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Long step path-following algorithm for the convex quadratic progamming in a Hilbert space

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Abstract

We develop an interior-point technique for solving quadratic programming problem in a Hilbert space.As an example we consider an application of these results to the linear-quadratic control problem with linear inequality constraints. It is shown that the Newton step in this situation is basically reduced to solving the standard linear-quadratic control problem.

1 Introduction

The development of the interior-point methodology has led to quite efficient algorithms for solving certain classes of optimization problems. We first of all should mention in this respect the standard convex finite dimensional quadratic programming problem (see e.g. [4]) and its semidefinite counterpart ([5]). The latter class of problems found various applications in control theory ([2]), ([9]). The next natural step in this direction in our opinion is to construct interior-point algorithms for infinite-dimensional problems. In this paper we consider a natural analogue of the convex quadratic programming problem in a Hilbert space. We then apply the obtained results to the linear-quadratic control problem with linear inequality constraints. The case of quadratic constraints can be treated in a similar way but requires the technique of self-concordant barriers ([8]). The performance of a Newton step is the most difficult part from the computational viewpoint. It is important therefore that it is reduced to the the solution of the standard linear-quadratic control problem. We believe that the technique developed in this paper will find numerous applications for control problems involving inequality constraints. Many proofs are omitted here and will be published elsewhere.

2 Duality, central trajectories and dynamical systems

Let $(H, \langle , \rangle)$ be a Hilbert space, X its closed vector subspace. Given $a_0, \dots a_m \in H, b_1, \dots, b_m \in \mathbb{R}$, and a symmetric bounded strictly positive operator $Q : H \to H$, consider the following optimization problem:

$$f(x) = \langle a_0, x\rangle + \frac{1}{2}\langle Qx, x\rangle \to \min, \tag{1}$$

$$\langle a_i, x\rangle \leq b_i, 1, 2, \dots, m, \tag{2}$$

$$x \in X. \tag{3}$$

In this paper we consider the case where H is infinite-dimensional. The problem (1) - (3) is probably one of the simplest nontrivial infinite-dimensional optimization problems. It possesses many interesting properties similar to its finite dimensional counterparts. In particular, certain simplex-types procedures are known for (1) - (3) (see e.g.. ([6]), ([7])). It is therefore natural to start developing infinite-dimensional interior-point methods with this problem. In this section we carry out necessary analysis of central trajectories and related dynamical systems. For a finite-dimensional counterpart see ([3]).

Denote the set determined by constraints (2), (3) by P. We assume throughout this paper that

$$\operatorname{int}(P) = \{x \in P : \langle a_i, x\rangle < b_i, i = 1, 2, \dots, m\} \tag{4}$$

is not empty.

Consider also the following (dual) problem:

$$h(x, \lambda) = -\frac{\langle Qx, x\rangle}{2} - \sum_{i=1}^{m} \lambda_i b_i \to \max, \tag{5}$$

$$a_0 + Qx + \sum_{i=1}^{m} \lambda_i a_i \in X^{\perp}, \lambda_i \geq 0, i = 1, 2, \dots, m \tag{6}$$

Here $X^{\perp} = \{x \in H : \langle x, y\rangle = 0 \text{ for every } y \in X\}$.

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Lemma 2.1 *Suppose that $x \in P$ and $(\bar{x}, \lambda_1, \ldots, \lambda_m)$ satisfies (6). Then*

$$f(x)-h(\bar{x},\lambda) = \frac{1}{2}\langle Q(x-\bar{x}), x-\bar{x}\rangle + \sum_{i=1}^{m} \lambda_i (b_i - \langle a_i, x\rangle) \geq 0.$$

Consider the problem:

$$f_\beta(x) =$$

$$\beta\left(\langle a_0, x\rangle + \frac{1}{2}\langle x, Qx\rangle\right) - \sum_{i=1}^{m} ln\,(b_i - \langle a_i, x\rangle) \to \min, \tag{7}$$

$$x \in \text{int}(P) \tag{8}$$

Proposition 2.1 *For any $\beta > 0$ the problem (7), (8) has a unique solution $x(\beta) \in \text{int}(P)$.*

Remark 1 In the finite-dimensional situation the curve $x(\beta), \beta > 0$ is usually called the central trajectory. Our intermediate goal is to prove that $x(\beta)$ converges to the optimal solution of (1) - (3) when $\beta \to +\infty$.

