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Least Squares Pole Assignment by Memory-less Output Feedback*

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Abstract

In this paper, a pole assignment problem of linear time invariant control system by memory-less output feedback is posed as a least squares poles assignment problem and analysed. The cost functions are appropriately modified so as to obtain existence of global minimum and convergence of the corresponding gradient flow. This approach is also extended to accommodate the output pole assignment problem with insensitivity against disturbances in system parameters. The relation between the modified cost functions and the original pole assignment is revealed. The proposed approach is compared with other existing approaches and illustrated by numerical results.

1 Introduction

The pole placement problem of linear systems has been an important issue for decades. Compared to the pole placement via state feedback or dynamic feedback, the same problem via output feedback is more complex. Different approaches from linear system theory, combinatorics, complex function theory and algebraic geometry are used to explore this problem and yet the results are not complete. See the survey paper [3] for some details. There are some recent works concerning system pole assignment by memory-less output feedback. For example, see [4], [5], [6], [7] and references cited therein. For a linear time invariant control system with n states, m inputs and l outputs, it is known that a necessary condition under which system poles can be assigned freely by output feedback is $ml \geq n$. This condition is sufficient if the inequality is strictly satisfied. See [4] for details. These approaches emphasize finding necessary and sufficient condition to the pole assignment problem by memory-less output feedback. Computation of the required feedback gain by these approaches is formidable.

In [5], [6], another approach is introduced to consider this problem as a least squares optimisation problem. There some cost functions were devised to force the closed-loop system poles as close to the target system poles as possible in a least square sense. This approach has several advantages. First, it converts the exact pole assignment problem into an optimisation problem which can be solved numerically by many mature software packages. Second, Even though exact pole assignment by output feedback may be not feasible, it can always provide a reasonable alternative which is optimal with respect to the least squares of the difference between system poles and target poles. Third, it may exclude complex and profound mathematical computation on Grassmannian manifold, which is not widely known to engineers. Furthermore, the corresponding eigenstructure is obtained automatically if the pole assignment problem is exactly solved.

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However, the cost function used in [5] has no compact sublevel sets. In fact, some numerical simulations have been conducted by us which shows that a trajectory of a negative gradient system does not converge to an equilibrium point. To remove this drawback, [6] restricts system to be symmetric so that the state transformation matrices belong to a compact manifold. Work remains to be done to cope with the drawback in general case.

The sum of determinants of the return difference of the closed-loop system at all poles is used as a cost function in [7]. This cost function is transformed on a Grassmannian manifold to a sum of n items. Each item is a product of two functions. One is a positive function of a feedback gain matrix and the target poles, and another is a determinant of a Hermitian projection matrix related to target poles plus another constant matrix. Then, an auxiliary cost function is constructed by replacing the positive functions with constant parameters. For this auxiliary cost function, sublevel sets are compact and hence the existence of global minimum and convergence of gradient system are guaranteed.

In this paper, a bound condition on state transformation matrix and its inverse matrix is imposed. This condition is always satisfied in practical settings. In fact, if a global minimum exists, the corresponding state transformation matrix and its inverse matrix can be bounded by some constant real matrix. Solely on the bases or by further removing some redundancy, two type of modifications are made to the cost function. The modified cost functions have the required properties such as the existence of global minimum, convergence of gradient flow, etc. Furthermore, the exact solution if it exists can also obtained as the global minimum of the modified cost functions.

For the pole assignment problem, another important issue is the robust property against parameter disturbance. Tens of papers are published to address the problem by static state feedback. See [8] and [9]. One of the critical techniques is that the eigenvectors corresponding to each closed-loop eigenvalue form a linear subspace and therefore can be easily parametrized. But this technique is not available to the problem by memory-less output feedback. In this paper, the proposed approach can also be adjusted to accommodate the robust pole assignment problem by memory-less output feedback.

2 Output Feedback Optimal Least Squares Pole assignment

Consider linear time invariant systems with output

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $x(t), y(t), u(t)$ are system state, output and input with dimension n, l, m respectively, A, B, C are constant matrices with appropriate dimensions. Without loss of generality, assume that B, C are of full column and row rank respectively. Given a set of n complex numbers in which those pure complex numbers are in conjugate pairs, the pole assignment problem by memory-less output feedback is to find a output feedback $u = Fy$ such that the closed-loop system poles coincides with the set of complex numbers.

2.1 Least squares pole assignment

Given a real $n \times n$ matrix Γ, the least squares optimal assignment problem by memory-less output feedback is posed as [5]:

Problem 1 *A pair (T, F), where T is a state transformation matrix and F is a memory-less output feedback gain matrix, is said to optimally (least squares) assign the closed-loop system poles to those* *of the matrix* Γ *if it minimises the following cost function:*

$$J(T,F) := \| T(A+BFC)T^{-1} - \Gamma \|_F^2 \tag{3}$$

where the matrix norm is Frobenious norm.

