\text{odd}} = 0$ can hold by a suitable b_k selection, first substitute $\bar{x}_0 = 0$, $\bar{x}_1 = A_0 \bar{x}_0 + B_0 \bar{u}_0 = B_0 \bar{u}_0$, $\bar{x}_2 = A_1 B_0 \bar{x}_0 + B_1 \bar{u}_1$, etc. Then, Eq. (12) can be re-organized as a matrix equation

$$[1 \quad 1 \quad \cdots \quad 1] \begin{bmatrix} \times & 0 & \dots & 0 \\ \times & \times & & 0 \\ \vdots & & \ddots & \\ \times & \times & \dots & \times \end{bmatrix} \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \\ \vdots \\ \bar{u}_T \end{bmatrix} = 0.$$

Denoting the lower triangular matrix as L and the row vector $[1 \quad 1 \quad \cdots \quad 1]$ as $\underline{\mathbf{1}}'$, then this holds if $\underline{\mathbf{1}}' L = 0$; that is, the sum of the columns of L is zero. Starting with the last column allows calculation of b_T, and then the second last column allows calculation of b_{T-1} in terms of b_T. Proceeding, allows calculation of b_k in terms of $b_{k+1}, \dots, b_T$.

Thus, using a backwards recursion, $\{b_k\}$ is calculated by solving successively (with forward substitutions)

$$\begin{aligned} &\alpha_T^B - b_T \beta^B = 0, \\ &(\alpha_T^A - b_T \beta_T^A) B_{T-1} + (\alpha_{T-1}^B - b_{T-1}\beta_{T-1}^B) = 0, \\ &\quad \vdots \\ &(\alpha_T^A - b_T \beta_T^A)(A_{T-1} A_{T-2} \cdots A_1 B_0) \\ &\quad + (\alpha_{T-1}^A - b_{T-1}\beta_{T-1}^A)(A_{T-2} \cdots A_1 B_0) + \cdots \\ &\quad + (\alpha_0 - b_0 \beta_0^B) = 0, \end{aligned} \tag{13}$$

for $b_T, b_{T-1}, b_{T-2}, \dots, b_0$. The equations have the form $\mathscr{L} b = d$ where $\mathscr{L}$ is lower triangular and $b' = [b_T \quad b_{T-1} \quad \cdots \quad b_0]$, which is readily solved for b.

3. Discrete time Riccati equation

In the continuous time LQR problem, a standard assumption is that the control weighting matrix $R(t)$ is strictly positive definite. In the paper by Chen et al. [3], it is shown that for full observation stochastic LQR problems with control-dependent diffusion terms, this assumption is not necessary. In fact, they derive necessary and sufficient conditions for the solvability of the associated Riccati equation and show that these conditions can be satisfied (and the associated LQR problem well posed) by control weighting matrices with negative eigenvalues. In this section, we examine the effect of the terms ΔA_k and ΔB_k on the well posedness of the LQR problem (1)–(2).

Recall that the LQR problem (1)–(2) is well posed if and only if $\Omega_k^c \geqslant 0$ for every k. Note once again that it is possible for the standard LQR problem (i.e. $\Delta A_k = 0$ and $\Delta B_k = 0$) to be well posed with either $Q_k < 0$ or $R_k < 0$ (but obviously not both). We show in this section that if $\Delta A_k \neq 0$ or $\Delta B_k \neq 0$, then Q_k and R_k can be made 'more negative'. That is, we can replace Q_k by $\bar{Q}_k \leqslant Q_k$ and R_k by $\bar{R}_k \leqslant R_k$ and with the associated problem still remaining well posed. Bounds on the allowable decrease are also derived for certain special cases.