Denote by $\pi : H \to X$ the orthogonal projection of H onto X. The point $x(\beta)$ is uniquely determined by the condition.

$$\nabla f_\beta(x(\beta)) \in X^\perp, \tag{9}$$

where we define the gradient $\nabla f_\beta(x)$ of the function f_β in the standard way:

$$Df_\beta(x).\xi = < \nabla f_\beta(x), \xi >, \quad \xi \in H.$$

In other words,

$$\pi\left(\beta a_0 + \beta Q x(\beta) + \sum_{i=1}^{m} \frac{a_i}{s_i(x)}\right) = 0 \tag{10}$$

where $s_i(x) = b_i - \langle a_i, x\rangle$. For any $\beta > 0$ consider the map $\psi_\beta : \text{int}(P) \to X$,

$$\psi_\beta(x) = \pi\left(\beta Q x + \sum_{i=1}^{m} \frac{a_i}{s_i(x)}\right) \tag{11}$$

We can rewrite (10) in the form:

$$\psi_\beta(x(\beta)) = -\beta\pi(a_0) \tag{12}$$

Proposition 2.2 *For each $\beta > 0$ the map ψ_β is a smooth isomorphism of $\text{int}(P)$ onto X such that ψ_β^{-1} depends smoothly on β.*

Corollary 2.1 *The solution $x(\beta)$ of the problem (7), (8) depends smoothly on β.*

Proof: By (12) $x(\beta) = \psi_\beta^{-1}(-\beta\pi(a_0))$. The result follows by Proposition 2.2.

We introduce now a family of Riemannian metrics g_β which are determined by Hessians of functions f_β. Namely,

$$g_\beta(x;\xi,\eta) = D^2 f_\beta(x)(\xi,\eta) \tag{13}$$

$\xi, \eta \in H$, x is such that $< a_i, x >< b_i, i = 1, 2, \ldots, m$. We clearly have:

$$g_\beta(x;\xi,\eta) = < \gamma_\beta(x)\xi, \eta >, \tag{14}$$

$$\gamma_\beta(x) = \beta Q + \sum_{i=1}^{m} \frac{a_i \otimes a_i}{s_i^2(x)}, \quad \beta > 0 \tag{15}$$

We can think of $\text{int}(P)$ as a sub-manifold of the open subset in H defined by conditions $< a_i, x >< b_i, i = 1, 2, \ldots, m$. Given a smooth function φ defined on an open neighborhood of $\text{int}(P)$ in H, denote by $\nabla_\beta \varphi(x) \in X$ the gradient of φ relative to the metric g_β. In other words, for every $\xi \in X, \text{int}(P)$,

$$D\varphi(x).\xi = < \nabla\varphi(x), \xi > = g_\beta(x;\xi,\nabla_\beta\varphi(x)), \quad \nabla_\beta\varphi(x) \in X. \tag{16}$$

■

Proposition 2.3

$$\nabla_\beta\varphi(x) = \gamma_\beta(x)^{-1}(\nabla\varphi(x) - \xi(x)), \tag{17}$$

where $\xi(x)$ is a unique vector in $X^\perp$ such that $\nabla_\beta\varphi(x) \in X$.

Proof: Suppose that $\xi(x) \in X^\perp$ is such that $\nabla_\beta\varphi(x) \in X$. We have:

$$\begin{aligned} g_\beta(x;\xi,\nabla_\beta\varphi(x)) &= \langle \gamma_\beta(x)\xi, \gamma_\beta(x)^{-1}(\nabla\varphi(x) - \xi(x))\rangle \\ &= \langle \xi, \nabla\varphi(x) - \xi(x)\rangle \end{aligned}$$

for $\xi \in X$. To prove that there exists a unique $\xi(x)$ satisfying this property consider the following optimisation problem:

$$\langle \gamma_\beta(x)^{-1}(\nabla\varphi(x) - \xi), (\nabla\varphi(x) - \xi)\rangle \to \min, \tag{18}$$

$$\xi \in X^\perp \tag{19}$$

As is well known (see e.g. [1]) this problem has a unique solution $\bar{\xi}$ which is characterized by the condition:

$$\gamma_\beta(x)^{-1}(\nabla\varphi(x) - \bar{\xi}) \in X.$$

Thus

$$\bar{\xi} = \xi(x).$$

■

Corollary 2.2 *Under assumptions of Proposition 2.2, we have:*

$$D\psi_\beta(x) \cdot \nabla_\beta\varphi(x) = \pi(\nabla\varphi(x)). \tag{20}$$

The next proposition describes a differential equation which has the central trajectory as its solution.