Problem 1 has already been analysed in [5]. All transformed closed-loop systems $T(A+BFC)T^{-1}$ form a smooth manifold which is an orbit of Lie group $GL(n,R)\times R^{m\times l}$, where $GL(n,R)$ is the general linear group containing all nonsingular $n\times n$ matrix with matrix product operation. The group action of $R^{m\times l}$ is a matrix summation operation. With respect to the normal Riemannian metric [5] the gradient is computed as

$$\begin{aligned}\text{grad}J(T,F) = -([T(A+BFC)T^{-1} - \Gamma, T^{-\top}(A+BFC)T^{\top}]T,\\ B^{\top}T^{\top}(T(A+BFC)T^{-1} - \Gamma)T^{-\top}C^{\top})\end{aligned} \tag{4}$$

where $T^{-\top} := (T^{-1})^{\top}$. However, the cost function $J(T,F)$ has no compact sublevel sets. It is easily seen that

$$J(T,F) = J(\lambda T, F)$$

for any nonzero constant real number λ, one can always construct a sequence in the sublevel set

$$\mathcal{S}(\varepsilon) := \{(T,F) \in GL(n,R)\times R^{m\times l} : J(T,F) \leq \varepsilon\},$$

say, $(kT_0, F_0), k = 1,2,\ldots,$ for any $(T_0,F_0) \in \mathcal{S}(\varepsilon)$ that is not convergent. If the difference of eigenvalues $\| T(A+BFC)T^{-1} - \Gamma \|_F$ is considered as a function of the triple $(A(T,F), B(T,F), C(T,F))$ which is the orbit of Lie group $GL(n,R)\times R^{m\times l}$. i.e.,

$$\hat{J}(A,B,C) = \| A - \Gamma \|_F^2 .$$

Then, $\hat{J}(A,B,C)$ does not have compact sublevel sets either, because

$$B(\lambda T, F) = \lambda TB \longrightarrow +\infty \text{ as } \lambda \longrightarrow +\infty.$$

Therefore, the technique used to prove Theorem 3.2 in Chapter 5 of [5] is not available to us here.

If we consider the negative gradient system of $J(T,F)$ defined as

$$(\dot{T}, \dot{F}) = -\text{grad}J(T,F), \tag{5}$$

Only the following weak results hold

Lemma 1 *To the system (5),*
(1). any trajectory has no finite escape time;
(2). along any trajectory of (5) the essential upper bound $\bar{U}(t)$ *of gradient norm on* $[0,\infty)$ *converges to zero, where* $\bar{U}(t)$ *is given by*

$$\bar{U}(t) := \{M \in R : \| \text{grad}J(T(t),F(t)) \|_F < M \textit{ almost everywhere in } [t,\infty).\}$$

Proof. Since the gradient vector is smooth, a trajectory of the negative gradient system exists at least locally. Assume it exists in $[0,t]$. Then,

$$\begin{aligned}\| (T(t),F(t)) \|_F^2 &= \| \int_0^t \text{grad}J(T(s),F(s))ds \|_F^2 \\ &\leq \int_0^t \| \text{grad}J(T(s),F(s)) \|_F^2 \, ds \int_0^t ds \\ = -t\int_0^t \frac{dJ}{ds}ds &= (J(T(0),F(0)) - J(T(t),F(t)))t \leq J(T(0),F(0))t\end{aligned}$$

which establishes the claim (1).

By noticing that

$$J(T(t), F(t)) = -\int_0^t \| \operatorname{grad} J(T(s), F(s)) \|_F^2 \, ds$$

is decreasing and positive one obtains the claim (2). □

In fact, the numerical computation of some examples conducted by us shows that the gradient along a trajectory may diverge. Fore details of the computation results of one example, see Section 4. Thus it motivates us to propose modified approaches in the following subsection.

2.2 Modified least squares pole assignment problems

In Subsection 2.1 it is observed that there exists a redundancy in the cost function $J(T, F)$. i.e., $J(T, F) = J(\lambda T, F)$. One may expect that by removing all redundancy in the cost function the local minimum can be unique and hence the global minimum. However, it is already known that the output feedback gain matrix which achieves the closed-loop system pole assignment is not unique. Example 5.2 in [3] shows that there are four output feedback gain matrices that assign the closed system poles to the open-loop system poles. Furthermore, these output feedback gain matrices are not proportional to each other.

If there is a global minimum of the cost function $J(T, F)$, the global minimum can always be achieved by a state transformation matrix $T^\star$ which is of unit Frobenious norm and its inverse norm is bounded by some constant positive real number M. Therefore it is reasonable to minimise the cost function $J(T, F)$ in a set that the state transformation matrices are of unit Frobenious norm and the norm of their inverse matrices is bounded, or in a set that the norm of both state transformation matrices and their inverse matrices is bounded. In the following, accordingly two types of modification are made to the minimisation problem.

First consider the set of all state transformation matrices with unit Frobenious norm denoted as:

$$\mathcal{U}_F := \{T \in GL(n, R) : \| T \|_F = 1\}.$$

It is known [5] that the $GL(n, R) \times R^{m \times l}$ is a orbit of itself considered as a Lie group by the following Lie group action

$$\begin{aligned} \beta : (GL(n, R) \times R^{m \times l}) \times (GL(n, R) \times R^{m \times l}) &\longrightarrow (GL(n, R) \times R^{m \times l}) \\ ((\bar{T}, \bar{F}), (T, F)) &\longmapsto (\bar{T}T, \bar{F} + F) \end{aligned}$$

This set is a smooth manifold with the manifold structure induced by Lie group action. Its tangent space at (T, F) can be calculated as

$$T_{(T,F)}(GL(n, R) \times R^{m \times l}) = \{(XT, L) : X \in R^{n \times n}, L \in R^{m \times l}\}$$

$\mathcal{U}_F$ is a subset of $GL(n, R)$ and has the following properties.