Before doing this however, we need to introduce some notation. Let $\mathscr{K} = \{(S_0, \dots, S_T) \,|\, S_j \in \mathbb{R}^{n\times n}$, symmetric$\}$, $\mathscr{Q} = \{(Q_0, \dots, Q_T) \,|\, Q_j \in \mathbb{R}^{n\times n}$, symmetric$\}$ and $\mathscr{P} = \{(R_1, \dots, R_T) \,|\, R_j \in \mathbb{R}^{m\times m}$, symmetric$\}$. Given a sequence $\bar{R}^c = (\bar{R}_1^c, \dots, \bar{R}_T^c) \in \mathscr{P}$ of control weights and $\bar{Q}^c = (\bar{Q}^c_0, \dots, \bar{Q}^c_T) \in \mathcal{Q}$ of state weights, the standard discrete time Riccati equation

$$S_k = A'_k S_{k+1} A_k - A'_k S_{k+1} B_k (\bar{R}^c_{k+1} + B'_k S_{k+1} B_k)^{-1} B'_k S_{k+1} A_k + \bar{Q}^c_k, \quad (14)$$
$$S_T = \bar{Q}^c_T,$$

gives rise to a sequence $(S_0, \dots, S_T) \in \mathcal{K}$. Hence, we can define a mapping $\psi : \mathcal{P} \times \mathcal{Q} \to \mathcal{K}$ which maps a sequence of control weights $\bar{R}^c = (\bar{R}^c_1, \dots, \bar{R}^c_T) \in \mathcal{P}$ and state weights $\bar{Q}^c = (\bar{Q}^c_0, \dots, \bar{Q}^c_T) \in \mathcal{Q}$ to the solution $\psi(\bar{Q}^c, \bar{R}^c) = (\psi_0(\bar{Q}^c, \bar{R}^c), \dots, \psi_T(\bar{Q}^c, \bar{R}^c)) \in \mathcal{K}$ of Eq. (14).

Suppose now that $\bar{Q}^c = Q^c \in \mathcal{Q}$ is given (and fixed) while $\bar{R}^c$ is the variable. In this case, we shall write $\psi(Q^c, \bar{R}^c)$ simply as $\psi(\bar{R}^c)$. It follows that the associated (standard) LQR problem is solvable if and only if

$$\bar{R}^c_{k+1} + B'_k \psi_{k+1}(\bar{R}^c) B_k \geqslant 0. \quad (15)$$

We begin by examining the case $\Delta A_k = 0$. Before stating our main results, we note the following.

Lemma 3.1. *Let $\Delta A_k = 0$ and $\Delta B_k \neq 0$. Let (Q^c_k, R^c_k) be given. Then Eqs.* (1) *and* (2) *are well posed if and only if there exists $\bar{R}^c_k$ such that*

$$R^c_{k+1} + \Delta B'_k \psi_{k+1}(\bar{R}^c) \Delta B_k = \bar{R}^c_{k+1}, \quad (16)$$

where $(Q^c_k, \bar{R}^c_k)$ satisfy Eqs. (14) *and* (15).

Proof. Obvious. □

Remark 3.1. In the continuous time case [3], the existence of a solution to the Riccati equation that satisfies a condition similar to the one in Lemma 3.1 is sufficient for well posedness of the LQR problem. However, it is not necessary for well posedness. This arises in the continuous time case because the Riccati equation may not have a solution. On the other hand, the Riccati equation associated with the discrete time problem always has a solution if pseudo-inverses are allowed.

Lemma 3.2. *If Eqs.* (1) *and* (2) *corresponding to (Q^c_k, R^c_k) are well posed, then Eqs.* (1) *and* (2) *with $(Q^c_k, \tilde{R}^c_k)$ are well posed for all $\tilde{R}^c_k \geqslant R^c_k$.*

Proof. Let $\Delta = \tilde{R}^c_k - R^c_k \geqslant 0$. Then for every feasible u_k, we have

$$\begin{aligned}
J_T &= E\left\{ \sum_{k=0}^{T-1} (x'_k Q^c_k x_k + u'_k \tilde{R}^c_{k+1} u_k) + x'_T Q^c_T x_T - x'_0 Q^c_0 x_0 \right\} \\
&= E\left\{ \sum_{k=0}^{T-1} (x'_k Q^c_k x_k + u'_k (R^c_{k+1} + \Delta_{k+1}) u_k) + x'_T Q^c_T x_T - x'_0 Q^c_0 x_0 \right\} \\
&= E\left\{ \sum_{k=0}^{T-1} (x'_k Q^c_k x_k + u'_k R^c_{k+1} u_k) + x'_T Q^c_T x_T - x'_0 Q^c_0 x_0 \right\} + E\left\{ \sum_{k=0}^{T-1} u'_k \Delta_{k+1} u_k \right\} \\
&\geqslant E\left\{ \sum_{k=0}^{T-1} (x'_k Q^c_k x_k + u'_k R^c_{k+1} u_k) + x'_T Q^c_T x_T - x'_0 Q^c_0 x_0 \right\} \\
&> 0
\end{aligned}$$

from which the result follows. □

We are now in the position to state our main result regarding the influence of the term ΔB_k on the problem (1)–(2).