Proposition 2.4 *Let* $\varphi(x) = -\langle a_0, x\rangle - \frac{1}{2}\langle Qx, x\rangle$. *Then*

$$\frac{dx(\beta)}{d\beta} = \nabla_\beta \varphi(x(\beta)), \quad \beta > 0 \tag{21}$$

Corollary 2.3 *Let*

$$f(x) = \langle a_0, x\rangle + \frac{1}{2}\langle Qx, x\rangle.$$

Then the function $f(x(\beta))$ *is a monotonically decreasing function of* β *for* $\beta > 0$.

Consider the following optimization problem:

$$\mathcal{X}_\beta(x, \lambda) = -\beta\left(\frac{\langle Qx, x\rangle}{2} + \sum_{i=1}^{m}\lambda_i b_i\right) + \sum_{i=1}^{m} ln\lambda_i \to \max, \tag{22}$$

$$a_0 + Qx + \sum_{i=1}^{m}\lambda_i a_i \in X^\perp, \lambda_i > 0, i = 1, 2, \ldots, m. \tag{23}$$

Proposition 2.5 *If* $x(\beta)$ *is the optimal solution to the problem (7), (8), then* $(x(\beta), \lambda(\beta))$ *with* $\lambda_i(\beta) = \frac{1}{\beta s_i(x(\beta))}$ *(see ,* $i = 1, 2, \ldots, m$, *is the optimal solution to the problem (22), (23).*

Proposition 2.6 *The function* $\alpha(\beta) = h(x(\beta), \lambda(\beta))$ *is a monotonically increasing function of* β *for* $\beta > 0$.

Proof: Set $\Delta(\beta) = \mathcal{X}_\beta(x(\beta), \lambda(\beta)), \beta > 0$. Let $\beta_1 > \beta_2 > 0$ Since $(x(\beta), \lambda(\beta))$ is an optimal solution to (22), (23), we have:

$$\begin{aligned}\Delta(\beta_1) &\geq \mathcal{X}_{\beta_1}(x(\beta_2), \lambda(\beta_2)),\\ \Delta(\beta_2) &\geq \mathcal{X}(x(\beta_1), \lambda(\beta_1)).\end{aligned}$$

Since $\mathcal{X}_\beta = \beta h + \sum_{i=1}^{m} ln\lambda_i$, adding up these two inequalities, we obtain:

$$(\beta_2 - \beta_1)(\Delta(\beta_2) - \Delta(\beta_1)) \geq 0.$$

Since $\beta_2 > \beta_1$, it implies $\Delta(\beta_2) \geq \Delta(\beta_1)$. ■

Corollary 2.4

$$f(x(\beta)) - h(x(\beta), \lambda(\beta)) = \frac{m}{\beta} \tag{24}$$

If x^* *is the optimal solution to the problem (1) - (3), then* $x(\beta) \to x^*, \beta \to +\infty$.

<u>Remark 2</u> One can show that $(x(\beta), \lambda(\beta))$ converges to the solution (x^*, λ^*) of the dual problem (5), (6), when $\beta \to +\infty$. We conclude.

Corollary 2.5

$$f(x^*) = h(x^*, \lambda^*)$$

.

Corollary 2.6 *If* $\beta_2 > \beta_1 > 0$, *then*

$$f(x(\beta_1)) - f(x(\beta_2)) \leq \frac{m}{\beta_1} - \frac{m}{\beta_2}. \tag{25}$$

3 A Large-Step Path-Following Algorithm

The idea of path-following algorithms is to produce a finite-step approximation to the central trajectory. Using the interpretation of points on the central trajectory as optimal solutions of the family of optimization problems (7), (8), one can reduce the construction of a path-following algorithm to the performance of Newton steps for functions of the form (7). The choice of logarithmic barriers enable us (in complete analogy with the finite-dimensional case [4]) to obtain a rather sharp bound on the number of Newton's steps. Many other interior-point algorithms can be generalized in this way (see e.g. [12]).