Lemma 2 *$\mathcal{U}_F$ is a connected submanifold in $GL(n, R)$. Its tangent space at T is calculated as*

$$T_T(\mathcal{U}_F) = \{XT : X \in R^{n \times n} \text{ and } tr(TT^\top X) = 0\}$$

Proof. First let us show $\mathcal{U}_F$ is connected. For any two matrices $T_1, T_2 \in \mathcal{U}_F \subset GL(n, R)$, there exists a matrix $X \in R^{n \times n}$ such that $T_2 = XT_1$. Given any $\lambda, 0 \leq \lambda \leq 1$, consider the following equation

$$\| (\lambda X + p(\lambda)I)T_1 \|_F = 1. \tag{6}$$

Its solution is

$$p(\lambda) = \{-\lambda tr(T_1^\top X^\top T_1 + T_1^\top X T_1) \pm [4 + \lambda^2(tr^2(T_1^\top X^\top T_1 + T_1^\top X T_1))]^{1/2}\}/2 \tag{7}$$

Since

$$tr(T_1^\top X^\top T_1 + T_1^\top X T_1) \leq 2[\| XT_1 \|_F \| T_1 \|_F] = 2,$$

it follows that $0 \leq p_1(\lambda) \leq 1$ where

$$p_1(\lambda) := -\lambda tr(T_1^\top X^\top T_1 + T_1^\top X T_1) + [4 + \lambda^2(tr^2(T_1^\top X^\top T_1 + T_1^\top X T_1))]^{1/2}\}/2,$$

It is obvious that $p_1(\lambda)$ is continuous and $p_1(1) = 0, p_1(0) = 1$. Let

$$T(\lambda) = (\lambda X + p_1(\lambda) I) T_1.$$

It is a continuous curve in $\mathcal{U}_F$ that connects T_1 and T_2. Therefore, $\mathcal{U}_F$ is connected.

Since $d(\| T \|_F) = d(tr(T^\top T))$ is a co-tangent vector field which is nonzero everywhere, the co-distribution spanned by $d(\| T \|_F)$ is 1-dimensional. Therefore, $\mathcal{U}_F$ is a $(n^2 - 1)$-dimensional submanifold of $GL(n, R)$.

The tangent space of $\mathcal{U}_F$ at T is calculated as

$$T_T(\mathcal{U}_F) = \{d(\| T \|_F)\}^\perp = \{XT : X \in R^{n\times n} \text{ and } tr(TT^\top X) = 0\}.$$

The proof is complete. □

To obtain a better convergent property of cost function, a bound condition can be imposed on $\| T^{-1} \|_2$ for the following reasons. First, it is already known that if $\| T^{-1} \|_2$ is large, the closed-loop system poles are very sensitive to system parameter disturbance. The detail will be revealed in next section or referred to [8] and [9]. Second, for any (T, F) that minimises the cost function, $\| T^{-1} \|_2$ is always bounded by some appropriate constance. Then, the following problem can be proposed to accommodate the system optimal pole assignment instead of Problem 1

Problem 2

$$\begin{array}{lll} \textit{minimise:} & \tilde{J}(T,F) = \| T(A + BFC)T^{-1} - \Gamma \|_F^2 & \\ \textit{subject to:} & T \in \mathcal{U}_F, \| T^{-1} \|_2 \leq M, \textit{ and } & F \in R^{m\times l} \end{array} \tag{8}$$

Another type of problem can also be posed similarly, however, with redundancy.

Problem 3

$$\begin{array}{ll} \textit{minimise:} & \tilde{J}(T,F) = \| T(A + BFC)T^{-1} - \Gamma \|_F^2 \\ \textit{subject to:} & T \in GL(n,R), \| T \|_2 \leq M, \| T^{-1} \|_2 \leq M, \textit{ and } \quad F \in R^{m\times l} \end{array} \tag{9}$$

Those are constrained optimisation problems. They can be converted into unconstrained optimisation problems by introducing additional penalty items in the cost functions. i.e., the cost functions can be defined as

$$J_1(T,F) := \tilde{J}(T,F) + p\{| M - \| T^{-1} \|_F^2 | - M + \| T^{-1} \|_F^2\}^2 \tag{10}$$

and

$$\begin{aligned} J_2(T,F) := \tilde{J}(T,F) &+ p_1\{| M - \| T \|_F^2 | - M + \| T \|_F^2\}^2 \\ &+ p_2\{| M - \| T^{-1} \|_F^2 | - M + \| T^{-1} \|_F^2\}^2 \end{aligned} \tag{11}$$

where p, p_1, p_2 are tuning parameters. Frobenious norm is used here instead of two norm for two reasons. One reason is that

$$\| T \|_2 \leq \| T \|_F \leq n \| T \|_2 .$$

A similar inequality is also available for T^{-1}. Another reason is that Frobenious norm is smooth, easier to calculate, and in harmony with the first term in the cost functions.

Now the following lemma holds:

Lemma 3 *The cost functions $J_i(T, F), i = 1, 2$ have compact sublevel sets in $\mathcal{U}_F \times R^{m\times l}$ or $GL(n, R) \times R^{m\times l}$ respectively. i.e., for any $\varepsilon > 0$, the sets*

$$\mathcal{S}_1(\varepsilon) := \{(T, F) \in \mathcal{U}_F \times R^{m\times l} : J_1(T, F) \leq \varepsilon\}$$

and

$$\mathcal{S}_2(\varepsilon) := \{(T, F) \in GL(n, R) \times R^{m\times l} : J_2(T, F) \leq \varepsilon\}$$

are compact.