Theorem 3.1. *Let (Q^c_k, R^c_k) be given. If $\Delta A_k = 0$, then the problem* (1)–(2) *corresponding to (Q^c_k, R^c_k) is well posed if and only if*

$$R^c_{k+1} \geqslant \bar{R}^c_{k+1} - \Delta B'_k \psi_{k+1}(\bar{R}^c) \Delta B_k \quad (17)$$

for some $\bar{R}^c_k$ such that $(\bar{Q}^c_k = Q^c_k, \bar{R}^c_k)$ satisfies Eqs. (14) *and* (15).

Proof. Suppose that Eqs. (1) and (2) is well posed for (Q^c_k, R^c_k). Then by Lemma 3.1, there exists $\bar{R}_k$ such that

$$R^c_{k+1} + \Delta B'_k \psi_{k+1}(\bar{R}^c) \Delta B_k = \bar{R}^c_{k+1} \quad (18)$$

and $(Q^c_k, \bar{R}^c_k)$ satisfy Eqs. (14) and (15). This implies Eq. (17).

Conversely, suppose that Eq. (17) holds for some $\bar{R}^c_k$ such that $(Q^c_k, \bar{R}^c_k)$ satisfies Eqs. (14) and (15). Then there exists some $\tilde{R}^c_k$ such that

$$R^c_{k+1} \geqslant \tilde{R}^c_{k+1} = \bar{R}^c_{k+1} - \Delta B'_k \psi_{k+1}(\bar{R}^c)\Delta B_k.$$

By Lemma 3.1, the problem (1)–(2) associated with $(Q^c_k, \tilde{R}^c_k)$ is well posed. Since $R^c_k \geqslant \tilde{R}^c_k$, it follows from Lemma 3.2 that Eqs. (1) and (2) with (Q^c_k, R^c_k) are well posed. □

Note in particular that if $\Delta B_k \neq 0$, then the control weighting matrices $\tilde{R}^c_k$ can be made 'more negative' and the problem (1)–(2) still remains well posed. That is, if $\Delta B_k \neq 0$, the matrices $\tilde{R}^c_k$ can be replaced by matrices R^c such that $R^c_k \leqslant \bar{R}^c_k$, and the problem (1)–(2) still remains well posed. The bound on this change is given by Eq. (17).

In the analysis above, we have assumed that $\bar{Q}^c$ is given and fixed. In fact, if we define the inner product $\langle \cdot,\cdot \rangle_{\mathscr{Q}} : \mathscr{Q} \times \mathscr{Q} \to \mathbb{R}$ on $\mathscr{Q}$ by

$$\langle Q^1, Q^2 \rangle = \sum_{k=0}^{T} \operatorname{tr}(Q^1_k \cdot Q^2_k) \tag{19}$$

and $\langle \cdot,\cdot \rangle : \mathscr{K} \times \mathscr{K} \to \mathbb{R}$ similarly, then it is easily shown that $\psi_{k+1}(\bar{Q}^c, \bar{R}^c)$ is continuous with respect to $\bar{Q}^c$. It follows then that $\bar{Q}^c_k$ can be made 'more negative' if $\Delta A_k = 0$ and $\Delta B_k \neq 0$. The allowable bounds on this change is still an open question.

Consider now the case when $\Delta A_k \neq 0$ but $\Delta B_k = 0$. Let $\bar{R}^c = R^c \in \mathscr{P}$ be fixed. Let $\psi : \mathscr{Q} \to \mathscr{K}$ be a mapping such that $\psi(\bar{Q}^c) = \psi(\bar{Q}^c, R^c)$ is the solution of the standard discrete time Riccati equation

$$\begin{aligned} S_k &= A'_k S_{k+1} A_k - A'_k S_{k+1} B_k (R^c_{k+1} + B'_k S_{k+1} B_k)^{-1} \\ &\quad B'_k S_{k+1} A_k + \bar{Q}^c_k, \\ S_T &= \bar{Q}^c_T. \end{aligned} \tag{20}$$

In this case, the associated (standard) LQR problem is solvable if and only if

$$R^c_{k+1} + B'_k \psi_{k+1}(\bar{Q}^c) B_k \geqslant 0. \tag{21}$$

In much the same way as the case $\Delta A_k = 0$, $\Delta B_k \neq 0$, the following result can be shown.