Let us introduce the Newton's direction for the function f_β. Namely,

$$p_\beta(x) = -\nabla_\beta f_\beta(x). \tag{26}$$

We have

$$p_\beta(x) = -\gamma_\beta(x)^{-1}\left(\beta a_0 + \beta Qx + \sum_{i=1}^{m}\frac{a_i}{s_i(x)} - \xi(x)\right), \tag{27}$$

where $\xi(x) \in X^\perp$ is such that $p_\beta(x) \in X$ (see (17).

Recall that (see (18), (19)) $p_\beta(X)$ is the solution to the following problem:

$$\langle\gamma_\beta(x)^{-1}(\nabla f_\beta(x) - \xi), (\nabla f_\beta(x) - \xi)\rangle \to \min, \tag{28}$$

$$\xi \in X^\perp \tag{29}$$

Let us introduce the quantity $\delta_\beta(x)$ which measures in a sense the distance between a point $x \in \text{int}(P)$ and the point $x(\beta)$ on the central trajectory:

$$\delta_\beta(x) = g_\beta(x; p_\beta(x), p_\beta(x))^{\frac{1}{2}}. \tag{30}$$

We will see that $\delta_\beta(x)$ appears in all our estimates.

Lemma 3.1 *One has:*

$$\delta_\beta(x)^2 = -\langle p_\beta(x), \nabla f_\beta(x)\rangle, x \in \text{int}(P). \tag{31}$$

Proof: Indeed,

$$\begin{aligned}\delta_\beta(x)^2 &= \langle p_\beta(x), \gamma_\beta(x)p_\beta(x)\rangle\\ &= \langle p_\beta(x), -\gamma_\beta(x)\gamma_\beta(x)^{-1}(\nabla f_\beta(x) - \xi(x))\rangle\\ &= -\langle p_\beta(x), \nabla f_\beta(x)\rangle, \textit{since} \langle p_\beta(x), \xi(x)\rangle = 0\end{aligned}$$

■

Proposition 3.1 *Suppose that* $x \in \text{int}(P)$ *and*

$$\langle\gamma_0(x)p_\beta(x), p_\beta(x)\rangle < 1.$$

Proof: Set in this proof $p = p_\beta(x)$. We have

$$\langle \gamma_0(x)p, p\rangle = \sum_{i=1}^{m} \frac{\langle a_i, p\rangle^2}{s_i{}^2(x)} < 1$$

Thus $|\langle a_i, p\rangle| < s_i(x), i = 1, 2, \dots, m$. In particular, $s_i(x+p) = s_i(x) - \langle a_i, p\rangle > 0, i = 1, 2, \dots, m$. Thus $x + p \in \text{int}(P)$. ■

Proposition 3.2 *Let $x \in \text{int}(P), \delta_\beta(x) < 1$. Then $x^+ = x + p_\beta(x) \in \text{int}(P)$ and, moreover, for $\beta > 0$*

$$\delta_\beta(x^+) \le \delta_\beta(x)^2. \tag{32}$$

Proposition 3.3 *If $x \in \text{int}(P)$ is such that $x + p_\beta(x) \in \text{int}(P)$, then*

$$f_\beta(x) - f_\beta\left(x + \bar{t}p_\beta(x)\right) \ge \delta_\beta(x) - ln(1 + \delta_\beta(x)), \tag{33}$$

where

$$\bar{t} = \frac{1}{1 + \delta_\beta(x)}.$$

Corollary 3.1 *Under the assumptions of Proposition 3.3, if $0 < \delta < \frac{1}{2}$, then $f_\beta(x + p) < f_\beta(x)$.*