Proof. For any sequence $\{(T_n, F_n)\}_{n=1}^{\infty}$ in the sublevel set $\mathcal{S}_1$, since $\mathcal{U}_F$ is bounded, it follows that there exist a convergent subsequence of $\{T_n\}_{n=1}^{\infty}$. Without loss of generality, we assume

$$\lim_{n\to\infty} T_n = T^\star.$$

Since

$$\| T^{-1} \|_F^2 \leq M + (\frac{\varepsilon}{2p})^{1/2},$$

it is clear that $T^\star$ is the limit of T_n and invertible. Therefore, there exist two positive numbers $M_1, N_1 > 0$, such that for any $n > N_1$,

$$\| T_n \|_F \leq M_1, \| T_n^{-1} \|_F \leq M_1.$$

Since for any $m \times l$ matrix Y,

$$\| Y \|_2 \leq \| Y \|_F \leq \min\{m, l\} \| Y \|_2,$$

it follows that

$$\begin{aligned} \| F_n \|_F \leq \min\{m, l\} \| (B^\top B)^{-1} B^\top BFCC^\top (CC^\top)^{-1} \|_2 \leq \\ \min\{m, l\} \| (B^\top B)^{-1} B^\top \|_2 \| C^\top (CC^\top)^{-1} \|_2 \| T_n^{-1}(T_n(A + BFC)T_n^{-1} - \Gamma)T_n - A + T_n^{-1}\Gamma T_n \|_2 \\ \leq \min\{m, l\} \| (B^\top B)^{-1} B^\top \|_2 \| C^\top (CC^\top)^{-1} \|_2 \{\| T_n^{-1} \tilde{J}_1(T, F) T_n \|_2 + \| T_n^{-1}\Gamma T_n \|_2 + \| A \|_2\} \quad (12) \end{aligned}$$

Therefore, F_n is also bounded. Hence there exists a convergent subsequence of $\{F_n\}_{n=1}^{\infty}$.

Now it follows the continuity of Frobenious norm that if $\{(T_n, F_n)\}_{n=1}^{\infty} \longrightarrow (T^\star, F^\star)$,

$$(T^\star, F^\star) \in \mathcal{S}_1(\varepsilon).$$

Similarly, one can proves that $\mathcal{S}_2(\varepsilon)$ is compact. □

By the compactness of sublevel sets, the existence of global minimum of the corresponding cost function is guaranteed. In order to obtain the gradient of cost function $\tilde{J}_1(T, F)$, we need to equip the manifold $\mathcal{U}_F \times R^{m\times l}$ with a proper Riemannian metric.

Given $(T, F) \in \mathcal{U}_F \times R^{m\times l}$, let σ_1 denote the following map:

$$\sigma_1 : R^{n\times n} \longrightarrow R$$
$$X \longmapsto \sigma_1(X) = tr(TT^\top X)$$

For any $X \in R^{n\times n}$, let

$$X = X^{\circ} + X^{\perp}$$

where

$$X^{\circ} \in \ker(\sigma_1), X^{\perp} \in [\ker(\sigma_1)]^{\perp}.$$

For $(X_1T, F_1), (X_2T, F_2) \in T_T(\mathcal{U}_F) \times R^{m\times l}$, the Riemannian can be defined as

$$\ll (X_1T, F_1), (X_2T, F_2) \gg := 2tr((X_1^{\perp})^{\top} X_2^{\perp}) + 2tr(F_1^{\top} F_2)$$

Now we have the following theorem.

Theorem 1 *For the cost function $J_1(T, F)$, the following results hold:*
(1). Its gradient is calculated as

$$\begin{aligned}\text{grad}J_1 \mid_{(T,F)} = (\mathcal{P}_1([T(A+BFC)T^{-1} - \Gamma, T^{-\top}(A+BFC)T^{\top}] - 2p(\mid M - \parallel T^{-1} \parallel_F^2 \mid - \\ M + \parallel T^{-1} \parallel_F^2)T^{-\top}T^{-1})T, B^{\top}T^{\top}[T(A+BFC)T^{-1} - \Gamma]T^{-\top}C^{\top})\end{aligned} \quad (13)$$

where $[A, B]$ is the matrix Lie bracket of any square matrix A, B defined as

$$[A, B] = AB - BA.$$

$\mathcal{P}_1$ is the projection operator from $R^{n\times n}$ to $[\ker(\sigma_1)]^{\perp}$. It can be calculated by the following formula:

$$vec(\mathcal{P}_1(X)) = \frac{tr(TT^{\top}X)}{\parallel TT^{\top} \parallel_F^2} vec(TT^{\top}) \quad (14)$$

(2). Equilibrium point set $\mathcal{E}_1$ is

$$\begin{aligned}\mathcal{E}_1 = \{(T, F) : \mathcal{P}_1([T(A+BFC)T^{-1} - \Gamma, T^{-\top}(A+BFC)T^{\top}] - 2p(\mid M - \parallel T^{-1} \parallel_F^2 \mid - \\ M + \parallel T^{-1} \parallel_F^2)T^{-\top}T^{-1})T = 0, B^{\top}T^{\top}[T(A+BFC)T^{-1} - \Gamma]T^{-\top}C^{\top} = 0\}.\end{aligned}$$

(3). The negative gradient system is defined as

$$(\dot{T}, \dot{F}) = -\text{grad}J_1 \mid_{(T,F)} .$$

Along its trajectory starting from any point $(T_0, F_0) \in \mathcal{U}_F \times R^{m\times l}$, the cost function decreases strictly at any non-equilibrium point.
(4). Along any trajectory, the gradient converges to a zero matrix.
(5). Any trajectory converges to a connected subset of $\mathcal{E}_1$.