Theorem 3.2. *Let (Q^c_k, R^c_k) be given. If $\Delta B_k = 0$, then the problem (1)–(2) corresponding to (Q^c_k, R^c_k) is well posed if and only if*

$$Q^c_k \geqslant \bar{Q}^c_k - \Delta A'_k \psi_{k+1}(\bar{Q}^c)\Delta A_k \tag{22}$$

for some $\bar{Q}^c_k$ such that $(R^c_k, \bar{Q}^c_k)$ satisfies Eqs. (20) *and* (21).

As in the case of Theorem 3.1, Theorem 3.2 shows how much 'more negative' the matrices $\bar{Q}^c_k$ can be made when $\Delta A_k \neq 0$ and $\Delta B_k = 0$. Furthermore, if we define an inner product $\langle \cdot,\cdot \rangle_{\mathscr{P}} : \mathscr{P} \times \mathscr{P} \to \mathbb{R}$ on $\mathscr{P}$ as we have for $\mathscr{Q}$ (see Eq. (19)), it is easily shown that $\psi_{k+1}(\bar{Q}^c, \bar{R}^c)$ is continuous with respect to $\bar{R}^c$. Therefore, if $\Delta A_k \neq 0$ and $\Delta B_K = 0, \bar{R}^c_k$ can be made 'more negative'. The allowable bounds on this change is still an open question. Similarly, the effect of both $\Delta A_k \neq 0$ and $\Delta B_k \neq 0$ is still unresolved.

4. State estimation

In this section, we first define a partially observed signal model. Next, we apply the known Kalman filter theory to yield a linear minimum variance state estimator, which is then linearized further so that the filter is linear in the states and control, and bilinear in the innovations (prediction errors) and the states/controls.

Consider the following partially observed model:

$$\begin{aligned} x_{k+1} &= (A_k + w^A_k \Delta A_k) x_k + (B_k + w^B_k \Delta B_k) u_k + w_k, \\ y_k &= (C_k + w^c_k \Delta C_k) x_k + v_k, \end{aligned} \tag{23}$$

where $y_k \in \mathbb{R}^p$. Here w^c_k, v_k are martingale increments, each orthogonal to $w^{\mathrm{A}}_k, w^{\mathrm{B}}_k, w_k$, and $E[v_k v'_k] = R_k$.

Linear conditional minimum variance state estimator: Applying standard filtering results [1] yields the estimator

$$\begin{aligned} \hat{x}_{k+1} &= A_k \hat{x}_k + B_k u_k + K_k(\hat{x}_k, u_k) v_k, \\ v_k &= y_k - C_k \hat{x}_k, \end{aligned} \tag{24}$$

where the gain $K_k(\hat{x}_k, u_k)$ is given in terms of a coupled matrix Riccati equation as follows:

$$K_k(\hat{x}_k, u_k) = L_k \Omega_k(\hat{x}_k, u_k)^{-1}, \tag{25}$$

with

$$\begin{aligned} L_k &= A_k \Sigma_k C'_k, \\ \Omega_k(\hat{x}_k, u_k) &= C_k \Sigma_k C'_k + R_k + \Delta C_k(\Sigma_k + \hat{x}_k \hat{x}'_k)\Delta C'_k, \end{aligned} \tag{26}$$

and

$$\Sigma_{k+1} = A_k \Sigma_k A_k' - L_k \Omega(\hat{x}_k, u_k)^{-1} L_k' + Q_k + \Delta A_k(\Sigma_k + \hat{x}_k \hat{x}_k')\Delta A_k' + \Delta B_k u_k u_k' \Delta B_k',$$
$$\Sigma_0 = E[x_0 x_0']. \tag{27}$$