Proposition 3.4 *Given $\beta > 0, x \in \text{int}(P)$ and $\delta_\beta(x) < 1$, consider the sequence $x_0 = x, x_1 = x_0 + p_\beta(x_0), \dots x_i = x_{i-1} + p_\beta(x_{i-1}), i \ge 2$. Then $x_i \in \text{int}(P)$ for all i and moreover*

$$\lim x_i = x(\beta), i \to +\infty.$$

Proposition 3.5 *Suppose that $x \in \text{int}(P), \delta_\beta(x) < 1$. Then*

$$f_\beta(x) - f_\beta(x(\beta)) \le \frac{\delta_\beta(x)^2}{1 - \delta_\beta(x)^2}. \tag{34}$$

Proposition 3.6 *If $\beta > 0, \delta = \delta_\beta(x) < 1, x \in \text{int}(P)$, then*

$$f(x) - f(x(\beta)) \le \frac{\delta(1+\delta)}{1-\delta}\frac{\sqrt{m}}{\beta}.$$

We now are in position to describe a version of the large-step path-following algorithm. Suppose we are given $\beta_0 > 0$ and $x_0 \in \text{int}(P)$ is such that $\delta_{\beta_0}(x_0) \le \frac{1}{2}$. Given $\theta > 0$, we perform what is called outer iterations $\beta_1 = (1+\theta)\beta_0, \dots \beta_i = (1+\theta)^i\beta_0$. Suppose we are able to construct a sequence of points $x_1, x_2, \dots \in \text{int}(P)$ such that $\delta_{\beta_i}(x_i) \le \frac{1}{2}$. Then:

Theorem 3.1 *Suppose $\epsilon > 0$ is given and*

$$i \ge \frac{ln\left(\frac{4m}{\epsilon\beta_0}\right)}{ln(1+\theta)} \tag{35}$$

Then

$$f(x_i) - f(x^*) \le \epsilon,$$

where x^* is an optimal solution to the problem (1) - (3).

It remains to describe a procedure for the updating x_i. Suppose we are given $x_i \in \text{int}(P)$ such that $\delta_{\beta_i}(x_i) \le \frac{1}{2}$. Set $\beta_{i+1} = (1+\theta)\beta_i$. We have to find $x_{i+1} \in \text{int}(P)$ such that $\delta_{\beta_{i+1}}(x_{i+1}) \le \frac{1}{2}$. To do that we perform several Newton steps for the function $f_{\beta_{i+1}}$ using x_i as a starting point. Each such a Newton step is called an inner iteration.

Theorem 3.2 *Each outer iteration requires at most*

$$\frac{11}{3} + 11\theta\left(\frac{3}{2}\sqrt{m} + \frac{\theta m}{\theta + 1}\right)$$

inner iterations.

We finally obtain.

Theorem 3.3 *An upper bound for the total number of Newton iterations is given by*

$$\frac{ln\left(\frac{4m}{\epsilon\beta_0}\right)}{\ln(1+\theta)}\left(\frac{11}{3} + 11\theta\left(\frac{3\sqrt{m}}{2} + \frac{\theta m}{\theta+1}\right)\right).$$

For the general case of convex linear-quadratic constraints we have ([8])

Theorem 3.4 *An upper bound for the total number of Newton iterations is given by:*

$$\frac{ln\left(\frac{4m}{\epsilon\beta_0}\right)}{ln(1+\vartheta)}\cdot\left(\frac{22}{3} + 22\vartheta\left(\frac{5\sqrt{m}}{2} + \frac{m\vartheta}{1+\vartheta}\right)\right).$$

4 Example

Consider the time-dependent linear-quadratic control problem with inequality constraints:

$$f(y,u) = \tag{36}$$

$$\frac{1}{2}\int_0^T \left[y(t)^T Q(t)y(t) + u(t)^T R(t)u(t)\right] dt \to \min,$$

$$\dot{y} = A(t)y + B(t)u, \quad y(0) = a \tag{37}$$

$$l_i(y,u) = \tag{38}$$

$$\int_0^T \left[a_i^T(y)y(t) + b_i^T(t)u(t)\right] dt \le C_i, i = 1, 2, \dots m.$$

Here $u \in L_2^m[0,T]$, and y is absolutely continuous function on $[0,T]$ with values in $\mathbb{R}^n$ such that $\dot{y} \in L_2^n[0,T]$. We use the standard notation $L_2^n[0,T]$ for the Hilbert space of measurable square integrable functions on $[0,T]$ with values in $\mathbb{R}^n$, $Q(t)$ (respectively, $R(t)$) is a symmetric positive definite n by n (respectively, p by p) matrix which depends continuously on $t \in [0,T]$; $A(t)$