The proof is standard. Therefore, it is omitted.

Remark If (T_1, F_1) exactly assigns system poles, then, there exists a positive number $M^{\star} > 0$ such that for any $M \geq M^{\star}$, $J_1(T, F) \geq J_1(T_1, F_1) = 0$. In fact, one can choose $M^{\star} = \parallel T_1^{-1} \parallel_F^2$. □

Similarly, let the Riemannian metric of $GL(n, R) \times R^{m\times l}$ be defined as:

$$\ll (X_1T, F_1), (X_2T, F_2) \gg := 2tr((X_1)^{\top} X_2) + 2tr(F_1^{\top} F_2)$$

We have the following theorem:

Theorem 2 *For the cost function $J_2(T,F)$, the following results hold:*
(1). Its gradient is calculated as

$$\begin{aligned}\text{grad}J_2\mid_{(T,F)} = (([T(A+BFC)T^{-1}-\Gamma, T^{-\top}(A+BFC)T^{\top}] + 2p_1(\mid M- \parallel T \parallel_F^2\mid + \\ \parallel T \parallel_F^2 - M)TT^{\top} - 2p_2(\mid M- \parallel T^{-1} \parallel_F^2\mid + \parallel T^{-1} \parallel_F^2 + M)T^{-\top}T^{-1})T, \\ B^{\top}T^{\top}(T(A+BFC)T^{-1}-\Gamma)T^{-\top}C^{\top})\end{aligned} \quad (15)$$

(2). Equilibrium point set $\mathcal{E}_2$ is

$$\begin{aligned}\mathcal{E}_2 = \{(T,F) : ([T(A+BFC)T^{-1}-\Gamma, T^{-\top}(A+BFC)T^{\top}] + 2p_1(\mid M- \parallel T \parallel_F^2\mid + \\ \parallel T \parallel_F^2 - M)TT^{\top} - 2p_2(\mid M- \parallel T^{-1} \parallel_F^2\mid + \parallel T^{-1} \parallel_F^2 + M)T^{-\top}T^{-1}) = 0, \\ B^{\top}T^{\top}(T(A+BFC)T^{-1}-\Gamma)T^{-\top}C^{\top} = 0\}.\end{aligned}$$

(3). The negative gradient system is defined as

$$(\dot{T},\dot{F}) = -\text{grad}J_2\mid_{(T,F)}.$$

Along its trajectory starting from any point $(T_0,F_0)\in GL(n,R)\times R^{m\times l}$, the cost function decreases strictly at any non-equilibrium point.
(4). Along any trajectory, the gradient converges to a zero matrix.
(5). Any trajectory converges to a connected subset of $\mathcal{E}_2$.

Remarks These two cost functions $J_1(T,F)$ and$J_2(T,F)$ may not have the same equilibrium point. However, it can be shown by simple computation that
(1). if $(T_1,F_1) = \arg\min J_1$, then, $\min J_2 \le J_2(T_1,F_1)$ given $M \ge 1$.
(2). if $(T_2,F_2) = \arg\min J_2$, then, $\min \bar{J}_1 \le J_1(T_2,F_2)$ where $\bar{J}_1$ is the same as J_1 but parameter M is chosen as $2M$. □

3 Robust Output Feedback Poles Assignment

If system parameters are not precisely known or subject to disturbances, the closed-loop system sensitivity should be take into consideration in system pole assignment problem. This is the so-called robust pole assignment problem. There are many papers devoted to address this problem in a state feedback context. For example, see [8] and [9]. In the state feedback settings, eigenvectors of the closed-loop system matrix form some linear subspace [8]. On the basis many algorithms can be designed to minimise the two norm or other norm of the state transformation matrix, which is a matrix consisting of the closed-loop eigenvectors. If only output feedback can be used, those eigenvectors no longer form linear spaces. Therefore, techniques as used in [8] and [9] are not available to the robust output feedback pole assignment problem. In this section, two optimisation problems will be proposed in the same way as in section 2 to handle the problem by output feedback.

Let (A,B,C) denote the nominal system parameters and $(\Delta A,\Delta B,\Delta C)$ denote parameter disturbances. Then,

$$\begin{aligned}&\parallel T(A+BFC)T^{-1} - T(A+\Delta A+(B+\Delta B)F(C+\Delta C))T^{-1} \parallel_2 \\ &= \parallel T\{\Delta A + \Delta BFC + BF\Delta C + \Delta BF\Delta C\}T^{-1} \parallel_2 \\ &\le \parallel T \parallel_F \parallel T^{-1} \parallel_F \{\parallel \Delta A \parallel_2 + \parallel F \parallel_2 (\parallel \Delta B \parallel_2 \parallel C \parallel_2 + \parallel B \parallel_2 \parallel \Delta C \parallel_2 + \parallel \Delta B \parallel_2 \parallel \Delta C \parallel_2)\}\end{aligned}$$

Therefore, the objective of insensitivity to parameter disturbance and pole assignment can be jointly accommodated in the minimisation problem of the following cost functions:

$$J_{r1}(T,F) = k_1 \parallel T(A+BFC)T^{-1} - \Gamma \parallel_F + k_2 \parallel T^{-1} \parallel_F^2 + k_3 \parallel F \parallel_F^2 \quad (16)$$

where $(T, F) \in \mathcal{U}_F \times R^{m\times l}$, and $k_1, k_2, k_3 \geq 0$ are tuning parameters and

$$J_{r2}(T,F) = k_1 \parallel T(A+BFC)T^{-1} - \Gamma \parallel_F + k_2 \parallel T \parallel_F^2 + k_3 \parallel T^{-1} \parallel_F^2 + k_4 \parallel F \parallel_F^2 \tag{17}$$

where $T \in GL(n, R)$ and $F \in R^{m\times l}$.