Here $\hat{x}_k$ is the best linear estimate conditioned on $\mathscr{Y}_{k-1}$, the σ-algebra generated by $y_0, \dots, y_{k-1}$, where best is in a minimum error variance sense. The associated conditional error covariance is

$$\Sigma_k = E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)' | \mathscr{Y}_{k-1}]. \tag{28}$$

In the derivation of Eqs. (24) and (28), the Projection Theorem is used, which tells us that

$$E[v_k | \mathscr{Y}_{k-1}] = 0, \qquad E[\hat{x}_k (x_k - \hat{x}_k)' | \mathscr{Y}_{k-1}] = 0. \tag{29}$$

Notice that the dependence of the noise on states and controls in our model (23) leads to an error covariance which depends on the past measurements (and controls), and in turn leads to a filter gain $K_k(\cdot,\cdot)$ which is dependent on the past measurements (and controls). Now this dependency of $K_k(\cdot,\cdot)$ on $\hat{x}_k, u_k$ is by no means affine, but in order to proceed to a control law based on the LQR theory of Section 2, we must linearize $K_k(\cdot,\cdot)$ in $\hat{x}_k$ and u_k.

A filter bilinear in the innovations: Consider a linearization of $K_k(\cdot,\cdot)$, via a Taylor expansion, for simplicity in the $p=1$ case

$$K_k(\hat{x}_k, u_k) = K_k + K_k^x \hat{x}_k + K_k^u u_k + \mathrm{o}(\|\hat{x}_k\|, \|u_k\|).$$

Neglecting the quadratic and higher-order terms in $\hat{x}_k, u_k$ leads to an approximate filter

$$\begin{aligned} \hat{x}_{k+1} &\approx A_k \hat{x}_k + B_k u_k + (K_k + K_k^x \hat{x}_k + K_k^u u_k) v_k \\ &= (A_k + K_k^x v_k)\hat{x}_k + (B_k + K_k^u v_k) u_k + K_k v_k. \end{aligned} \tag{30}$$

5. State estimate feedback

The approach taken in an LQG control design is taken here, namely to consider the state estimator (24) (or in our case the approximation (30)) as a state space signal model with state $\hat{x}_k$, and to re-organize the control performance index J_T of Eq. (2) in terms of $\hat{x}_k$, rather than x_k. Noting Eqs. (24) and (28) we have a re-organization of J_T as

$$J_T = \sum_{k=0}^{T-1} [\hat{x}_k' Q_k^c \hat{x}_k + u_k' R_{k+1}^c u_k + \mathrm{tr}(Q_k^c \Sigma_k)]. \tag{31}$$

Actually, Σ_k is perhaps best written as $\Sigma_k(\hat{x}_k \hat{x}_k', u_k u_k')$ since it is dependent on $\hat{x}_k \hat{x}_k'$ and $u_k u_k'$. Now a Taylor series expansion leads to

$$\Sigma_k \approx \Sigma_k^0 + \Sigma_k^x \hat{x}_k \hat{x}_k' + \Sigma_k^u u_k u_k', \tag{32}$$

being linear in $\hat{x}_k \hat{x}_k'$ and $u_k u_k'$. Thus Eq. (31) under Eq. (32) becomes

$$J_T \approx \sum_{k=0}^{T-1} [\hat{x}_k'(Q_k^c + \Sigma_k^x)\hat{x}_k + u_k'(R_{k+1}^c + \Sigma_k^u) u_k + \mathrm{tr}(Q_k^c \Sigma_k^0)]. \tag{33}$$

Now the optimization of Eq. (33) under Eq. (30) can be tackled using the optimal LQR results of Section 2 with $w^A = w^B = w$. Thus,

$$u_k^{\mathrm{opt}} \approx K_k^c \hat{x}_k + b_k, \tag{34}$$

where $\hat{x}_k$ is derived from the filter (30). Also K_k^c are derived from an approximate specialization of Eqs. (3) and (4) in which $\Delta A_k = K_k^x$, $\Delta B_k = K_k^u$. The term b_k is derived by solving the algebraic equations (13) in turn for $b_T, b_{T-1}, \dots, b_0$ being backward recursions.

6. Conclusion

The LQG approach to partially observed stochastic models leads to useful suboptimal state estimate linear feedback controllers when the models are bilinear in the noise and a linearization of certain equations is analysed.

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