(respectively, $B(t)$) is n by n (respectively n by p) matrix which depends continuously on $t \in [0,T]$; a_i, b_i are continuous vector-valued functions on $[0,T]$ with values $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively; T is a fixed positive number, $C_i, 1, 2, \ldots m$, a are some real numbers. The problem (36) - (38) is, of course, of the form (1) - (3) with $H = L_2^n[0,T] \times L_2^m[0,T], X = \{(y,u) \in L^2 : y$ is absolutely continuous on $[0,T], y(0) = 0$ and (37) holds true$\}$.

Theorem 3.3 gives a rather sharp estimate for the number of Newton steps for out path-following algorithm. From the computational viewpoint it is crucial to see that the performance of a Newton step is a reasonable problem. In what follows we frequently omit the explicit dependence of various functions on time t.

Set $x = (y, u)$. We have for $\beta > 0$

$$\begin{aligned} f_\beta(x) &= \beta f(y,u) - \sum_{i=1}^{m} ln(C_i - l_i(x)), \\ s_i(x) &= C_i - l_i(x), i = 1, 2, \ldots m. \end{aligned}$$

An easy computation shows that:

$$\nabla f_\beta(y,u) = \begin{bmatrix} \beta Qy + \sum_{i=1}^m \frac{a_i}{s_i(x)} \\ \beta Ru + \sum_{i=1}^m \frac{b_i}{s_i(x)} \end{bmatrix}, \quad (39)$$

$$\gamma_\beta(y,u)\begin{bmatrix} \xi \\ r \end{bmatrix} = \begin{bmatrix} \beta Q\xi + \sum_{i=1}^m \frac{l_i(\xi,r)}{s_i(x)^2} \\ \beta Rr + \sum_{i=1}^m \frac{l_i(\xi,r)}{s_i(x)^2} \end{bmatrix} \quad (40)$$

$$x = (y,u) \in \mathrm{int}(P), (\xi, r) \in H,$$

$$X^\perp = \{\begin{bmatrix} \dot{p} + A^T p \\ B^T p \end{bmatrix} : p \textit{ is absolutely}$$

$$\textit{continuous on } [0,T], p(T) = 0 \textit{ and } \dot{p} \in L_2^n[0,T]\}. \quad (41)$$

Recall that for the Newton direction $p_\beta(x)$ at the point $x = (y,u) \in H$ we have the formula:

$$\begin{aligned} p_\beta(x) &= -\gamma_\beta(x)^{-1}(\nabla f_\beta(x) - \mu(x)), \\ \mu(x) &\in X^\perp \end{aligned}$$

If $p_\beta(x) = \begin{bmatrix} \xi \\ r \end{bmatrix} \in X$, then using (40), (41)

$$\dot{\xi} = A\xi + Br, \xi(0) = 0, \quad (42)$$

$$\dot{p} + A^T p - \bar{y} = \beta Q\xi + \sum_{i=1}^{m} \frac{l_i(\xi,r)a_i}{s_i(x)^2}, \quad (43)$$

$$B^T p - \bar{u} = \beta Rr + \sum_{i=1}^{m} \frac{l_i(\xi,r)b_i}{s_i(x)^2}, \quad (44)$$

$$p(T) = 0 \quad (45)$$

where $\begin{bmatrix} \bar{y} \\ \bar{u} \end{bmatrix} = \nabla f_\beta(x)$ (see (39).

To solve (42) - (45) one can use the procedure quite similar to the standard one ([11]). Denote $l_i(\xi, r)$ by $d_i, i = 1, 2, \ldots m$. We are looking for the solution of (42) - (45) in the form:

$$p(t) = K(t)\xi(t) + \rho(t). \quad (46)$$

From (44)

$$r = \frac{R^{-1}}{\beta}\left(B^T p - \bar{u} - \sum_{i=1}^{m} \frac{d_i b_i}{s_i^2(x)}\right). \quad (47)$$

Denote $B(t)R^{-1}(t)B^T(t)$ by $L(t)$.