The following results can be obtained by the same techniques in Section 2.

Lemma 4 *Cost functions $J_{ri}, i = 1, 2$ have compact sublevel sets. Therefore, the global minima of cost functions can be reached in any non-empty sublevel sets.*

Theorem 3 *For cost function $J_{ri}, i = 1, 2$, the following results hold:*
(1). Their gradients with respect to the corresponding Riemannian metrics are calculated as

$$\begin{aligned}\mathrm{grad}J_{r1} \mid_{(T,F)} = (&\mathcal{P}_1(k_1[T(A+BFC)T^{-1} - \Gamma, T^{-\top}(A+BFC)T^{\top}] - k_2T^{-\top}T^{-1})T,\\ &B^{\top}T^{\top}[T(A+BFC)T^{-1} - \Gamma]T^{-\top}C^{\top} + k_3F) \qquad (18)\\ \mathrm{grad}J_{r2} \mid_{(T,F)} = (&k_1[T(A+BFC)T^{-1} - \Gamma, T^{-\top}(A+BFC)T^{\top}] + k_2TT^{\top} - k_3T^{-\top}T^{-1})T,\\ &k_1B^{\top}T^{\top}[T(A+BFC)T^{-1} - \Gamma]T^{-\top}C^{\top} + k_4F) \qquad (19)\end{aligned}$$

(2). Equilibrium point set $\mathcal{E}_{r1}$ and $\mathcal{E}_{r2}$ are

$$\begin{aligned}\mathcal{E}_{r1} = \{(T,F) : &\mathcal{P}_1(k_1[T(A+BFC)T^{-1} - \Gamma, T^{-\top}(A+BFC)T^{\top}] - k_2T^{-\top}\\ &\cdot T^{-1}) = 0, [T(A+BFC)T^{-1} - \Gamma]T^{-\top} + k_3F = 0\}.\\ \mathcal{E}_{r2} = \{(T,F) : &k_1[T(A+BFC)T^{-1} - \Gamma, T^{-\top}(A+BFC)T^{\top}] + k_2TT^{\top} - k_3T^{-\top}T^{-1}) = 0,\\ &k_1B^{\top}T^{\top}[T(A+BFC)T^{-1} - \Gamma]T^{-\top}C^{\top} + k_4F = 0\}.\end{aligned}$$

(3). The negative gradient systems are defined as

$$(\dot{T}, \dot{F}) = -\mathrm{grad}J_{r1} \mid_{(T,F)},$$

$$(\dot{T}, \dot{F}) = -\mathrm{grad}J_{r2} \mid_{(T,F)} .$$

Along their trajectories starting from any point $(T_0, F_0) \in \mathcal{U}_F \times R^{m\times l}$ or $(T_0, F_0) \in GL(n, R) \times R^{m\times l}$, the cost functions decrease strictly at any non-equilibrium point.
(4). Along any trajectory, the gradients converges to a zero matrix.
(5). Any trajectory of gradient system of J_{ri} converges to a connected subset of $\mathcal{E}_{ri}, i = 1, 2$.

To justify these results, we have the following lemma.

Lemma 5 *If the system poles can be exactly assigned by output feedback, then, by fixing tuning parameters k_2, k_3 in J_{r1} or k_2, k_3, k_4 in J_{r2} and letting $k_1 \longrightarrow +\infty$, the global minimal point has a subsequence that converges to a point $(T^\star, F^\star)$ which minimises the cost functions while exactly assignment system poles.*

Proof. If k_2, k_3 are fixed, the cost function $J_{r1}(T, F)$ is nondecreasing with respect to k_1. For any $(\bar{T}, \bar{F})$ that exactly assigns system poles,

$$\min_{T\in\mathcal{U}_F, F\in R^{m\times l}} J_{r1}(T,F) \leq C_1 := k_2 \parallel \bar{T}^{-1} \parallel_F + \parallel \bar{F} \parallel_F$$

holds for any positive k_1, k_2, k_3. Let $\{k_1^i\}_{i=1}^{\infty}$ be any sequence that tends to infinite, and the corresponding global minimal points of $J_{r1}(T, F)$ be denoted as (T_i, F_i). For any $i > 0$, (T_i, F_i) belongs to the sublevel set

$$\mathcal{S}_i(C_1) := \{(T,F) \in \mathcal{U}_F \times R^{m\times l} : J_{r1}(T,F) \leq C_1\}$$

Since

$$\mathcal{S}_i(C_1) \subseteq \mathcal{S}_{i-1}(C_1) \subseteq \mathcal{S}_1(C_1),$$

from the compactness of $\mathcal{S}_1(C_1)$ it follows that there exist a convergent subsequence of (T_i, F_i). Without loss of generality, assume that $(T_i, F_i) \longrightarrow (T^\star, F^\star) \in \mathcal{S}_1(C_1)$. By $J_{r1}(T_i, F_i) \leq C_1$ for any $i > 1$ we know that

$$\lim_{i\to\infty} \| T_i(A + BF_iC)T_i^{-1} - \Gamma \|_F = 0.$$

The optimality of $(T^\star, F^\star)$ follows by noticing that $(\bar{T}, \bar{F})$ can be any point which exactly assigns system poles.

The proof of the result corresponding to J_{r2} follows in exactly the same way. □

4 Numerical Examples

In the previous two sections, several cost functions and their gradient formulas are given to optimally assign system poles. A bound condition is imposed on the state transformation matrix and its inverse matrix to obtain a convergence property of cost function. If a cost index term related to a robustness against system parameter disturbances is introduced, the bound condition is not required to obtain the convergence of gradient flow. In this section, the proposed method is illustrated for the system that is considered in Example 5.2 in [3].

Consider Example 5.2 in [3]. where

$$A = \begin{pmatrix} 1 & 2 & -1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 2 & 3 & 1 \\ -1 & 1 & 2 & 3 & -1 & 1 \\ 3 & 2 & 1 & -3 & -1 & -2 \\ -1 & -3 & -2 & -1 & 1 & -3 \\ -2 & -1 & 1 & 3 & 2 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

In order to show the convergence property of gradient system, the optimal solution is obtained by directly calculating numerical solutions to these negative gradient systems. We use ODE45 command in Matlab to compute numerical solutions of several gradient systems.

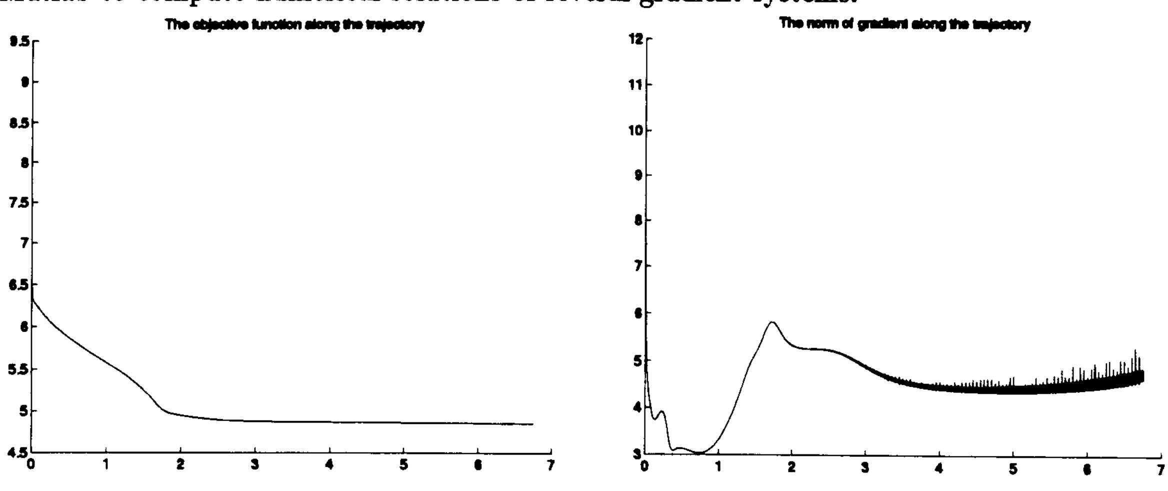

Figure 1. The computation results of gradient system of $J(T, F)$

In Figure 1, the function $log(log(J(T,F))$ and $log(\|\text{ grad}J\ \|_F^2)$ are plotted versus time t. The cost function is seen to be decreasing. However, one may observe that the algorithm is not stable and the norm of the gradient does not converge to zero.

In Figure 2 and Figure 3, the gradient systems for indexes J_1 and J_2 are computed. The costs $log(log(J_1))$, gradJ_1, $log(log(J_2))$, and gradJ_2 are plotted versus t. In these figures, not only cost functions are shown to be decreasing along the trajectories of gradient systems, but also Frobenious norms of gradients are shown to be converging to zero "exponentially".