Substituting (46), (47) into (42), (43), we obtain:

$$\dot{K} + KA + A^T K + \frac{1}{\beta}KLK - \beta Q = 0, K(T) = 0, \quad (48)$$

$$\dot{\xi} = \left(A + \frac{LK}{\beta}\right)\xi + \frac{1}{\beta}L\rho + \beta(t), \xi(0) = 0, \quad (49)$$

$$\dot{\rho} = -\left(A^T + \frac{KL}{\beta}\right)\rho + \alpha(t), \rho(T) = 0, \quad (50)$$

where

$$\begin{aligned} \alpha(t) &= \bar{y} + \sum_{i=1}^{m} \frac{d_i a_i(t)}{s_i^2(x)} + \frac{KBR^{-1}}{\beta}(\bar{u} + \sum_{i=1}^{m}\left(\frac{b_i}{s_i^2(x)}\right) d_i), \\ \beta(t) &= -B\left(\frac{1}{\beta}R^{-1}\bar{u} + \frac{1}{\beta}\sum_{i=1}^{m}\left(\frac{R^{-1}b_i}{s_i^2(x)}\right) d_i\right). \end{aligned}$$

Observe that the Riccati equation (48) does not depend on unknown constants $d_i, \ldots d_m$. Thus one can find $K(t)$ from (48).

Let $\phi(t,T)$ be the fundamental solution to

$$\dot{\rho} = -\left(A^T + \frac{KL}{\beta}\right)\rho.$$

Then $\psi(t,T) = [\phi(t,T)^T]^{-1}$ is the fundamental solution to

$$\dot{\xi} = \left(A + \frac{LK}{\beta}\right)\xi.$$

From (49), (50), we have

$$\begin{aligned} \rho(t) &= \int_T^t \phi(t,\tau)\alpha(\tau)d\tau \quad (51) \\ \xi(t) &= \int_0^t \psi(t,\tau)\left(\frac{1}{\beta}(L\rho)(\tau) + \beta(\tau)\right)d\tau. \quad (52) \end{aligned}$$

From (51), (52), (46) and (47), we immediately conclude:

$$\begin{aligned} r(t) &= \mu(t) + \sum_{i=1}^{m} d_i\mu_i(t) \\ \xi(t) &= v(t) + \sum_{i=1}^{m} d_i v_i(t) \end{aligned}$$

for the known functions μ, v, μ_i, v_i. Recalling the definition of d_i, we obtain

$$d_i = \sum_{j=1}^{m} f_{ij} d_j + g_i, i = 1, 2, \ldots m, \qquad (53)$$

where

$$\begin{aligned} f_{ij} &= \int_0^T \left[a_i^T(t)v_j(t) + b_i^T(t)\mu_j(t)\right] dt, \\ g_i &= \int_0^T \left[a_i^T(t)v(t) + b_i^T(t)\mu(t)\right] dt. \end{aligned}$$

This is a system of linear algebraic equations which can be easily solved.

We see that the performance of the Newton's step involves in our situation three essential steps:

(a) The numerical integration of the Riccati equation (48)

(b) Finding the fundamental matrix for the linear system (50)

(c) Solving a system of m by m linear algebraic equations (53).

The first two steps are essential for solving the standard time-dependent LQ-problem. We see that the performance of the Newton step in our situation is essentially equivalent to solving the standard LQ-problem plus the system of linear algebraic equations. Similar results hold for a more general case of linear-quadratic inequality constraints ([8]).

5 Concluding Remarks

In the present paper we have considered a long-step path-following algorithm for the infinite-dimensional quadratic programming problem. We found an estimate for the number of Newton steps which coincides with the best known ([4] in the finite-dimensional case. We then considered an example of the linear-quadratic time-dependent problem with linear inequality constraints. We have shown that in this concrete situation the performance of the Newton step is essentially equivalent to solving the standard LQ-problem (without inequality constraints). Numerical experiments in the finite-dimensional situation show that in the case of long-step path-following algorithms the number of Newton steps grows very slowly with m (the number of inequality constraints). If this trend proves to be true in the infinite-dimensional situation, we will have a powerful tool for solving various optimal control problems involving inequality constraints on control and the state space constraints.

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