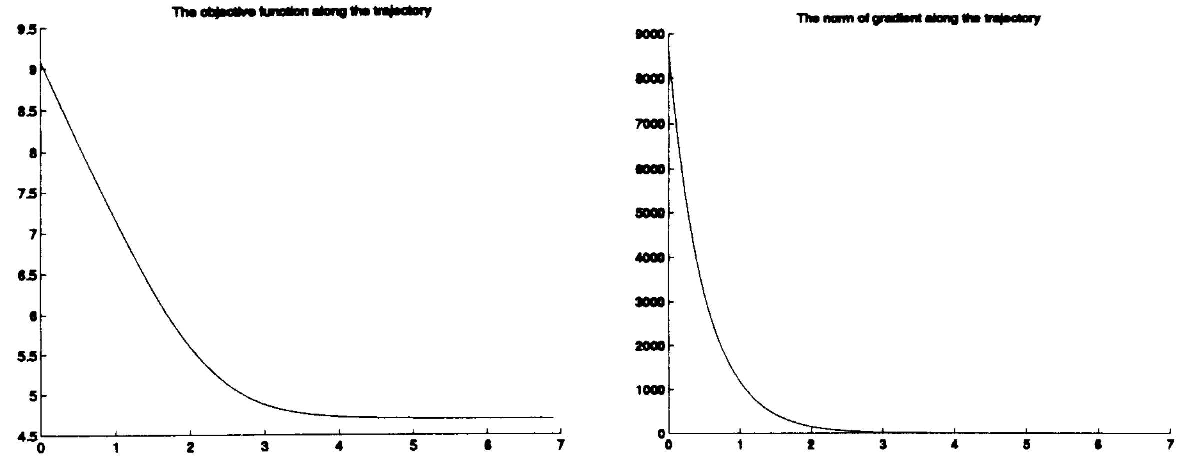

Figure 2. The computation results of gradient system of $J_1(T,F)$

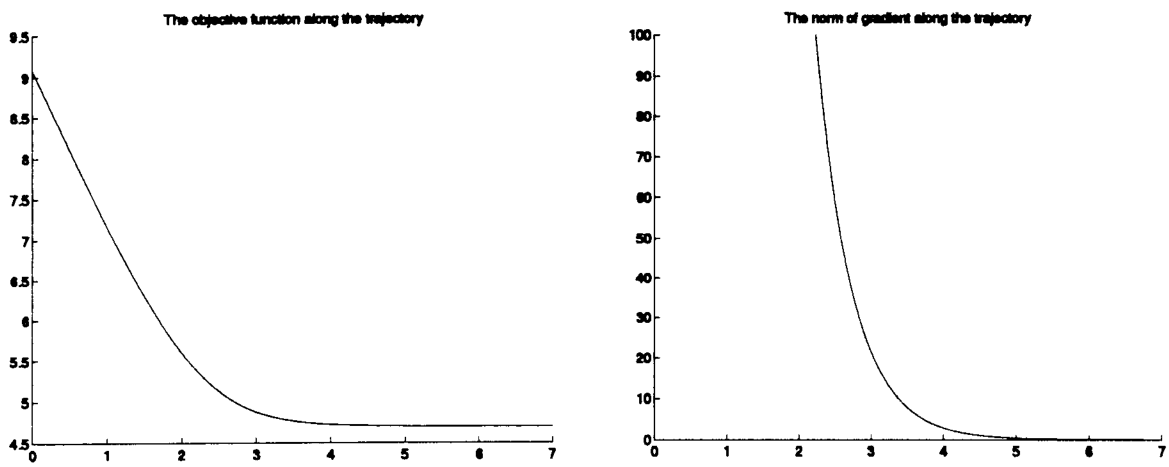

Figure 3. The computation results of gradient system of $J_2(T,F)$

If system is large, say, n states, m inputs and l outputs, the dimension of the gradient system is (n^2+ml). Therefore, for large systems, it is computationally expensive to solve the group of ordinary differential equations. Newton method involves computation of the inversion of Hessian which is also computationally expensive if the number of variables is large. Besides, the Hessian is not always invertible. One can use a truncated Newton method, a conjugate gradient method or a gradient method to keep computation and memory at each iterate low. These computational methods are addressed by many standard books. For example, see [10]. Details are not explicitly addressed in this paper.

147

5 Conclusion

In this paper, several cost functions for optimally assign the closed-loop system poles in a least squares sense were constructed by penalising a state transformation matrix and its inverse matrix to be bounded or a state transformation matrix and its inverse matrix, and an output feedback gain matrix to be small. Gradient with respect to two types of Riemannian metric were given and convergence property of gradient systems and existence of global minima were obtained. Numerical example were also given to illustrate the effectiveness of the proposed approach.

Compared to existing works such as [6], [5], [7], the proposed approach is attractive in the following aspects. First, systems considered here are not restricted to a subclass of linear time invariant control system such as the class of symmetric systems. Second, the existence of global minimum and convergence of gradient system trajectory are guaranteed. Third, cost functions constructed are closely related to original pole assignment problem. In fact, if an exact solution of output pole assignment problem is feasible, it can be approximated by the methods proposed in this paper. Fourth, insensitivity issues against system parameter disturbances can also be cast in the same frame work.

It is worth noting that only local minima are achieved here. Some computation methods can be applied to search for global minimum. But these methods are usually expensive in computation. It needs further research to find an economical method to search for a global optimal solution.

References

[1] W. M. Wonham, On pole assignment in multi-input, controllable linear systems, IEEE Trans. Automatic. Contr., Vol. AC-12, No. 6, pp. 660-665, 1967.

[2] T. J. Owens and J. O'reilly, Parametric static-feedback control with response insensitivity, Int. J. Contr., Vol 45, pp. 791-809, 1987.

[3] C. I. Byrnes. Pole placement by output feedback. Three Decades of Mathematical Systems Theory, volume 135 of Lecture Notes in Control and Information Sciences, pages 31-78. Springer-Verlag, 1989.

[4] X. Wang. Pole placement by static output feedback. Journal of Mathematical Systems, Estimation and Control, Vol. 2, No. 2, pp. 205-218, 1992.

[5] Uwe Helmke and John B. Moore, Optimisation and Dynamical Systems, Springer-Verlag, London, 1994.

[6] R. E. Mahony, U. Helmke, System assignment and pole placement for symmetric realisations. Journal of Mathematical Systems, Estimation, and Control, No. 5, pp. 267-270, 1995.

[7] K. Hüper and U. Helmke, Geometrical methods for pole assignment algorithms. Proc. 34th IEEE Conference on Decision and Control. New Orleans, Dec. 1995.

[8] J. Kautsky, N. K. Nichols and P. Van Dooren, Robust pole assignment in linear state feedback, Internat. J. Control. 41(1985), pp. 1129-1155.

[9] R. Byers and S. G. Nash, Approaches to robust pole assignment, Internat. J. Control 45 (1989), pp 97-107.

[10] J. E. Dennis and R. B. Schnabel, Numerical Methods for Unconstrained Optimization and Nonlinear Equations, Prentice-hall, Englewood Cliffs, New Jersey, 1983.